Controllability for a pseudoparabolic equation

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We are interested in the null controllability of the following pseudoparabolic equation:

$$\left|\begin{array}{lll} u_t - \Delta u - \Delta u_t = f 1_\omega & \text{in} & Q := (0,T)\times\Omega, \\ u = 0 & \text{on} & \Sigma := (0,T)\times\partial\Omega, \\ u(0) = u_0 & \text{in} & \Omega. \end{array}\right. \tag{1}$$

More precisely, given u_0, find f such that the solution of (1) satisfies:

$$u(T) = 0.$$

Here, $\Omega \subset \mathbb{R}^{\mathbb{N}}$ and $\omega \subset \Omega$ is a nonempty subset.

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Equation (1) is an example of the general class of equations of Sobolev type, sometimes referred to as of Sobolev-Galpern type. These are characterized by having mixed time and space derivatives appearing in the highest order terms of the equation.

Some properties of (1)

- There is a one parameter group $\{E(t);\ t \in \mathbb{R}\}$.
- For every $k \in \mathbb{N}$, the space $H_0^1(\Omega) \cap H^k(\Omega)$ is invariant under $E(t)$.
- In space,the solution is just as smooth as the initial data allow it to be.
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A nice decomposition

Equation (1) can be decomposed as

$$\left| \begin{array}{lll} u - \Delta u = v & \text{in} & Q, \\ v_t + v - u = f 1_\omega & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ v(0) = u_0 - \Delta u_0 & \text{in} & \Omega. \end{array} \right. \qquad (2)$$

We have:

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Lack of controllability with fixed control

For a given (fixed) $\omega \subset \Omega$, system (1) fails to be controllable.

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Remedy: Moving Control

Make the control in the second equation of (2) move (i.e., $\omega = \omega(t)$) or, equivalently, replace the ODE by a transport equation.

The set $\omega(t)$ covers the whole domain Ω in its motion.

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This strategy has been previously used for other models:

P. Martin, L. Rosier, P. Rouchon, Null Controllability of the Structurally Damped Wave Equation with Moving Control, SIAM J. Control Optim., 51 (1)(2013), 660–684.

F. W. Chaves-Silva, L. Rosier, and E. Zuazua, Null controllability of a system of viscoelasticity with a moving control, J. Math. Pures Appl. (9), 101 (2014), 198–222.

L. Rosier, B.-Y. Zhang, Unique continuation property and control for the Benjamin-Bona-Mahony equation on a periodic domain, J. Differential Equations 254 (2013), 141–178.

With moving controls, equation (1) and system (2) read

$$\left| \begin{array}{lll} u_t - \Delta u - \Delta u_t = f 1_{\omega(t)} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u(0) = u_0 & \text{in} & \Omega \end{array} \right. \tag{3}$$

and

$$\left| \begin{array}{lll} u - \Delta u = v & \text{in} & Q, \\ v_t + v - u = f 1_{\omega(t)} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ v(0) = v_0 & \text{in} & \Omega, \end{array} \right. \tag{4}$$

respectively.

And, still, we want to find f such that $u(T) = v(T) = 0$.

1-d case

For the $1 - d$ case, using moment method, Q. Tao et al.[1], showed the null controllability of the pseudoparabolic equation (3) .

[1] Q. Tao, H. Gao, Z. Yao, Null controllability of a pseudo-parabolic equation with moving control, J. Math. Analysis and Appl., 418 (2)(2014)

N-dimensional case

The adjoint system of (2) reads

$$\left| \begin{array}{ll} z - \Delta z = \xi & \text{in } \; Q, \\ -\xi_t + \xi = z & \text{in } \; Q, \\ z = 0 & \text{on } \; \Sigma, \\ \xi(T) = \xi_T & \text{in } \; \Omega. \end{array} \right. \tag{5}$$

Null controllability of (2) is equivalent to

$$||\xi(0)||^2_{L^2(\Omega)} \leq C \int_0^T \int_{\omega(t)} |\xi|^2 dxdt. \tag{6}$$

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Carleman inequality

Theorem (C-S & Souza-2015)

Given $\xi_T \in L^2(\Omega)$, the solution (z, ξ) of system (5) satisfies:

$$\begin{aligned}
&\int_0^T\int_\Omega \rho_1(x,t)(|\nabla z|^2 + |z|^2)dxdt + \int_0^T\int_\Omega \rho_2(x,t)|\xi|^2 dxdt \\
&+ \int_0^T\int_\Omega \rho_3(t)(|\nabla z_t|^2 + |z_t|^2)dxdt \\
&\leq C\int_0^T\int_{\omega(t)} \rho_4(x,t)|\xi|^2 dxdt, \qquad (7)
\end{aligned}$$

where ρ_i, $i = 1, \ldots, 4$ are appropriate weights.

Idea of the proof

Our strategy to prove the Carleman inequality is based on the use of Carleman inequalities for the Laplace operator and the ODE + Elliptic estimates.

Three main difficulties appear:

1. Carleman inequalities for the Laplace operator and ODE equations with a moving control region[2];
2. We must have the same weight functions in the Carleman for both equations[3].
3. Eliminate a local integral of z.

Fortunately, we can handle all difficulties.

[2] F. W. Chaves-Silva, L. Rosier, and E. Zuazua, Null controllability of a system of viscoelasticity with a moving control, J. Math. Pures Appl. (9), 101 (2014), 198–222.

[3] P. Albano, D. Tataru, Carleman estimates and boundary observability for a coupled parabolic-hyperbolic system, Electron. J. Differential Equations, 22 (2000), 1–15.

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Controllability Result I

Theorem (C-S & Souza-2015)

Given $v_0 \in L^2(\Omega)$, there exists $f \in L^2(Q)$ such that the associated solution (u, v) of (4) satisfies:

$$v(T) = u(T) = 0.$$

Controllability Result II

Corollary

Given $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$*, there exists* $f \in L^2(Q)$ *such that the associated solution* u *of* (3) *satisfies*

$$u(T) = 0.$$

Generalized Benjamin-Bona-Mahony equation

Similar ideas (but not the same!) can be used to study the controllability of

$$\left| \begin{array}{lll} u_t - \Delta u_t - div(\phi(u)) = f 1_{\omega(t)} & \text{in} & Q, \\ u = 0 & \text{on} & \Sigma, \\ u(0) = u_0 & \text{in} & \Omega, \end{array} \right. \tag{8}$$

when $div(\phi(u)) = uA$, for a constant vector field A.

In particular, for the $1-d$ case, this gives an alternate proof of the controllability of the linearized BBM using moving control[4] [5].

[4] S. Micu, *On the controllability of the linearized Benjamin-Bona-Mahony equation, SIAM J. Control Optim.*, **39** (2001), 1677–1696.

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Open problem

Controllability results for general Sobolev-Galpern type equations

$$M\partial_t u + Lu = f,$$

where M and L are elliptic operators?

Thank you!