國立陽明交通大學

NATIONAL YANG MING CHIAO TUNG UNIVERSITY

Determination of mellin moments of the pion LCDA using the HOPE method

THE HOPE COLLABORATION

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with
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Outline

- Motivation: The pion electromagnetic form factor
- Pion light cone distribution amplitude (LCDA)
- Lattice determination of second Mellin moment
- Progress towards lattice determination of fourth Mellin moment
- Conclusion

Characteristic Features

$$\mathcal{L}_{\text{QCD}} = \overline{q}(i\not{D} - m)q - \frac{1}{4} G^{a}_{\mu\nu} G^{\mu\nu}_{a}$$

- Simple to write!

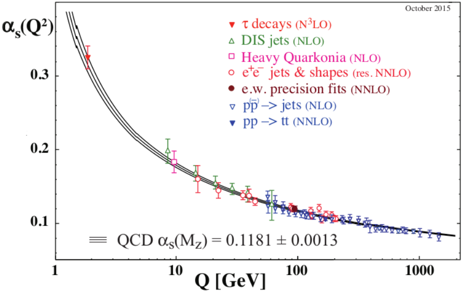


Figure 1: PDG, 2015

How do we reconcile the observed spectrum and structure of hadrons with QCD?

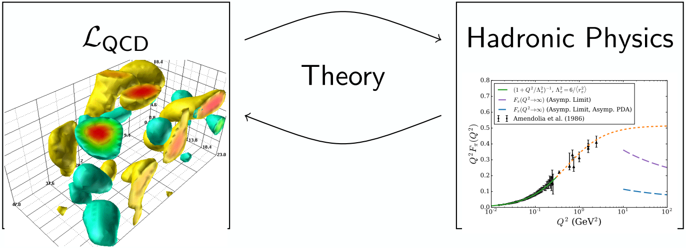
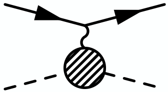


Figure 2: Image of gauge field configuration taken from J. Charvetto. See `arXiv:1903.08308`

The Pion as a testbed

- Pion complex composite object.
- Quarks charged under $U(1)$.
- Electron clean probe of quark dynamics.
- Lorentz decomposition for spin-0:

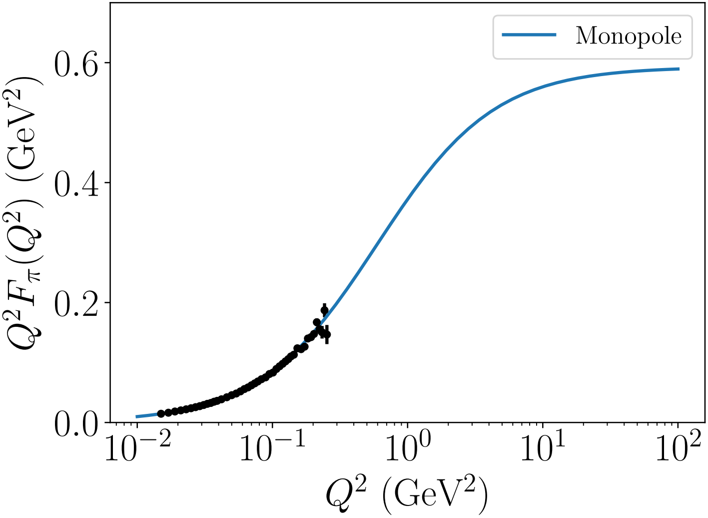


$$= L_\mu \langle \pi(p_2) | J^\mu(q) | \pi(p_1) \rangle = F_\pi(q^2) L \cdot (p_1 + p_2)$$

- VMD (space-like):

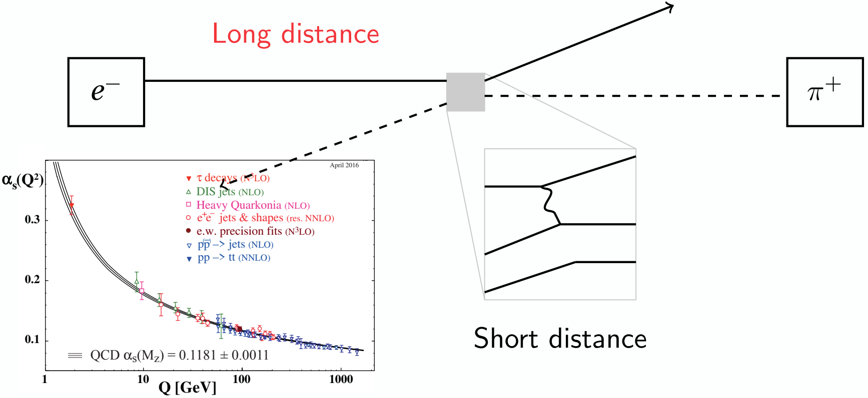
$$F_\pi(Q^2) = \frac{1}{1 + \frac{Q^2}{\Lambda^2}} = \frac{1}{1 + \frac{\langle r^2 \rangle}{6} Q^2} \approx \frac{m_\rho^2}{Q^2 + m_\rho^2}$$

$F_\pi(Q^2)$



$e + \pi$ Scattering at Large Q^2

- Cross section combination of long distance and short distance physics.



$$A = \int_0^1 \frac{d\xi}{\xi} f(\xi, \mu^2) H(\xi P) = (\text{long distance}) \otimes (\text{short distance})$$

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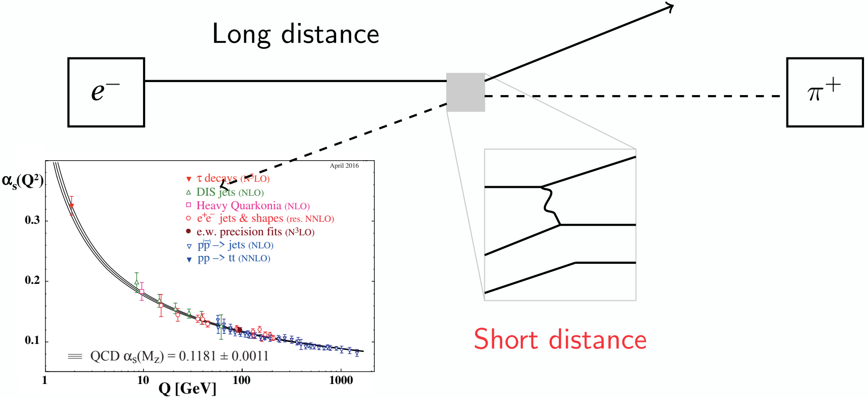

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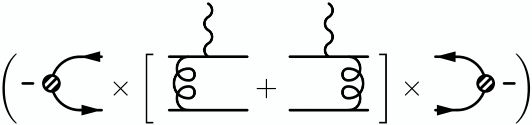
- $\phi_M(x, \mu^2)$: Light Cone Distribution Amplitude (LCDA): Probability amplitude
- Factorization theorem:

$$F_\pi(Q^2) \underset{\text{large } Q^2}{=} \int_0^1 dxdy \phi_{\overline{M}}(y, Q^2) T_H(x, y, Q^2) \phi_M(x, Q^2)$$

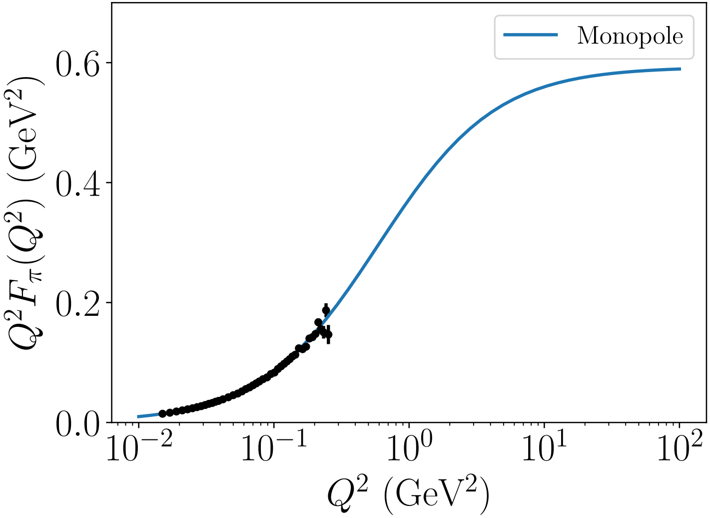
$$\underset{\text{large } Q^2}{=} \int_0^1 dxdy$$



$$\underset{\text{large } Q^2}{=} \frac{16\pi\, \alpha_S(Q^2)}{Q^2} f_\pi^2 \omega_\phi^2(Q^2) \rightarrow \frac{16\pi\, \alpha_S(Q^2)}{Q^2} f_\pi^2$$

$$\omega_\phi(Q^2) = \frac{1}{3}\int_0^1 dx \, \frac{\phi(x, Q^2)}{x}$$

A New Experimental Approach: JLab (2008)



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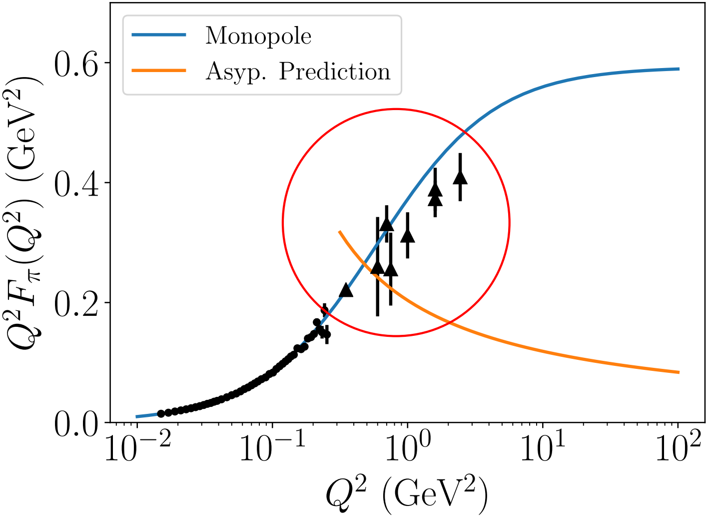
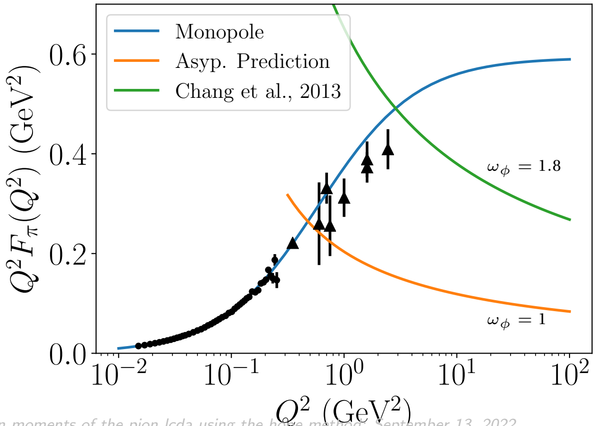


Insight from SDE

$$F_\pi(Q^2) \underset{\text{large } Q^2}{=} \frac{16\pi\alpha_S(Q^2)}{Q^2} f_\pi^2 \omega_\phi^2(Q^2) \quad , \quad \omega_\phi(Q^2) = \frac{1}{3}\int_0^1 dx\, \frac{\phi(x,Q^2)}{x}$$

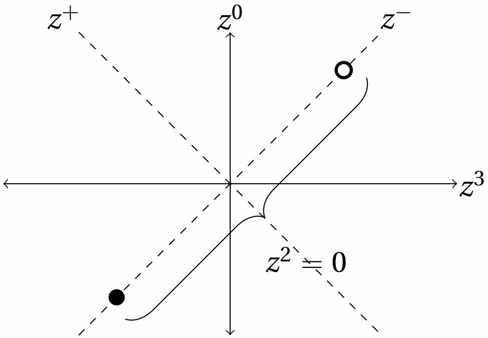


WHAT IS $\phi_\pi(x, \mu^2 \sim 2\ \mathsf{GeV}^2)$?

Calculating the Pion LCDA

- Only *ab-initio* method to calculate non-perturbative QCD: Lattice QCD.
- Problem: LCDA defined as

$$\left\langle \Omega \left| \bar{\psi}(z_-)\gamma_\mu\gamma_5 W[z_-,-z_-]\psi(-z_-) \right| \pi(\mathbf{p}) \right\rangle = ip_\mu f_\pi \int_{-1}^{1} d\xi \mathrm{e}^{i\xi p^+ \cdot z_-} \phi_\pi(\xi,\mu^2)$$

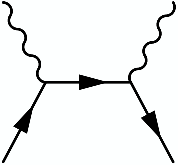


Options

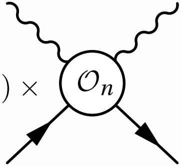
- Calculate Mellin moments directly:

$$\langle \xi^n \rangle = \int_{-1}^{1} d\xi \, \xi^n \phi_\pi(\xi, \mu^2)$$

  - G. S. Bali et al., JHEP 2019.
  - V. M. Braun, et al., PRD 2015.
- Utilize Factorization Theorem
  - X. Ji, PRL 2013.
  - A. V. Radyushkin, PRD 2017.
  - Ma, Y.-Q., Qiu, J.-W. PRD, 2018.
- Match hadronic matrix element to OPE
  - V. Braun and D. Müller, EPJC 2008.
  - W. Detmold and C. J. D. Lin, PRD 2006.
  - Chambers et al, PRL 2017



$$= \sum_n C_W^{(n)}(Q^2, \mu^2) \times$$



$$+ \mathcal{O}(1/Q)$$

- $C_W^{(n)}\left(Q^2, \mu^2\right)$ Wilson Coefficients
- Twist-2 operators:

$$\mathcal{O}_{2,n}^{\mu_1 \ldots \mu_n}(\mu) = \psi \gamma^{\{\mu_1}(iD^{\mu_2}) \ldots (iD^{\mu_n\}})\psi - \mathrm{tr}$$

- Matrix elements related to moments

$$\langle \Omega | \, \mathcal{O}_{2,n}^{\mu_1 \ldots \mu_n}(\mu) \, | \pi(\mathbf{p}) \rangle = f_\pi \left\langle \xi^{n-1} \right\rangle p^{\mu_1} \ldots p^{\mu_n}$$

OPE for Hadronic Matrix Element

- Consider matrix element

$$V^{\mu\nu}(p,q) = \int d^4 z e^{iq\cdot z} \langle\Omega| T\{J^\mu(z/2)J^\nu(-z/2)\} |\pi(\mathbf{p})\rangle$$

- Perform operator product expansion:

$$Q^2 = -q^2 \qquad \text{large scale}$$
$$\omega = \frac{1}{x} = \frac{2p\cdot q}{Q^2} \qquad \text{expansion parameter}$$

$$V^{\mu\nu}_{\text{OPE}}(p,q) = K[1 + \omega^2 \left\langle \xi^2 \right\rangle + \dots] + \underbrace{\mathcal{O}(\alpha_S)}_{\text{Perturbative corrections}} + \underbrace{\mathcal{O}(1/Q^3)}_{\text{Higher twist}}$$

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Heavy-quark OPE

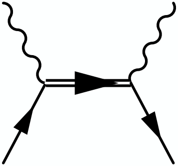
- Replace

$$J^\mu \to J^\mu_\Psi = \overline{\Psi}(x)\Gamma^\mu\psi(x) + \overline{\psi}(x)\Gamma^\mu\Psi(x)$$

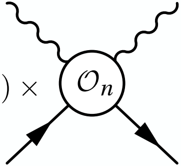
- With hierachy of scales:

$$\Lambda_{\text{QCD}} \ll m_\Psi \sim Q \ll \frac{1}{a}$$

- OPE becomes



$$= \sum_n C_W^{(n)}(Q^2, m_\Psi^2, \mu^2) \times$$



$$+ \mathcal{O}(1/Q)$$

$$V^{\mu\nu}_{\text{HOPE}}(p,q) = K[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \ldots] + \underbrace{\mathcal{O}(\alpha_S)}_{\text{Perturbative corrections}} + \underbrace{\mathcal{O}(1/\tilde{Q}^3)}_{\text{Higher twist}}$$

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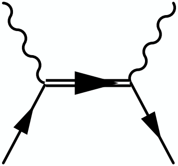

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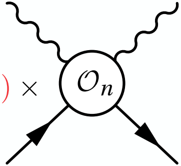

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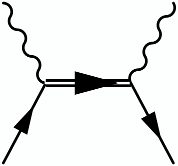

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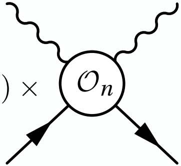

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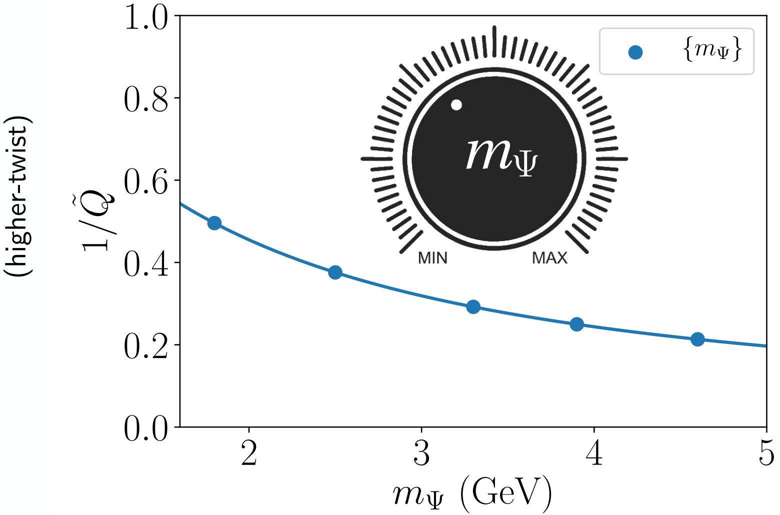
Advantage of Heavy Quark OPE



Recap

- Pion LCDA $\phi_\pi(\xi, \mu^2)$ important in description of $F_\pi(Q^2)$
- Long-range sensitive: non-perturbative.
- HOPE method allows for determination of moments:

$$\langle \xi^n \rangle (\mu^2) = \int_{-1}^{1} d\xi \, \xi^n \phi(\xi, \mu^2)$$

- HOPE Method:

$$\boxed{V^{\mu\nu}_{\text{LQCD}}(p, q; a) = \int d^4 z e^{iq\cdot z} \langle \Omega | T\{J^\mu_\Psi(z/2) J^\nu_\Psi(-z/2)\} | \pi(\mathbf{p}) \rangle}$$

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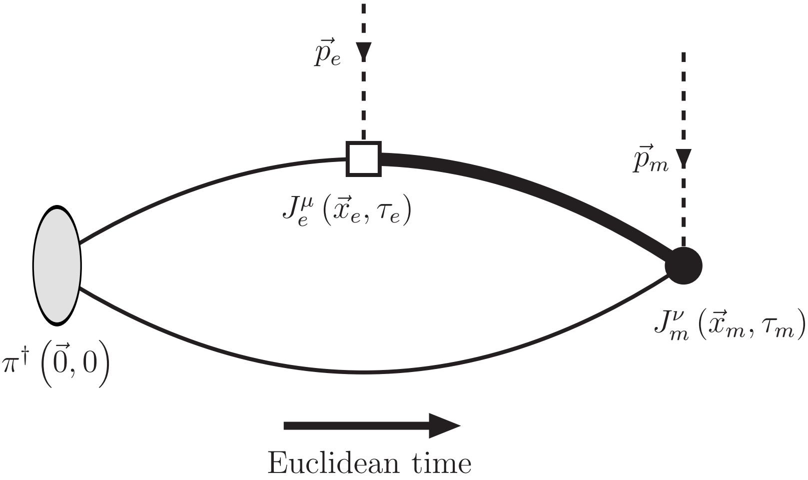
$$V^{\mu\nu}_{\mathsf{HOPE}}(p,q;a) = K[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \dots]$$

$$\langle \xi^2 \rangle$$

CALCULATING $\langle \xi^2 \rangle$

Calculating the LCDA on the Lattice

$$C_3^{\mu\nu}(\tau_e, \mathbf{p}_e, \tau_m, \mathbf{p}_m) = \int d^3x_e d^3x_m e^{i\mathbf{p}_e\cdot\mathbf{x}_e + i\mathbf{p}_m\cdot\mathbf{x}_m} \langle\Omega| T\{J_\Psi^\mu(x_e) J_\Psi^\nu(x_m) \mathcal{O}_\pi^\dagger(0)\} |\Omega\rangle$$



Calculating the LCDA on the Lattice

- Large Euclidean time

$$C_3^{\mu\nu}(\tau_e, \tau_m; \mathbf{p}_e, \mathbf{p}_m) \sim R^{\mu\nu}(\tau; \mathbf{p}, \mathbf{q}) \frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(\tau_e+\tau_m)/2} ,$$

where

$$R^{\mu\nu}(\tau; \mathbf{p}, \mathbf{q}) = \int d^3z e^{i\mathbf{q}\cdot\mathbf{z}} \langle\Omega| T\{J^\mu(z/2)J^\mu(-z/2)\} |\pi(\mathbf{p})\rangle$$
$$= \int \frac{dq_4}{2\pi} V^{\mu\nu}(p, q)$$

- identify

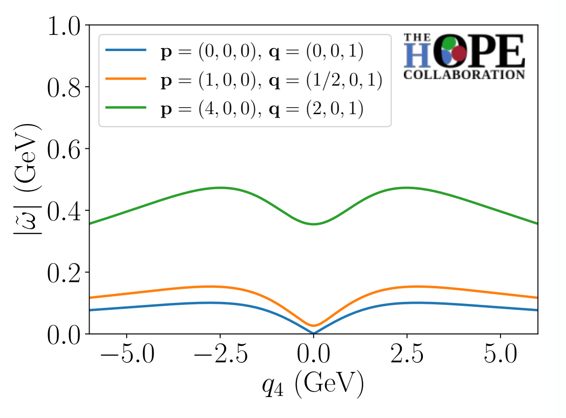
$$p_E = (iE_\pi(\mathbf{p_e} + \mathbf{p}_m), \mathbf{p}_e + \mathbf{p}_m), \qquad q_E = (q_4, (\mathbf{p}_e - \mathbf{p}_m)/2)$$

- OPE proportional to

$$V^{\mu\nu}(p,q) \approx K[1+\langle\xi^2\rangle\,\tilde{\omega}^2$$

$$\tilde{\omega} = \frac{2p\cdot q}{\tilde{Q}^2} = \frac{1}{\tilde{x}}$$

- Evaluate HOPE for $|\tilde{\omega}| < 1$.
- $\mathbf{p} = (1,0,0)$



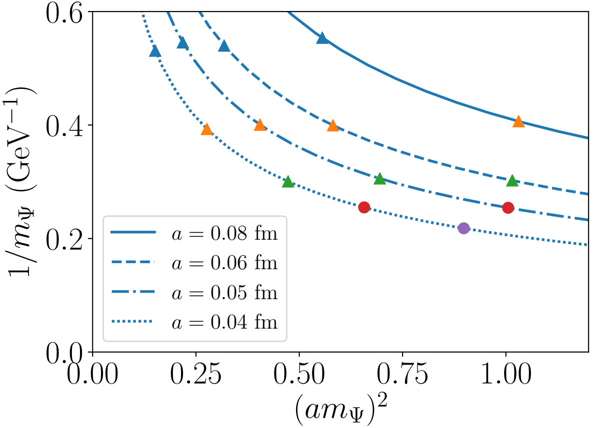
Ensembles Used

$L^3 \times T$	a (fm)	N_{cfg}	N_{src}	N_{Ψ}
$24^3 \times 48$	0.0813	650	12	2
$32^3 \times 64$	0.0600	450	10	3
$40^3 \times 80$	0.0502	250	6	4
$48^3 \times 96$	0.0407	341	10	5

- Quenched approximation with $m_\pi = 550$ MeV
- Wilson-clover fermions with non-perturbatively tuned c_{SW}
- With clover term, results fully $O(a)$ improved
  - Axial current renormalizes multiplicatively: $A^\mu \to A^\mu Z_A(1 + \tilde{b}_A a\tilde{m}_q)$
  - This only affects overall normalization (not $\langle \xi^2 \rangle$)

Higher-twist/Lattice Artifacts tradeoff

$$m_\Psi \lessapprox \frac{1}{a}$$

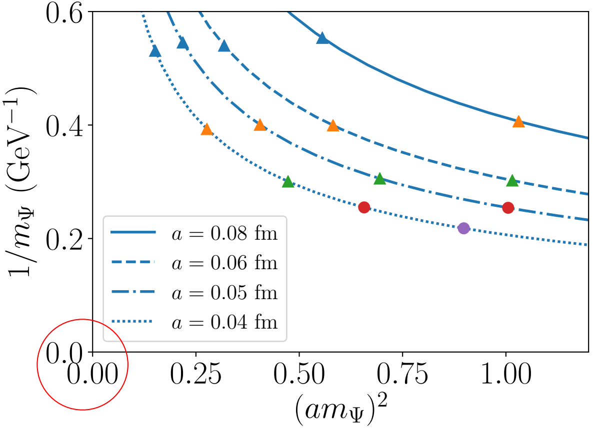


Higher-twist/Lattice Artifacts tradeoff

$$m_\Psi \lesssim \frac{1}{a}$$

Larger twist artifacts

Smaller twist artifacts

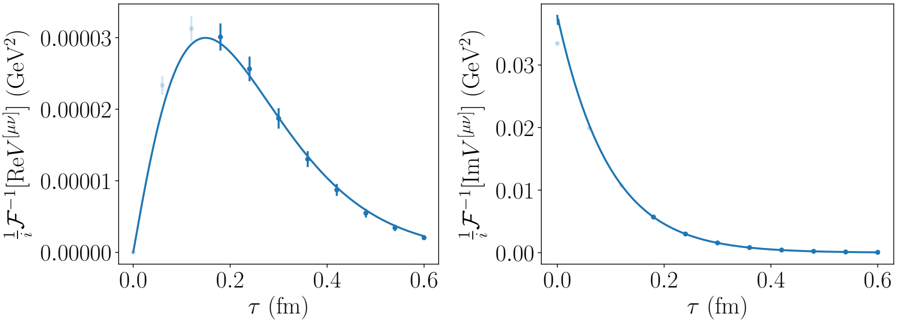


Smaller lattice artifacts

Larger lattice artifacts

Extraction of Parameters at $a = 0.06$ fm

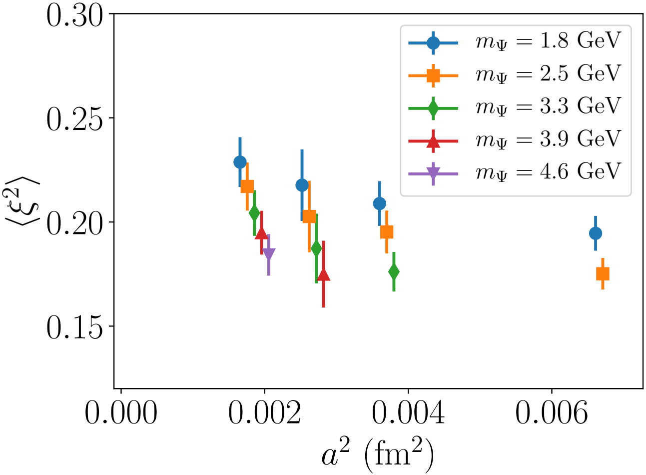
$$R^{[\mu\nu]}(\tau, \mathbf{p}, \mathbf{q}) = \int \frac{dq_4}{(2\pi)} e^{iq_4\tau} V^{[\mu\nu]}(p, q)$$



- $\mathbf{p} = (1, 0, 0),\ \mathbf{q} = (-1, 0, -2)$
- Extract parameters: $f_\pi,\ m_\Psi,\ \langle \xi^2 \rangle$

Fits to Various Ensembles

$$\langle \xi^2 \rangle (\mu^2; a, m_\Psi) = \langle \xi^2 \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2$$



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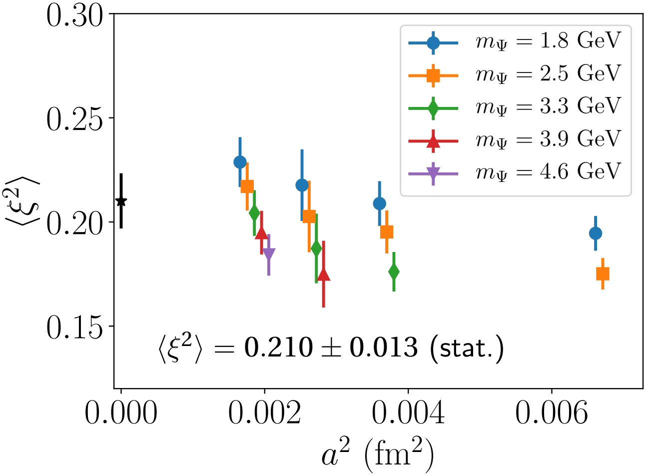
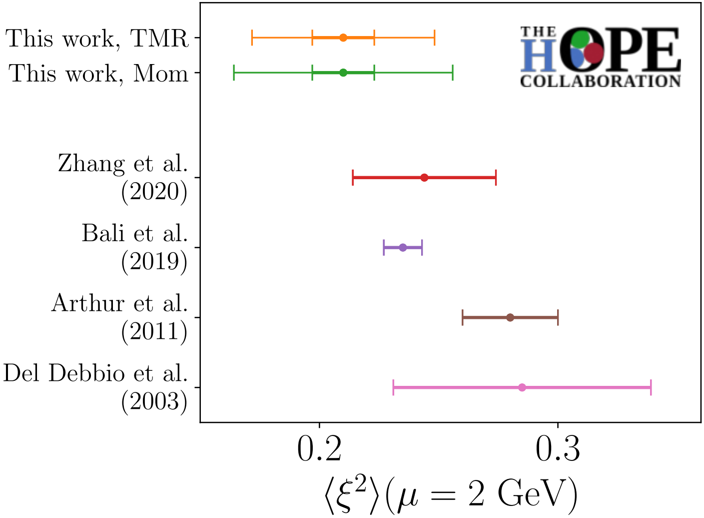


Combined Uncertainty

$$\begin{aligned}
\langle \xi^2 \rangle = 0.210 &\pm 0.013 \text{ (statistical)} \\
&\pm \mathbf{0.016} \text{ (continuum)} \\
&\pm \mathbf{0.025} \text{ (higher twist)} \\
&\pm 0.002 \text{ (excited states)} \\
&\pm 0.0002 \text{ (finite volume)} \\
&\pm 0.014 \text{ (unphysically heavy pion)} \\
&\pm 0.002 \text{ (fit range)} \\
&\pm 0.008 \text{ (running coupling)} \\
\hline
\langle \xi^2 \rangle = 0.210 &\pm 0.036 \text{ (total, exc. quenching)}
\end{aligned}$$

Comparison to Literature



Calculating $\langle \xi^4 \rangle$

Recap

$$V^{\mu\nu}_{\mathsf{LQCD}}(p, q; a) = \int d^4 z e^{iq\cdot z} \langle\Omega| T\{J^{\mu}_{\Psi}(z/2) J^{\nu}_{\Psi}(-z/2)\} |\pi(\mathbf{p})\rangle$$

$$V^{\mu\nu}_{\mathsf{HOPE}}(p, q; a) = K[1 + \tilde{\omega}^2 \left\langle \xi^2 \right\rangle + \tilde{\omega}^4 \left\langle \xi^4 \right\rangle + \ldots]$$

$$\left\langle \xi^2 \right\rangle, \ \left\langle \xi^4 \right\rangle$$

Recap

$$V^{\mu\nu}_{\mathsf{LQCD}}(p, q; a) = \int d^4 z e^{iq\cdot z} \langle\Omega| \, T\{J^{\mu}_{\Psi}(z/2) J^{\nu}_{\Psi}(-z/2)\} \, |\pi(\mathbf{p})\rangle$$

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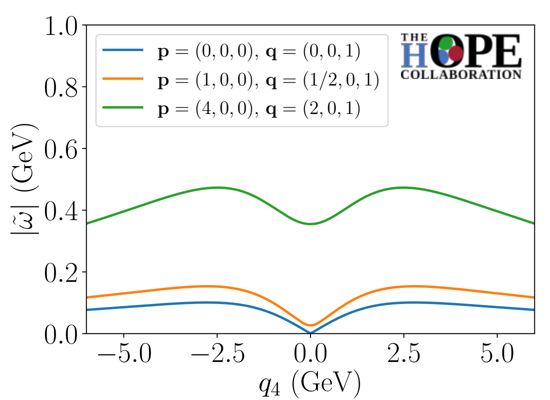
$$\boxed{\langle\xi^2\rangle\,,\ \ \langle\xi^4\rangle}$$

Optimizing Kinematics

$$V^{\mu\nu}(p,q) = K[1 + \tilde{\omega}^2 \langle \xi^2 \rangle + \tilde{\omega}^4 \langle \xi^4 \rangle + \ldots], \qquad \tilde{\omega} = \frac{2p \cdot q}{\tilde{Q}^2}$$

- Choose $\mathbf{p} = (2,0,0) \times 2\pi/L$



Lattice Details

$L^3 \times T$	a (fm)	N_{cfg}	N_Ψ
$24^3 \times 48$	0.0813	6500	2
$32^3 \times 64$	0.0600	4500	3
$40^3 \times 80$	0.0502	$\mathcal{O}(5000)$	4
$48^3 \times 96$	0.0407	$\mathcal{O}(5000)$	5

- Quenched approximation with $m_\pi = 550$ MeV
- Wilson-clover fermions with non-perturbatively tuned c_{SW}
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Still to come...

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Ratio Method

- Excited state dependent only on sum $t_e + t_m$.

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q}) \frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2} + \ldots,$$

- Define $t_+ = t_e + t_m$, $t_- = t_e - t_m$.

$$C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) = R^{\mu\nu}(t_-; \mathbf{p}, \mathbf{q}) \frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})t_+/2} + \ldots,$$

- Consider two sets of time; (t_e, t_m) and $(t'_e, t'_m) = (t_e + \delta, t_m - \delta)$

$$\begin{aligned} t'_+ &= (t_e + \delta) + (t_m - \delta) = t_e + t_m = t_+ \\ t'_- &= (t_e + \delta) - (t_m - \delta) = t_e - t_m + 2\delta \neq t_- \end{aligned}$$

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Ratio Method

- Construct ratio with fixed t_+, varying t_- $(\delta = -1)$

$$\mathcal{R} = \frac{C_3^{\mu\nu}(t_e - 1, t_m + 1; \mathbf{p}_e, \mathbf{p}_m)}{C_3^{\mu\nu}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m)}$$
$$= \frac{R^{\mu\nu}(t_e - t_m - 2; \mathbf{p}, \mathbf{q})}{R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q})} \frac{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e + t_m)/2}}{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e + t_m)/2}} \left[1 + \dots\right]$$

- Need two t_e, ie t_e and $t_e - 1$
- No need for 2-point data!
- No renormalization required.
- No f_π

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Operator Optimization

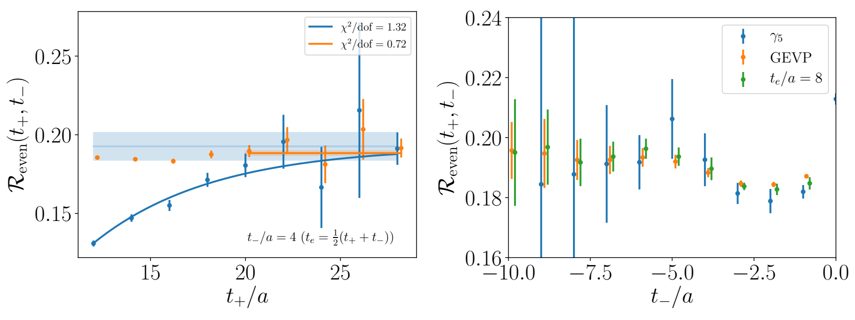
- Momentum smearing (Bali et al)
- Variational analysis:

$$\mathcal{O}_\pi(x) = c_1\mathcal{O}_1(x) + c_2\mathcal{O}_2(x), \quad \mathcal{O}_1(x) = \overline{\psi}\gamma_5\psi, \quad \mathcal{O}_2(x) = \overline{\psi}\gamma_4\gamma_5\psi$$

$$\begin{aligned} C^{\mu\nu}_{3,\mathsf{GEVP}}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) &= c_1 C^{\mu\nu}_{3,\gamma_5}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) \\ &+ c_2 C^{\mu\nu}_{3,\gamma_4\gamma_5}(t_e, t_m; \mathbf{p}_e, \mathbf{p}_m) \end{aligned}$$

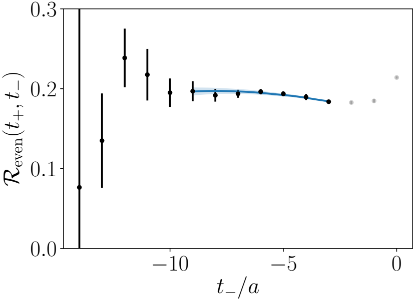
Excited State Contamination

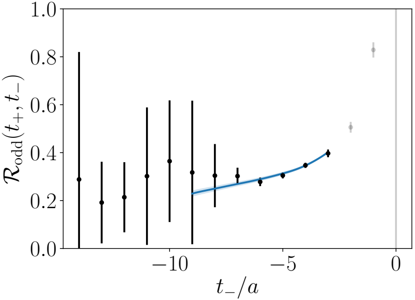
$$\mathcal{R} == \frac{R^{\mu\nu}(t_e - t_m - 2; \mathbf{p}, \mathbf{q})}{R^{\mu\nu}(t_e - t_m; \mathbf{p}, \mathbf{q})} \frac{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}}{\frac{Z_\pi(\mathbf{p})}{2E_\pi(\mathbf{p})} e^{-E_\pi(\mathbf{p})(t_e+t_m)/2}} \left[1 + \dots\right]$$



Example at fixed lattice spacing

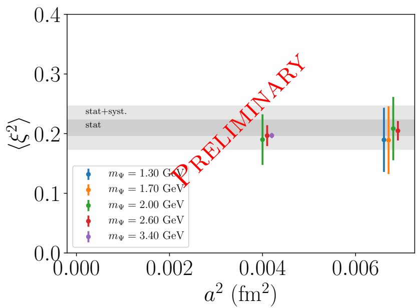
$$\mathcal{R}(t_-, \mathbf{p}, \mathbf{q}, m_\psi, \langle \xi^2 \rangle, \langle \xi^2 \rangle; a) = \mathcal{R}_{\mathsf{HOPE}}(t_-, \mathbf{p}, \mathbf{q}, m_\psi, \langle \xi^2 \rangle, \langle \xi^2 \rangle) + \mathcal{O}(a^2)$$

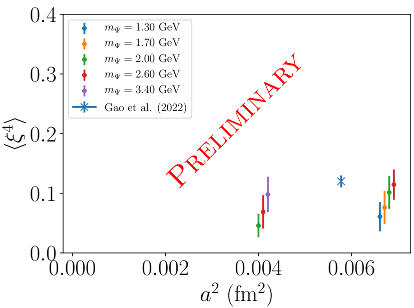




$L/a = 24$, $m_\Psi = 2.0$ GeV, $\langle \xi^2 \rangle = 0.17 \pm 0.04$, $\langle \xi^4 \rangle = 0.07 \pm 0.02$

Status of Calculation





$$\langle \xi^n \rangle (a, m_\Psi) = \langle \xi^n \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2$$

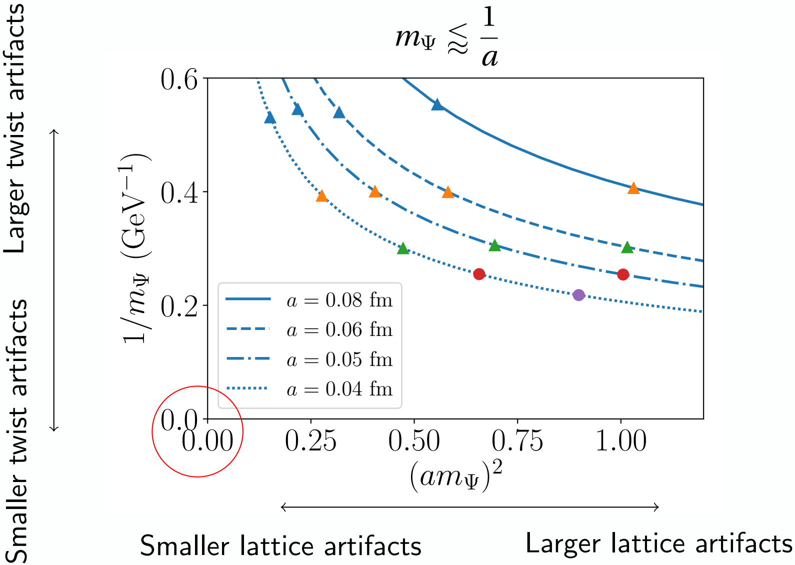
Further Work & Conclusions

- First numerical determination of LCDA using HOPE.
  - $\langle \xi^2 \rangle = 0.210 \pm 0.036$ in good agreement with other groups.
- Introduced ratio method: no renormalzation.
  - $\langle \xi^4 \rangle$: 2 lattice spacings, 3 heavy quark masses; still taking data. Result seems in reasonable agreement with Gao (2022).

Backup Slides

Higher-twist/Lattice Artifacts tradeoff

$$m_\Psi \lessapprox \frac{1}{a}$$



- In asymptotically free theory, quark counting argument predicts

$$F_{\pi}(Q^2) \sim Q^{-2}, \qquad F_K(Q^2) \sim Q^{-2}$$

DEEP ELASTIC PROCESSES OF COMPOSITE PARTICLES IN FIELD THEORY AND ASYMPTOTIC FREEDOM*

A.V. Radyushkin**

** The investigation has been performed (and completed in June 1977) at the Laboratory of Theoretical Physics, JINR, Dubna, Russian Federation*

English translation and comments: October 2004

*** Present address: Physics Department, Old Dominion University, Norfolk, VA 23529, USA and Theory Group, Jefferson Lab, Newport News, VA 23606, USA*

This is an English translation of my 1977 Russian preprint. It contains the first explicit definition of the pion distribution amplitude (DA), the expression for the pion form factor asymptotics in terms of the pion DA, and formulates the pQCD parton picture for hard exclusive processes.
Abstract of the original paper:
The large Q^2 behavior of the pion electromagnetic form factor is explicitly calculated in the non-Abelian gauge theory to demonstrate a field-theoretical approach to the deep elastic processes of composite particles. The approach is equivalent to a new type of parton model.

HYSICAL REVIEW D VOLUME 22, NUMBER 9 1 NOVEMBER 1980

Exclusive processes in perturbative quantum chromodynamics

G. Peter Lepage
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York 14853

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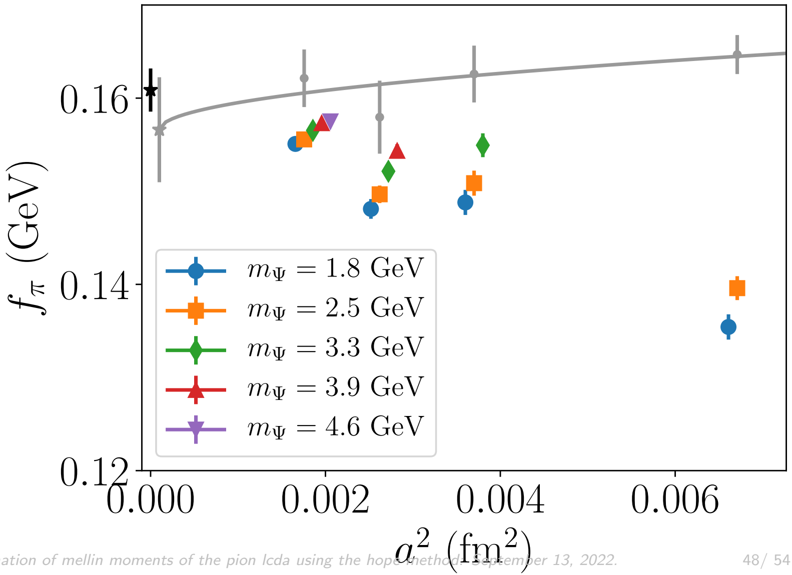
Stanford Linear Accelerator Center, Stanford University, Stanford, California 94305

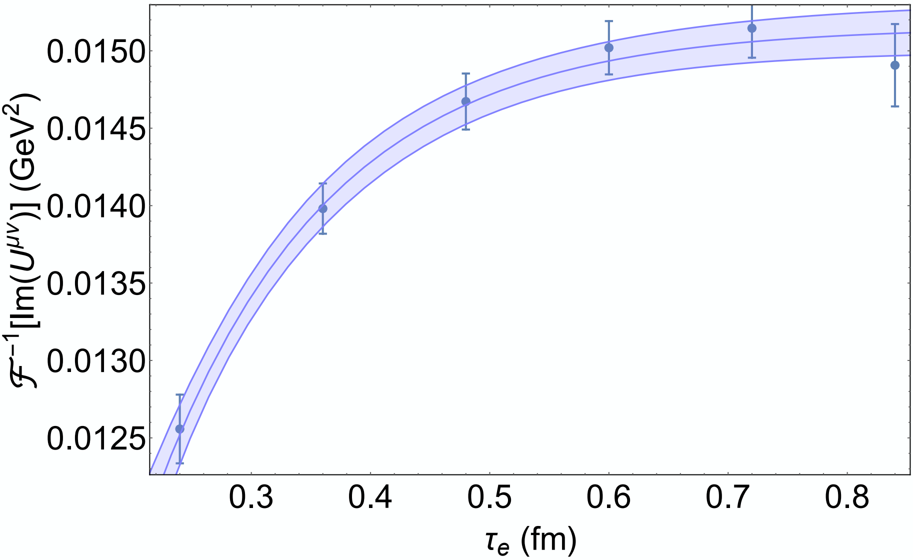
(Received 27 May 1980)

We present a systematic analysis in perturbative quantum chromodynamics (QCD) of large-momentum-transfer exclusive processes. Predictions are given for the scaling behavior, angular dependence, helicity structure, and normalization of elastic and inelastic form factors and large-angle exclusive scattering amplitudes for hadrons and photons. We prove that these reactions are dominated by quark and gluon subprocesses at short distances, and thus that the dimensional-counting rules for the power-law falloff of these amplitudes with momentum transfer are rigorous predictions of QCD, modulo calculable logarithmic corrections from the behavior of the hadronic wave functions at short distances. These anomalous-dimension corrections are determined by evolution equations for process-independent meson and baryon "distribution amplitudes" $\phi(x_i, Q)$ which control the valence-quark distributions in high-momentum-transfer exclusive reactions. The analysis can be carried out systematically in powers of $\alpha_s(Q^2)$, the QCD running coupling constant. Although the calculations are most conveniently carried out using light-cone perturbation theory and the light-cone gauge, we also present a gauge-independent analysis and relate the distribution amplitude to a gauge-invariant Bethe-Salpeter amplitude.

f_π Determination



Excited States

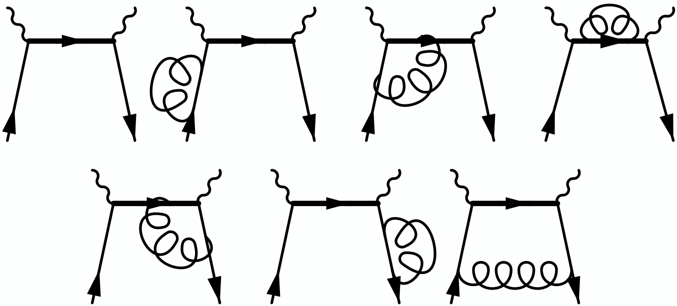


- $\tau_m - \tau_e$ fixed at 0.06 fm

$$V^{\mu\nu}(p,q) = \frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta}{\tilde{Q}^2} \sum_{\text{even}} \tilde{\omega}^n C_W^{(n)}(\tilde{Q}^2/\mu^2, \alpha_S(\mu^2)) \langle \xi^n \rangle (\mu^2) + \mathcal{O}(1/\tilde{Q}^3)$$

- Heavy quark: have to recalculate these coefficients.

$$T_q^{\mu\nu}(p,q) = \int d^4 z e^{iq\cdot z} \langle 0| T\{J^\mu(z/2) J^\nu(-z/2)\} \left| u(p_1,\uparrow); \overline{d}(p_2,\downarrow) \right\rangle$$



HOPE: Order-α_S Improved

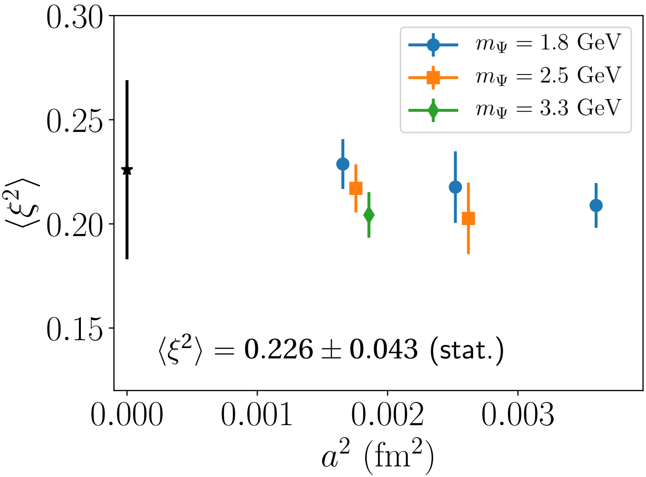
- We fit to

$$V^{\mu\nu}(p,q) = \frac{2if_\pi \epsilon^{\mu\nu\alpha\beta} q_\alpha p_\beta}{\tilde{Q}^2} \sum_{\text{even}} C_W^{(n)}\left(\tilde{Q}^2, \mu, \tau\right) \langle \xi^n \rangle \left[\frac{\zeta^n C_n^2(\eta)}{2^n(n+1)}\right]$$

- where $\tau = Q^2/\tilde{Q}^2$, $\zeta = \sqrt{p^2 q^2}/\tilde{Q}^2$, $\eta = p \cdot q/\sqrt{p^2 q^2}$

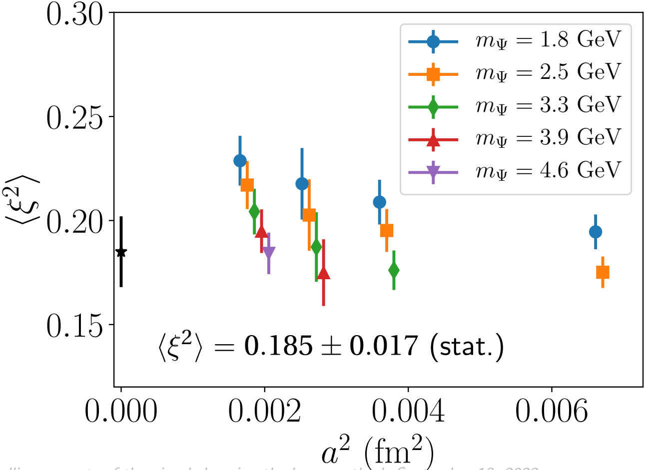
Uncertainty in Continuum Extrapolation

- Original fit restricted am_Ψ to < 1.05
- Could take a more conservative threshold, e.g. $am_\Psi < 0.7$



Uncertainty in Higher-Twist Effects

$$\langle \xi^2 \rangle (\mu^2; a, m_\Psi) = \langle \xi^2 \rangle + \frac{A}{m_\Psi} + Ba^2 + Ca^2 m_\Psi + Da^2 m_\Psi^2 + \frac{E}{m_\Psi^2}$$



Remaining Uncertainties

- Excited state contamination: estimated at 1%
- Finite volume effects: $m_\pi L = 5.4 \Rightarrow \frac{1}{m_\pi L} e^{-m_\pi L} = \mathbf{0.08}\%$
- Unphysical pion mass ($m_\pi = 550$ MeV): Likely a $\sim \mathbf{5}\%$ error (V. M. Braun et al., hep-lat/1503.03656)
- Fit range: Excluding $\tau = 3a$ as well gives discrepancy of 1%
- Wilson coefficients: Performing fit at $\mu = 4$ GeV and running back to 2 GeV gives discrepancy of 4%
- Quenching: Formally uncontrolled, typically around 10–20%

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