On the asymptotic stabilization of a generalized hyperelastic-rod wave equation

Fabio ANCONA

University of Padova

Summer School & Workshop
PDEs: Optimal Design and Numerics'

Centro de Ciencias de Benasque Pedro Pasqual
August 23 - Septemer 4, 2015

Outline

1. General setting and physical motivations
2. A stabilization problem
3. A semigroup of exponentially decaying weak solns
4. Hyperelastic-rod wave equation with more general source

Nonlinear dispersive wave equation

$$\partial_t u - \partial^3_{txx} u + \partial_x \left(\frac{g(u)}{2} \right) = \gamma \left(2\partial_x u \, \partial^2_{xx} u + u \, \partial^3_{xxx} u \right),$$

- $t \geq 0$ time
- $x \in \mathbb{R}$ space (one-dimensional)
- $u(t, x) \in \mathbb{R}$ unknown (one-dimensional)
- $\gamma > 0$ is a given constant
- $g : \mathbb{R} \to \mathbb{R}$ smooth map

The models

$$g(u) = 3u^2$$

Hyperelastic-rod wave equation

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u \partial^2_{xx} u + u \partial^3_{xxx} u \right)$$

- finite length, small amplitude waves
- $u(t,x)$: radial deformation in cylindrical compressible hyperelastic rod
- γ: constant depending on the material and on the pre-stress of the rod

- Dai (1998 - 1998)
- Dai & Huo (2002)

$$g(u) = 2\kappa u + 3u^2, \quad \gamma = 1$$

Camassa-Holm equation

$$\partial_t u - \partial^3_{txx} u + 3u\partial_x u + 2\kappa\partial_x u = \left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u\right)$$

Unidirectional Shallow Water Waves

$\Updownarrow$

Depth of the water $\ll$ Length of the waves

- $u(t, x)$: wave velocity above the bottom
- flat bottom
- $\kappa > 0$: water depth

- Camassa & Holm (1993)
- Johnson (2002)

No (global in time) classical solutions

$$\partial_t u - \partial^3_{txx} u + \partial_x \left(\frac{g(u)}{2} \right) - \gamma \left(2\partial_x u \, \partial^2_{xx} u + u \, \partial^3_{xxx} u \right) = 0 \qquad \text{(GHR)}$$

Remark

Solutions to (GHR)

- *may produce wave breaking: spatial derivatives of sol'ns become unbounded in finite time*
- *experience presence of peakons and antipeakons: solitary waves (travelling waves decaying at infinity) with discontinuous first derivative*
- *when peakons and antipeakons collide annihilating each other two scenarios are possible:*
  - *conservative sol'ns (a switching phenomena occurs: the waves pass through each other $\Longrightarrow$ energy is preserved)*
  - *dissipative sol'ns (total annihilation at collision time: null solution after annihilation $\Longrightarrow$ loss of energy)*

No (global in time) classical solutions

$$\partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = 0 \qquad \text{(GHR)}$$

Remark

Solutions to (GHR)

- *may produce wave breaking: spatial derivatives of sol'ns become unbounded in finite time*
- *experience presence of peakons and antipeakons: solitary waves (travelling waves decaying at infinity) with discontinuous first derivative*
- *when peakons and antipeakons collide annihilating each other two scenarios are possible:*
  - *conservative sol'ns (a switching phenomena occurs: the waves pass through each other $\implies$ energy is preserved)*
  - *dissipative sol'ns (total annihilation at collision time: null solution after annihilation $\implies$ loss of energy)*

No (global in time) classical solutions

$$\partial_t u - \partial^3_{txx} u + \partial_x \left(\frac{g(u)}{2} \right) - \gamma \left(2\partial_x u \, \partial^2_{xx} u + u \, \partial^3_{xxx} u \right) = 0 \qquad \text{(GHR)}$$

Remark

Solutions to (GHR)

- *may produce wave breaking: spatial derivatives of sol'ns become unbounded in finite time*
- *experience presence of peakons and antipeakons: solitary waves (travelling waves decaying at infinity) with discontinuous first derivative*
- *when peakons and antipeakons collide annihilating each other two scenarios are possible:*
  - *conservative sol'ns (a switching phenomena occurs: the waves pass through each other $\Longrightarrow$ energy is preserved)*
  - *dissipative sol'ns (total annihilation at collision time: null solution after annihilation $\Longrightarrow$ loss of energy)*

No (global in time) classical solutions

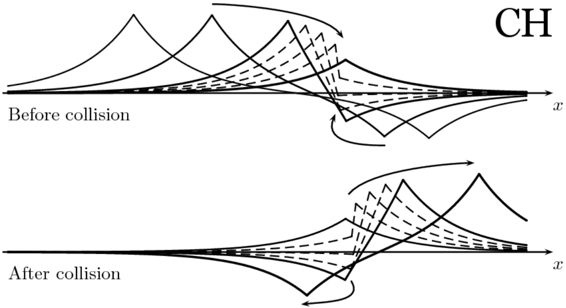
$$\partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = 0 \qquad \text{(GHR)}$$

Remark

Solutions to (GHR)

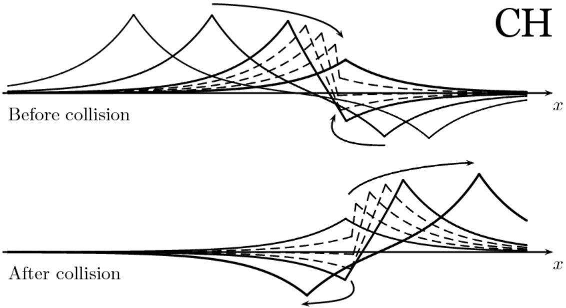
- *may produce wave breaking: spatial derivatives of sol'ns become unbounded in finite time*
- *experience presence of peakons and antipeakons: solitary waves (travelling waves decaying at infinity) with discontinuous first derivative*
- *when peakons and antipeakons collide annihilating each other two scenarios are possible:*
    - *conservative sol'ns (a switching phenomena occurs: the waves pass through each other $\Longrightarrow$ energy is preserved)*
    - *dissipative sol'ns (total annihilation at collision time: null solution after annihilation $\Longrightarrow$ loss of energy)*

Peakons and antipeakons collision for conservative sol'ns:



$\Longrightarrow$ look for weak solutions

Peakons and antipeakons collision for conservative sol'ns:



$\Longrightarrow$ look for weak solutions

The generalized hyperelastic-rod wave equation

$$\partial_t u - \partial^3_{txx} u + \partial_x\left(\tfrac{g(u)}{2}\right) - \gamma\left(2\partial_x u\,\partial^2_{xx} u + u\,\partial^3_{xxx} u\right) = 0 \qquad \text{(GHR)}$$

rewritten as

$$(1-\partial^2_{xx})\partial_t u + \gamma(1-\partial^2_{xx})(u\partial_x u) + \partial_x\left(\frac{g(u)-\gamma\left(u^2-(\partial_x u)^2\right)}{2}\right) = 0$$

is formally equivalent to the elliptic-hyperbolic system

$$\begin{cases} \partial_t u + \gamma\,\partial_x\left(\tfrac{u^2}{2}\right) + \partial_x P = 0, \\ -\partial^2_{xx} P + P = \dfrac{g(u)-\gamma\left(u^2-(\partial_x u)^2\right)}{2}. \end{cases} \qquad \text{(E-H)}$$

Since

$$\frac{e^{-|x|}}{2}$$

is the Green's function of the Helmholtz operator $-\partial_{xx}^2 + 1$, one can recast the elliptic-hyperbolic system (E-H) as a balance law with a nonlocal source term

$$\partial_t u + \gamma\, \partial_x \left(\tfrac{u^2}{2}\right) + \partial_x P[u] = 0,$$

with

$$P[u] \doteq \frac{e^{-|x|}}{2} * \left(\frac{g(u) - \gamma\big(u^2 - (\partial_x u)^2\big)}{2} \right).$$

Notice:

$$u \in H^1(\mathbb{R}) \qquad \Longrightarrow \qquad P[u] \in H^1(\mathbb{R})$$

Weak solutions

Thus we say that a Lipschitz continuous map

$$t \mapsto u(t) \in H^1(\mathbb{R}), \qquad t \geq 0,$$

is a weak solution of the elliptic-hyperbolic system if it satisfies the equality between functions of $L^2(\mathbb{R})$

$$\frac{d}{dt} u = -\gamma u \partial_x u - \partial_x P[u], \qquad \text{for a.e. } t \geq 0,$$

Asymptotic Stabilization

Hyperelastic-rod wave equation

$$g(u) = 3u^2$$

Problem

Find an operator

$$f : H^1(\mathbb{R}) \longrightarrow H^{-1}(\mathbb{R})$$

such that for every initial condition $u_0 \in H^1(\mathbb{R})$ the solution of the Cauchy problem

$$\begin{cases} \partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma\left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u\right) + f[u] \\ u(0,x) = u_0(x) \end{cases}$$

decays as $t \longrightarrow \infty$, i.e.,

$$\lim_{t \longrightarrow \infty} \|u(t)\|_{H^1} = 0.$$

Goal

Damp the waves on hyperelastic rods

- source term $f[u] \equiv$ external force

Literature (on control problems)

- O. Glass (2008): compactly supported, source type feedback, H^2 sol'ns
- V. Perrollaz (2010): boundary feedback, H^2 solutions

H^1 Weak solutions

- Exhibit unbounded and discontinuos spatial derivatives
- Solitary waves: peakons and antipeakons
- Interactions of peakons and antipeakons may occur

Goal

Damp the waves on hyperelastic rods

- source term $f[u] \equiv$ external force

Literature (on control problems)

- O. Glass (2008): compactly supported, source type feedback, H^2 sol'ns
- V. Perrollaz (2010): boundary feedback, H^2 solutions

H^1 Weak solutions

- Exhibit unbounded and discontinuos spatial derivatives
- Solitary waves: peakons and antipeakons
- Interactions of peakons and antipeakons may occur

Goal

Damp the waves on hyperelastic rods

- source term $f[u] \equiv$ external force

Literature (on control problems)

- O. Glass (2008): compactly supported, source type feedback, H^2 sol'ns
- V. Perrollaz (2010): boundary feedback, H^2 solutions

H^1 Weak solutions

- Exhibit unbounded and discontinuos spatial derivatives
- Solitary waves: peakons and antipeakons
- Interactions of peakons and antipeakons may occur

Our feedback law

$$f[u] = -\lambda(1 - \partial_{xx}^2)u, \qquad \lambda > 0$$

$$\partial_t u - \partial_{txx}^3 u + 3u\partial_x u = \gamma\left(2\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u\right) - \lambda(1 - \partial_{xx}^2)u$$

$$\Updownarrow$$

$$\begin{cases} \partial_t u + \gamma u \partial_x u + \partial_x P = -\lambda u \\ -\partial_{xx}^2 P + P = \frac{3-\gamma}{2} u^2 + \frac{\gamma}{2}(\partial_x u)^2 \end{cases}$$

$$\Updownarrow$$

$$\frac{d}{dt} u = -\gamma u \partial_x u - \partial_x P[u] - \lambda u$$

$$P[u] \doteq \frac{e^{-|x|}}{2} * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2}(\partial_x u)^2\right)$$

Our feedback law

$$f[u] = -\lambda(1 - \partial_{xx}^2)u, \qquad \lambda > 0$$

$$\partial_t u - \partial_{txx}^3 u + 3u\partial_x u = \gamma\left(2\partial_x u \partial_{xx}^2 u + u\partial_{xxx}^3 u\right) - \lambda(1 - \partial_{xx}^2)u$$

$$\Updownarrow$$

$$\begin{cases} \partial_t u + \gamma u \partial_x u + \partial_x P = -\lambda u \\ -\partial_{xx}^2 P + P = \frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \end{cases}$$

$$\Updownarrow$$

$$\frac{d}{dt} u = -\gamma u \partial_x u - \partial_x P[u] - \lambda u$$

$$P[u] \doteq \frac{e^{-|x|}}{2} * \left(\frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2\right)$$

Formal Energy Estimate

$$\begin{cases} \partial_t u + \gamma u \partial_x u + \partial_x P = -\lambda u \\ -\partial_{xx}^2 P + P = \frac{3-\gamma}{2} u^2 + \frac{\gamma}{2} (\partial_x u)^2 \end{cases}$$

$$\Downarrow$$

$$\partial_t \left(\frac{u^2 + (\partial_x u)^2}{2} \right) + \partial_x \left(\frac{\gamma}{2} u (\partial_x u)^2 - \frac{1-\gamma}{2} u^3 + uP \right) = -\lambda \left(u^2 + (\partial_x u)^2 \right)$$

The total energy

$$E(t) := \|u(t,\cdot)\|_{H^1(\mathbb{R})}^2 = \int_{\mathbb{R}} \left(u(t,x)^2 + (\partial_x u(t,x))^2 \right) dx$$

satisfies the following ordinary differential equation

$$\frac{d}{dt} E(t) = -2\lambda E(t)$$

and therefore

$$E(t) = E(0) e^{-2\lambda t}, \qquad t \geq 0.$$

Definition 1 (Weak Dissipative Solutions)

A function $u : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a weak dissipative solution of the Cauchy problem

$$\begin{cases} \partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u \partial^2_{xx} u + u\partial^3_{xxx} u\right) u - \lambda(1 - \partial^2_{xx})u \\ u(0, x) = u_0(x) \end{cases}$$

if

- $u = u(t, x)$ is Hölder continuous;
- $u(t, \cdot) \in H^1(\mathbb{R})$ at every $t \in [0, \infty)$;
- $t \mapsto u(t, \cdot)$ is Lipschitz continuous from $[0, \infty)$ into $L^2(\mathbb{R})$, satisfies the initial condition and the following equality between functions in $L^2(\mathbb{R})$:

$$\frac{d}{dt} u = -\gamma u \partial_x u - \partial_x P[u] - \lambda u, \qquad \text{for a.e. } t \in [0, \infty).$$

- Oleinik type inequality: there exists $C = C(\|u_0\|_{H^1})$ s.t.

$$\partial_x u(t, x) \leq C(1 + t^{-1}) \qquad t > 0$$

The Main Result (F.A. & G.M. Coclite, 2014)

Let $\gamma,\ \lambda > 0$ be fixed. There exists a semigroup

$$S : [0, \infty) \times H^1(\mathbb{R}) \longrightarrow H^1(\mathbb{R}), \qquad (t, u_0) \mapsto S_t u_0$$

such that the following properties hold.

- For every $u_0 \in H^1(\mathbb{R})$, $u(t,x) \doteq S_t(u_0)(x)$ is a weak dissipative solution of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + 3u\partial_x u = \gamma \left(2\partial_x u \partial^2_{xx} u + u \partial^3_{xxx} u\right) u - \lambda(1 - \partial^2_{xx})u, \\ u(0,x) = u_0(x). \end{cases}$$

- $E(t) \le E(0)e^{-2\lambda t},\ t \ge 0.$
- For every $\{u_{0,n}\}_n \subset H^1(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$

$$u_{0,n} \longrightarrow u_0 \text{ in } H^1(\mathbb{R}) \implies S(u_{0,n}) \longrightarrow S(u_0) \text{ in } L^\infty_{loc}((0,\infty) \times \mathbb{R}).$$

Remark

- *Semigroup of solutions*
  - *no uniqueness of weak solutions established so far (within dissipative sol'ns)*
  - *solitons interaction may occur*
- *Oleinik type estimate*
  - $\partial_x u$ *is bounded from above*
  - $\partial_x u$ *may go to* $-\infty$
- S *is not continuous as a map with values in* H^1 *(even* $t \mapsto S_t u_0$ *may fail to be continuous as a map with values in* H^1 *due to the complete annihilation of peakons and antipeakons of the same strength when they collide).*
- *Energy exponential decay*

General strategy

- We introduce a new set of independent and dependent variables which yield a semilinear system of ODEs in a Banach space.
- Local existence of solutions for the semilinear system as fixed points of a contraction.
- An energy estimate gives the global existence of solutions for the semilinear system.
- Continuous dependence with respect to the initial conditions for the semilinear system.
- We come back to the original variables and prove our result.

- Semigroup of dissipative solns for CH
  Bressan & Constantin (Anal. & Appl. - 2007)

New indipendent variable

Energy variable $\xi \in \mathbb{R}$.

Assume $u_0 \in H^1(\mathbb{R})$. The map

$$y \in \mathbb{R} \longmapsto \int_0^y \left(1 + (\partial_x u_0)^2\right) dx$$

is continuous, increasing, and goes to $\pm\infty$ as $y \longrightarrow \pm\infty$. So we can define implicitly the function $y_0 = y_0(\xi)$ by the relation

$$\int_0^{y_0(\xi)} \left(1 + (\partial_x u_0)^2\right) dx = \xi, \qquad \xi \in \mathbb{R}.$$

- ξ plays the role of a Lagrangian variable (it is constant along characteristics)

New dependent variables

Characteristic curve $t \longmapsto y(t,\xi)$

$$\partial_t y(t,\xi) = \gamma u(t, y(t,\xi)), \qquad y(0,\xi) = y_0(\xi).$$

Notation

$$u(t,\xi) := u(t, y(t,\xi)), \qquad P(t,\xi) := P(t, y(t,\xi)).$$

New variables $v = v(t,\xi)$ **and** $q = q(t,\xi)$

$$v := 2\arctan(\partial_x u), \qquad q := (1 + (\partial_x u)^2)\partial_\xi y.$$

- v is bounded ($v \to -\pi$ as $\partial_x u \to -\infty$)
- $q \geq 0$

The semilinear system for $u = u(t,\xi),\ v = v(t,\xi),\ q = q(t,\xi)$

$$\begin{cases} \partial_t u = -\partial_x P - \lambda u \\ \partial_t v = \left(\frac{3-\gamma}{2}u^2 - P\right)(1+\cos(v)) - \gamma \sin^2\left(\frac{v}{2}\right) - \lambda \sin(v) \\ \partial_t q = \left(\frac{3-\gamma}{2}u^2 - P + \frac{\gamma}{2}\right)\sin(v) q - 2\lambda \sin^2\left(\frac{v}{2}\right) q \\ \\ u(0,\xi) = u_0(y_0(\xi)) \\ v(0,\xi) = 2\arctan(\partial_x u_0(y_0(\xi))) \\ q(0,\xi) = 1 \end{cases}$$

can be regarded as an ODE in the Banach space

$$X \doteq H^1(\mathbb{R}) \times \left(L^2(\mathbb{R}) \cap L^\infty(\mathbb{R})\right) \times L^\infty(\mathbb{R}).$$

The nonlocal term $P(t,\xi) \doteq P[u,v,q](t,\xi)$

$$
\begin{aligned}
P(t,\xi) = &\frac{1}{2}\int_{\mathbb{R}} e^{-\left|\int_{\xi}^{\xi'} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s)ds\right|} \times \\
&\times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2}\sin^2\left(\frac{v(t,\xi')}{2}\right)\right) \times \\
&\times q(t,\xi')d\xi',
\end{aligned}
$$

$$
\begin{aligned}
\partial_x P(t,\xi) = &\frac{1}{2}\int_{\mathbb{R}} e^{-\left|\int_{\xi}^{\xi'} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s)ds\right|} \times \\
&\times \operatorname{sign}(\xi-\xi') \times \\
&\times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2}\sin^2\left(\frac{v(t,\xi')}{2}\right)\right) \times \\
&\times q(t,\xi')d\xi'.
\end{aligned}
$$

In order to obtain global dissipative solutions, a modification of the system for u, v, q is needed.

Assume that, along a given characteristic $t \longmapsto y(t,\xi)$, the wave breaks at a first time $t = \tau(\xi)$. Arguing as for the Burgers equation and reminding that $\partial_x u$ satisfies an Oleinik type inequality, the wave break means $\partial_x u(t,\xi) \longrightarrow -\infty$, as $t \longrightarrow \tau(\xi)^-$.

For all $t \geq \tau(\xi)$ we then set $v(t,\xi) \equiv -\pi$ and remove the values of $u(t,\xi)$, $v(t,\xi)$, $q(t,\xi)$ from the computation of nonlocal terms P and $\partial_x P$.

In order to obtain global dissipative solutions, a modification of the system for u, v, q is needed.

Assume that, along a given characteristic $t \longmapsto y(t, \xi)$, the wave breaks at a first time $t = \tau(\xi)$. Arguing as for the Burgers equation and reminding that $\partial_x u$ satisfies an Oleinik type inequality, the wave break means $\partial_x u(t, \xi) \longrightarrow -\infty$, as $t \longrightarrow \tau(\xi)^-$.

For all $t \geq \tau(\xi)$ we then set $v(t, \xi) \equiv -\pi$ and remove the values of $u(t, \xi)$, $v(t, \xi)$, $q(t, \xi)$ from the computation of nonlocal terms P and $\partial_x P$.

In order to obtain global dissipative solutions, a modification of the system for u, v, q is needed.

Assume that, along a given characteristic $t \longmapsto y(t, \xi)$, the wave breaks at a first time $t = \tau(\xi)$. Arguing as for the Burgers equation and reminding that $\partial_x u$ satisfies an Oleinik type inequality, the wave break means $\partial_x u(t, \xi) \longrightarrow -\infty$, as $t \longrightarrow \tau(\xi)^-$.

For all $t \geq \tau(\xi)$ we then set $v(t, \xi) \equiv -\pi$ and remove the values of $u(t, \xi)$, $v(t, \xi)$, $q(t, \xi)$ from the computation of nonlocal terms P and $\partial_x P$.

The dissipative semilinear system for $u = u(t,\xi)$, $v = v(t,\xi)$, $q = q(t,\xi)$

$$\begin{cases} \partial_t u = -\partial_x P - \lambda u \\ \partial_t v = \begin{cases} \left(\frac{3-\gamma}{2}u^2 - P\right)(1+\cos(v)) - \gamma \sin^2\left(\frac{v}{2}\right) - \lambda \sin(v) & \text{if } v > -\pi \\ 0 & \text{if } v \leq -\pi \end{cases} \\ \partial_t q = \begin{cases} \left(\frac{3-\gamma}{2}u^2 - P + \frac{\gamma}{2}\right)\sin(v) q - 2\lambda \sin^2\left(\frac{v}{2}\right) q & \text{if } v > -\pi \\ 0 & \text{if } v \leq -\pi \end{cases} \\ u(0,\xi) = u_0(y_0(\xi)) \\ v(0,\xi) = 2\arctan(\partial_x u_0(y_0(\xi))) \\ q(0,\xi) = 1 \end{cases}$$

Notice: r.h.s. ODE is discontinuous

The dissipative nonlocal terms $P, \partial_x P$ are also discontinuous

$$
\begin{aligned}
P(t,\xi) = & \frac{1}{2} \int_{\{v(\xi')>-\pi\}} e^{-\left| \int_{\{\xi' \le \xi,\ v(\xi')>-\pi\}} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s) ds \right|} \times \\
& \times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2} \sin^2\left(\frac{v(t,\xi')}{2}\right) \right) \times \\
& \times q(t,\xi') d\xi',
\end{aligned}
$$

$$
\begin{aligned}
\partial_x P(t,\xi) = & \frac{1}{2} \int_{\{v(\xi')>-\pi\}} e^{-\left| \int_{\{\xi' \le \xi,\ v(\xi')>-\pi\}} \cos^2\left(\frac{v(t,s)}{2}\right) q(t,s) ds \right|} \times \\
& \times \operatorname{sign}(\xi - \xi') \times \\
& \times \left(\frac{3-\gamma}{2} u(t,\xi')^2 \cos^2\left(\frac{v(t,\xi')}{2}\right) + \frac{\gamma}{2} \sin^2\left(\frac{v(t,\xi')}{2}\right) \right) \times \\
& \times q(t,\xi') d\xi'.
\end{aligned}
$$

Remark

- *We regard the dissipative semilinear system as an ODE on the Banach space $X \doteq L^\infty(\mathbb{R}; \mathbb{R}^3)$.*
- *The r.h.s. of the ODE is discontinuous and discontinuity occurs along the plane $v = -\pi$.*
- *The second equation of the system implies that v approaches the value $-\pi$ transversally, i.e. $\partial_t v \approx -\gamma \implies$ r.h.s of ODE is transversal to plane $v = -\pi$. This transversality condition guarantees well-posedness of the system.*

Global existence and uniqueness for the semilinear system

Local existence and uniqueness

- General thms on directionally continuous ODEs in functional spaces do not apply (r.h.s. has unbounded variation in the direction of a cone in L^∞)
- Ad hoc analysis for discontinuous ODEs with non local terms as in: Bressan & Shen - 2006

Global existence

- Energy estimate
- Global bound on the total energy

$$E(t) = \|u(t,\cdot)\|^2_{H^1(\mathbb{R})} = \int_{\mathbb{R}} \left(u(t,x)^2 + (\partial_x u(t,x))^2 \right) dx$$

$$= \int_{\{v(t,\xi)>-\pi\}} \left(u^2(t,\xi) \cos^2\left(\frac{v(t,\xi)}{2}\right) + \sin^2\left(\frac{v(t,\xi)}{2}\right)\right) q(t,\xi)d\xi$$

Stability for the semilinear system

Remark: local existence by fixed point argument in L^∞ yields continuous dependence of sol'ns w.r.t. convergence of initial data in L^∞.

However: introducing a suitable distance functional Γ s.t.

$$\Gamma\big((u,v,q),\,(\tilde u,\tilde v,\tilde q)\big) \geq \big\|u-\tilde u\big\|_{L^\infty(\mathbb{R})},$$

$$u_n \longrightarrow u \qquad \text{in } H^1(\mathbb{R}) \qquad \Longrightarrow \qquad \Gamma\big((u_n,v_n,q_n),\,(u,v,q)\big) \longrightarrow 0,$$

and providing a priori bounds on the time increase of Γ along two sol'ns of semilinear dissipative system we derive

Theorem

Let $\{u_{0,n}\}_n \subset H^1(\mathbb{R})$ and $u_0 \in H^1(\mathbb{R})$. If

$$u_{0,n} \longrightarrow u_0 \qquad \text{in } H^1(\mathbb{R}),$$

then

$$u_n \longrightarrow u \qquad \text{in } L^\infty((0,T)\times\mathbb{R}) \text{ for every } T>0,$$

where u_n and u are (1st components of) the sol'ns of the semilinear dissipative system with initial data $u_{0,n}$ and u_0, respectively.

Global Dissipative Solutions in the Original Variables $u = u(t,x)$, $P = P(t,x)$

Let (u, v, q) be the solution of the semilinear system.
Define

$$y(t,\xi) = y_0(\xi) + \int_0^t u(\tau,\xi)d\tau.$$

For each fixed ξ, the function $t \longmapsto y(t,\xi)$ solves

$$\partial_t y(t,\xi) = \gamma u(t,\xi), \qquad y(0,\xi) = y_0(\xi).$$

We set

$$u(t,x) = u(t,\xi) \quad \text{if } y(t,\xi) = x.$$

Clearly

$$u(0,x) = u_0(x) \quad x \in \mathbb{R}.$$

The facts

- the energy estimate on $\|u(t,\cdot)\|_{H^1(\mathbb{R})}$
- the image of the singular set where $v = -\pi$ has measure zero (in the x-variable), i.e.,

$$\text{meas}(\{y(t,\xi);\, v(t,\xi) = -\pi\}) = 0 \qquad \forall\, t > 0$$

give that

- $u = u(t,x)$ is Hölder continuous
- $t \longmapsto u(t,\cdot) \in L^2(\mathbb{R})$ is Lipschitz continuous
- $\frac{d}{dt}u = -\gamma u \partial_x u - \partial_x P - \lambda u.$

Moreover

- Oleinik type inequality holds

Therefore

u is a weak dissipative solution of the hyperelastic rod wave equation.

The facts

- the energy estimate on $\|u(t,\cdot)\|_{H^1(\mathbb{R})}$
- the image of the singular set where $v = -\pi$ has measure zero (in the x-variable), i.e.,

$$\mathrm{meas}(\{y(t,\xi);\, v(t,\xi) = -\pi\}) = 0 \qquad \forall\, t > 0$$

give that

- $u = u(t,x)$ is Hölder continuous
- $t \longmapsto u(t,\cdot) \in L^2(\mathbb{R})$ is Lipschitz continuous
- $\frac{d}{dt}u = -\gamma u \partial_x u - \partial_x P - \lambda u.$

Moreover

- Oleinik type inequality holds

Therefore

u is a weak dissipative solution of the hyperelastic rod wave equation.

The facts

- the energy estimate on $\|u(t,\cdot)\|_{H^1(\mathbb{R})}$
- the image of the singular set where $v = -\pi$ has measure zero (in the x-variable), i.e.,

$$\mathrm{meas}(\{y(t,\xi);\, v(t,\xi) = -\pi\}) = 0 \qquad \forall\, t > 0$$

give that

- $u = u(t,x)$ is Hölder continuous
- $t \longmapsto u(t,\cdot) \in L^2(\mathbb{R})$ is Lipschitz continuous
- $\frac{d}{dt} u = -\gamma u \partial_x u - \partial_x P - \lambda u.$

Moreover

- Oleinik type inequality holds

Therefore

u is a weak dissipative solution of the hyperelastic rod wave equation.

Generalized hyperelastic-rod wave equation with source

$$\partial_t u - \partial^3_{txx} u + \partial_x \left(\frac{g(u)}{2} \right) = \gamma \left(2 \partial_x u \, \partial^2_{xx} u + u \, \partial^3_{xxx} u \right) + f(t, x, u),$$

- $(t, x) \in \mathbb{R}^+ \times \mathbb{R}$
- $u(t, x) \in \mathbb{R}$
- $\gamma > 0$
- $g : \mathbb{R} \to \mathbb{R}$ smooth map, $|g(u)| \leq M |u|^2 \quad \forall\, u$
- $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ smooth map,

$$|f(\cdot, \cdot, u)|, \ |\partial_t f(\cdot, \cdot, u)| \leq L |u|, \qquad |\partial_u f(\cdot, \cdot, u)| \leq L \qquad \forall\, u$$

Applications: Analyze controllability and stabilizability problems where $f(t, x, u)$ is treated as a distributed control.

Generalized hyperelastic-rod wave equation with source

$$\partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) = \gamma\left(2\partial_x u\,\partial^2_{xx} u + u\,\partial^3_{xxx} u\right) + f(t,x,u),$$

- $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$
- $u(t,x) \in \mathbb{R}$
- $\gamma > 0$
- $g : \mathbb{R} \to \mathbb{R}$ smooth map, $|g(u)| \leq M|u|^2 \quad \forall\, u$
- $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ smooth map,

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \leq L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \leq L \qquad \forall\, u$$

Applications: Analyze controllability and stabilizability problems where $f(t,x,u)$ is treated as a distributed control.

The generalized hyperelastic-rod wave equation with source

$$\partial_t u - \partial^3_{txx} u + \partial_x\left(\tfrac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \qquad (\mathrm{GHR})_f$$

rewritten as

$$(1-\partial^2_{xx})\partial_t u + \gamma(1-\partial^2_{xx})(u\partial_x u) + \partial_x\left(\frac{g(u)-\gamma\left(u^2-(\partial_x u)^2\right)}{2}\right) = f(t,x,u)$$

is formally equivalent to the elliptic-hyperbolic system

$$\begin{cases} \partial_t u + \gamma\,\partial_x\left(\tfrac{u^2}{2}\right) + \partial_x P = F, \\ -\partial^2_{xx} P + P = \frac{g(u)-\gamma\left(u^2-(\partial_x u)^2\right)}{2}, \\ -\partial^2_{xx} F + F = f(t,x,u). \end{cases} \qquad (\mathrm{E\text{-}H})_f$$

Definition 2 (Weak Dissipative Solutions)

A function $u : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a weak dissipative solution of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R}, t \geq 0,$$

if

- $u = u(t,x)$ is Hölder continuous, $u(t,\cdot) \in H^1(\mathbb{R})\ \forall\, t \geq 0$;
- $\exists\, P, F$, with $P(t), F(t) \in W^{1,\infty}\ \ \forall\, t \geq 0$, so that u is a distr. sol'n of (E-H)$_f$:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\partial_t\varphi + \gamma\frac{u^2}{2}\partial_x\varphi + P\partial_x\varphi + F\varphi\right) dtdx + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0,$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-P\partial^2_{xx}\varphi + P\varphi - \frac{g(u)}{2}\varphi - \frac{\gamma}{2}(u^2 - (\partial_x u)^2)\varphi\right) dtdx = 0, \qquad \forall\, \varphi \in C_c^\infty$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-F\partial^2_{xx}\varphi + F\varphi - f(t,x,u)\varphi\right) dtdx = 0,$$

- Oleinik type inequality: $\forall\, T > 0$, $\exists\, C = C(\|u_0\|_{H^1}, \gamma, g, f, T)$ s.t.

$$\partial_x u(t,x) \leq C(1 + t^{-1}) \qquad t \in]0,T], \quad x \in \mathbb{R}.$$

Definition 2 (Weak Dissipative Solutions)

A function $u : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a weak dissipative solution of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x \left(\frac{g(u)}{2} \right) - \gamma \left(2\partial_x u \, \partial^2_{xx} u + u \, \partial^3_{xxx} u \right) = f(t, x, u) \\ u(0, x) = u_0(x) \end{cases} \qquad x \in \mathbb{R}, t \geq 0,$$

if

- $u = u(t, x)$ is Hölder continuous, $u(t, \cdot) \in H^1(\mathbb{R}) \ \forall \ t \geq 0$;
- $\exists \ P, F$, with $P(t), F(t) \in W^{1,\infty} \ \forall \ t \geq 0$, so that u is a distr. sol'n of (E-H)$_f$:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\partial_t \varphi + \gamma \frac{u^2}{2} \partial_x \varphi + P \partial_x \varphi + F \varphi \right) dtdx + \int_{\mathbb{R}} u_0(x) \varphi(0, x) dx = 0,$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-P \partial^2_{xx} \varphi + P \varphi - \frac{g(u)}{2} \varphi - \frac{\gamma}{2} (u^2 - (\partial_x u)^2) \varphi \right) dtdx = 0, \qquad \forall \ \varphi \in C^\infty_c$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-F \partial^2_{xx} \varphi + F \varphi - f(t, x, u) \varphi \right) dtdx = 0,$$

- Oleinik type inequality: $\forall \ T > 0, \ \exists \ C = C(\|u_0\|_{H^1}, \gamma, g, f, T)$ s.t.

$$\partial_x u(t, x) \leq C(1 + t^{-1}) \qquad t \in]0, T], \quad x \in \mathbb{R}.$$

Definition 2 (Weak Dissipative Solutions)

A function $u : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a weak dissipative solution of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R}, t \geq 0,$$

if

- $u = u(t,x)$ is Hölder continuous, $u(t,\cdot) \in H^1(\mathbb{R})$ $\forall\, t \geq 0$;
- $\exists\, P, F$, with $P(t), F(t) \in W^{1,\infty}$ $\forall\, t \geq 0$, so that u is a distr. sol'n of (E-H)$_f$:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\partial_t\varphi + \gamma\frac{u^2}{2}\partial_x\varphi + P\partial_x\varphi + F\varphi\right) dtdx + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0,$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-P\partial^2_{xx}\varphi + P\varphi - \frac{g(u)}{2}\varphi - \frac{\gamma}{2}(u^2 - (\partial_x u)^2)\varphi\right) dtdx = 0, \qquad \forall\, \varphi \in C_c^\infty$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-F\partial^2_{xx}\varphi + F\varphi - f(t,x,u)\varphi\right) dtdx = 0,$$

- Oleinik type inequality: $\forall\, T > 0$, $\exists\, C = C(\|u_0\|_{H^1}, \gamma, g, f, T)$ s.t.

$$\partial_x u(t,x) \leq C(1 + t^{-1}) \qquad t \in]0,T], \quad x \in \mathbb{R}.$$

Definition 2 (Weak Dissipative Solutions)

A function $u : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a weak dissipative solution of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R}, t \geq 0,$$

if

- $u = u(t,x)$ is Hölder continuous, $u(t,\cdot) \in H^1(\mathbb{R})\ \forall\, t \geq 0$;
- $\exists\, P, F$, with $P(t), F(t) \in W^{1,\infty}\ \ \forall\, t \geq 0$, so that u is a distr. sol'n of (E-H)$_f$:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\partial_t\varphi + \gamma\frac{u^2}{2}\partial_x\varphi + P\partial_x\varphi + F\varphi\right) dtdx + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0,$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-P\partial^2_{xx}\varphi + P\varphi - \frac{g(u)}{2}\varphi - \frac{\gamma}{2}(u^2 - (\partial_x u)^2)\varphi\right) dtdx = 0, \qquad \forall\, \varphi \in C_c^\infty$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-F\partial^2_{xx}\varphi + F\varphi - f(t,x,u)\varphi\right) dtdx = 0,$$

- Oleinik type inequality: $\forall\, T > 0, \exists\, C = C(\|u_0\|_{H^1}, \gamma, g, f, T)$ s.t.

$$\partial_x u(t,x) \leq C(1 + t^{-1}) \qquad t \in]0, T], \quad x \in \mathbb{R}.$$

Definition 2 (Weak Dissipative Solutions)

A function $u : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ is a weak dissipative solution of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x \left(\frac{g(u)}{2}\right) - \gamma \left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R}, t \geq 0,$$

if

- $u = u(t,x)$ is Hölder continuous, $u(t,\cdot) \in H^1(\mathbb{R})\ \forall\, t \geq 0$;
- $\exists\, P, F$, with $P(t), F(t) \in W^{1,\infty}\ \ \forall\, t \geq 0$, so that u is a distr. sol'n of (E-H)$_f$:

$$\int_0^\infty \int_{\mathbb{R}} \left(u\partial_t\varphi + \gamma \frac{u^2}{2}\partial_x\varphi + P\partial_x\varphi + F\varphi\right) dtdx + \int_{\mathbb{R}} u_0(x)\varphi(0,x)dx = 0,$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-P\partial^2_{xx}\varphi + P\varphi - \frac{g(u)}{2}\varphi - \frac{\gamma}{2}(u^2 - (\partial_x u)^2)\varphi\right) dtdx = 0, \qquad \forall\, \varphi \in C^\infty_c$$

$$\int_0^\infty \int_{\mathbb{R}} \left(-F\partial^2_{xx}\varphi + F\varphi - f(t,x,u)\varphi\right) dtdx = 0,$$

- Oleinik type inequality: $\forall\ T > 0$, $\exists\ C = C(\|u_0\|_{H^1}, \gamma, g, f, T)$ s.t.

$$\partial_x u(t,x) \leq C(1 + t^{-1}) \qquad t \in]0,T], \quad x \in \mathbb{R}.$$

The Main Result (F.A. & G.M. Coclite, 2015)

Assume:

- $\gamma > 0$
- $g : \mathbb{R} \to \mathbb{R}$ smooth map, $|g(u)| \leq M|u|^2 \quad \forall\, u$
- $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ smooth map,

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \leq L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \leq L \qquad \forall\, u$$

Then:

- For every $u_0 \in H^1(\mathbb{R})$, there exists is a weak dissipative solution $u(t,x)$ of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R},\ t \geq 0.$$

Moreover (higher integrability):

$$\partial_x u \in L^p_{loc}([0,\infty) \times \mathbb{R}) \qquad \forall\, 1 \leqslant p < 3$$

Goal: Analyze stabilizability properties for distributed control f supported on a subset of $\mathbb{R}$ (damp the waves of hyperelastic rods by an external force f).

The Main Result (F.A. & G.M. Coclite, 2015)

Assume:

- $\gamma > 0$
- $g : \mathbb{R} \to \mathbb{R}$ smooth map, $|g(u)| \leq M|u|^2 \quad \forall\, u$
- $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ smooth map,

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \leq L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \leq L \qquad \forall\, u$$

Then:

- For every $u_0 \in H^1(\mathbb{R})$, there exists is a weak dissipative solution $u(t,x)$ of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R},\ t \geq 0.$$

Moreover (higher integrability):

$$\partial_x u \in L^p_{loc}([0,\infty) \times \mathbb{R}) \qquad \forall\, 1 \leqslant p < 3$$

Goal: Analyze stabilizability properties for distributed control f supported on a subset of $\mathbb{R}$ (damp the waves of hyperelastic rods by an external force f)

The Main Result (F.A. & G.M. Coclite, 2015)

Assume:

- $\gamma > 0$
- $g : \mathbb{R} \to \mathbb{R}$ smooth map, $|g(u)| \leq M|u|^2 \quad \forall\, u$
- $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ smooth map,

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \leq L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \leq L \qquad \forall\, u$$

Then:

- For every $u_0 \in H^1(\mathbb{R})$, there exists is a weak dissipative solution $u(t,x)$ of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R},\ t \geq 0.$$

Moreover (higher integrability):

$$\partial_x u \in L^p_{loc}([0,\infty) \times \mathbb{R}) \qquad \forall\ 1 \leqslant p < 3$$

Goal: Analyze stabilizability properties for distributed control f supported on a subset of $\mathbb{R}$ (damp the waves of hyperelastic rods by an external force f)

The Main Result (F.A. & G.M. Coclite, 2015)

Assume:

- $\gamma > 0$
- $g : \mathbb{R} \to \mathbb{R}$ smooth map, $|g(u)| \leq M|u|^2 \quad \forall\, u$
- $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ smooth map,

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \leq L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \leq L \qquad \forall\, u$$

Then:

- For every $u_0 \in H^1(\mathbb{R})$, there exists is a weak dissipative solution $u(t,x)$ of

$$\begin{cases} \partial_t u - \partial^3_{txx} u + \partial_x\left(\frac{g(u)}{2}\right) - \gamma\left(2\partial_x u\, \partial^2_{xx} u + u\, \partial^3_{xxx} u\right) = f(t,x,u) \\ u(0,x) = u_0(x) \end{cases} \qquad x \in \mathbb{R},\ t \geq 0.$$

Moreover (higher integrability):

$$\partial_x u \in L^p_{loc}([0,\infty) \times \mathbb{R}) \qquad \forall\, 1 \leqslant p < 3$$

Goal: Analyze stabilizability properties for distributed control f supported on a subset of $\mathbb{R}$ (damp the waves of hyperelastic rods by an external force f)

Consider the elliptic-parabolic approximation of the (E-H)$_f$ system:

$$\begin{cases} \partial_t u_\varepsilon + \gamma u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = F_\varepsilon + \varepsilon\, \partial_{xx}^2 u_\varepsilon, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \frac{g(u_\varepsilon) - \gamma\left({u_\varepsilon}^2 - (\partial_x u_\varepsilon)^2\right)}{2}, \\ -\partial_{xx}^2 F_\varepsilon + F_\varepsilon = f(t, x, u_\varepsilon), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \qquad \text{(E-H)}_{f,\varepsilon}$$

which is equivalent to the fourth order equation

$$\partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + \partial_x \left(\frac{g(u_\varepsilon)}{2}\right) = \gamma \left(2\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon\right) + f(t, x, u_\varepsilon) + \varepsilon\left(\partial_{xx}^2 u_\varepsilon - \partial_{xxxx}^4 u_\varepsilon\right).$$

Well-posedness of (E-H)$_{f,\varepsilon}$

- Existence and uniqueness of smooth sol'ns with $u_\varepsilon(0, \cdot) \in H^1(\mathbb{R})$
- Lipschitz continuity w.r.t. γ, g, f and initial data $u_\varepsilon(0, \cdot)$.

[Coclite, Holden and Karlsen, 2005]

Consider the elliptic-parabolic approximation of the $(\text{E-H})_f$ system:

$$\begin{cases} \partial_t u_\varepsilon + \gamma u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = F_\varepsilon + \varepsilon\, \partial^2_{xx} u_\varepsilon, \\ -\partial^2_{xx} P_\varepsilon + P_\varepsilon = \frac{g(u_\varepsilon) - \gamma\left({u_\varepsilon}^2 - (\partial_x u_\varepsilon)^2\right)}{2}, \\ -\partial^2_{xx} F_\varepsilon + F_\varepsilon = f(t, x, u_\varepsilon), \end{cases} \qquad t > 0, \ \ x \in \mathbb{R}, \qquad (\text{E-H})_{f,\varepsilon}$$

which is equivalent to the fourth order equation

$$\partial_t u_\varepsilon - \partial^3_{txx} u_\varepsilon + \partial_x\left(\tfrac{g(u_\varepsilon)}{2}\right) = \gamma\left(2\partial_x u_\varepsilon \partial^2_{xx} u_\varepsilon + u_\varepsilon \partial^3_{xxx} u_\varepsilon\right) + f(t, x, u_\varepsilon) + \varepsilon\left(\partial^2_{xx} u_\varepsilon - \partial^4_{xxxx} u_\varepsilon\right).$$

Well-posedness of $(\text{E-H})_{f,\varepsilon}$

- Existence and uniqueness of smooth sol'ns with $u_\varepsilon(0,\cdot) \in H^1(\mathbb{R})$
- Lipschitz continuity w.r.t. γ, g, f and initial data $u_\varepsilon(0,\cdot)$.

[Coclite, Holden and Karlsen, 2005]

Consider the elliptic-parabolic approximation of the $(\text{E-H})_f$ system:

$$\begin{cases} \partial_t u_\varepsilon + \gamma u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = F_\varepsilon + \varepsilon\, \partial_{xx}^2 u_\varepsilon, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \frac{g(u_\varepsilon) - \gamma\left({u_\varepsilon}^2 - (\partial_x u_\varepsilon)^2\right)}{2}, \\ -\partial_{xx}^2 F_\varepsilon + F_\varepsilon = f(t, x, u_\varepsilon), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \qquad (\text{E-H})_{f,\varepsilon}$$

which is equivalent to the fourth order equation

$$\partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + \partial_x \left(\frac{g(u_\varepsilon)}{2}\right) = \gamma \left(2\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon\right) + f(t, x, u_\varepsilon) + \varepsilon\left(\partial_{xx}^2 u_\varepsilon - \partial_{xxxx}^4 u_\varepsilon\right).$$

Well-posedness of $(\text{E-H})_{f,\varepsilon}$

- Existence and uniqueness of smooth sol'ns with $u_\varepsilon(0, \cdot) \in H^1(\mathbb{R})$
- Lipschitz continuity w.r.t. γ, g, f and initial data $u_\varepsilon(0, \cdot)$.

[Coclite, Holden and Karlsen, 2005]

Consider the elliptic-parabolic approximation of the $(\text{E-H})_f$ system:

$$\begin{cases} \partial_t u_\varepsilon + \gamma u_\varepsilon \partial_x u_\varepsilon + \partial_x P_\varepsilon = F_\varepsilon + \varepsilon\, \partial_{xx}^2 u_\varepsilon, \\ -\partial_{xx}^2 P_\varepsilon + P_\varepsilon = \frac{g(u_\varepsilon) - \gamma\left({u_\varepsilon}^2 - (\partial_x u_\varepsilon)^2\right)}{2}, \\ -\partial_{xx}^2 F_\varepsilon + F_\varepsilon = f(t, x, u_\varepsilon), \end{cases} \qquad t > 0, \quad x \in \mathbb{R}, \qquad (\text{E-H})_{f,\varepsilon}$$

which is equivalent to the fourth order equation

$$\partial_t u_\varepsilon - \partial_{txx}^3 u_\varepsilon + \partial_x \left(\frac{g(u_\varepsilon)}{2}\right) = \gamma \left(2\partial_x u_\varepsilon \partial_{xx}^2 u_\varepsilon + u_\varepsilon \partial_{xxx}^3 u_\varepsilon\right) + f(t, x, u_\varepsilon) + \varepsilon\left(\partial_{xx}^2 u_\varepsilon - \partial_{xxxx}^4 u_\varepsilon\right).$$

Well-posedness of $(\text{E-H})_{f,\varepsilon}$

- Existence and uniqueness of smooth sol'ns with $u_\varepsilon(0, \cdot) \in H^1(\mathbb{R})$
- Lipschitz continuity w.r.t. γ, g, f and initial data $u_\varepsilon(0, \cdot)$.

[Coclite, Holden and Karlsen, 2005]

Compactness - Existence of solutions

Given $u_0 \in H^1(\mathbb{R})$, consider:

$$\{u_{\varepsilon,0}\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}), \quad \|u_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \ \varepsilon > 0, \quad u_{\varepsilon,0} \to u_0 \text{ in } H^1(\mathbb{R}).$$

Let $u_\varepsilon : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ be:

$$\text{solution of (E-H)}_{f,\varepsilon} \quad \text{with} \quad u_\varepsilon(0,\cdot) = u_{\varepsilon,0}$$

Prove compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ and show $\exists\, u \in L^\infty_{loc}([0,\infty);\, H^1(\mathbb{R}))$ s.t.

- strong convergence:

$$u_\varepsilon \quad \longrightarrow \quad u \ \text{ in } \ L^\infty([0,T];\, H^1(\mathbb{R})) \qquad \forall\, T > 0,$$

- u is a weak dissipative solution of (E-H)$_f$

Compactness - Existence of solutions

Given $u_0 \in H^1(\mathbb{R})$, consider:

$$\{u_{\varepsilon,0}\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}), \quad \|u_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \ \varepsilon > 0, \quad u_{\varepsilon,0} \to u_0 \ \text{in} \ H^1(\mathbb{R}).$$

Let $u_\varepsilon : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ be:

$$\text{solution of (E-H)}_{f,\varepsilon} \quad \text{with} \quad u_\varepsilon(0,\cdot) = u_{\varepsilon,0}$$

Prove compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ and show $\exists\, u \in L^\infty_{loc}([0,\infty); H^1(\mathbb{R}))$ s.t.

- strong convergence:

$$u_\varepsilon \quad \longrightarrow \quad u \ \text{in} \ L^\infty([0,T]; H^1(\mathbb{R})) \qquad \forall\, T > 0,$$

- u is a weak dissipative solution of (E-H)$_f$

Compactness - Existence of solutions

Given $u_0 \in H^1(\mathbb{R})$, consider:

$$\{u_{\varepsilon,0}\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}), \quad \|u_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \ \varepsilon > 0, \quad u_{\varepsilon,0} \to u_0 \ \text{in} \ H^1(\mathbb{R}).$$

Let $u_\varepsilon : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ be:

$$\text{solution of (E-H)}_{f,\varepsilon} \quad \text{with} \quad u_\varepsilon(0,\cdot) = u_{\varepsilon,0}$$

Prove compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ and show $\exists\, u \in L^\infty_{loc}([0,\infty);\, H^1(\mathbb{R}))$ s.t.

- strong convergence:

$$u_\varepsilon \quad \longrightarrow \quad u \ \text{in} \ L^\infty([0,T];\, H^1(\mathbb{R})) \qquad \forall\, T > 0,$$

- u is a weak dissipative solution of (E-H)$_f$

Compactness - Existence of solutions

Given $u_0 \in H^1(\mathbb{R})$, consider:

$$\{u_{\varepsilon,0}\}_{\varepsilon>0} \subset C^\infty(\mathbb{R}), \quad \|u_{\varepsilon,0}\|_{H^1(\mathbb{R})} \leq \|u_0\|_{H^1(\mathbb{R})}, \ \varepsilon > 0, \quad u_{\varepsilon,0} \to u_0 \text{ in } H^1(\mathbb{R}).$$

Let $u_\varepsilon : [0,\infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ be:

solution of (E-H)$_{f,\varepsilon}$ with $u_\varepsilon(0,\cdot) = u_{\varepsilon,0}$

Prove compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ and show $\exists\, u \in L^\infty_{loc}([0,\infty); H^1(\mathbb{R}))$ s.t.

- strong convergence:

$$u_\varepsilon \quad \longrightarrow \quad u \text{ in } L^\infty([0,T]; H^1(\mathbb{R})) \qquad \forall\, T > 0,$$

- u is a weak dissipative solution of (E-H)$_f$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \le L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \le L \qquad \forall\, u$$

Energy estimates:

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} + 2\varepsilon e^{2Lt}\int_0^t e^{-2Ls}\,\|\partial_x u_\varepsilon(s,\cdot)\|^2_{H^1(\mathbb{R})}\,ds \le e^{2Lt}\,\|u_0\|^2_{H^1(\mathbb{R})} \qquad \forall\, t\ge 0.$$

Oleĭnik type estimates: $\forall\, T>0,\ \exists\, C_T = C_T(\|u_0\|_{H^1},\gamma,g,L,T)$ s.t.

$$\partial_x u_\varepsilon(t,x) \le \frac{2}{\gamma t} + C_T \qquad \forall\, t\in\,]0,T],\ x\in\mathbb{R}.$$

Higher integrability: $\forall\, a,b,T>0,\ \exists\, K_{a,b,T} = K_{a,b,T}(\|u_0\|_{H^1},a,b,T)$ s.t.

$$\Big\|\partial_x u_\varepsilon\Big\|_{L^p([0,T]\times[a,b])} \le K_{a,b,T} \qquad \forall\, 1\leqslant p<3.$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \leq L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \leq L \qquad \forall\ u$$

Energy estimates:

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} + 2\varepsilon e^{2Lt}\int_0^t e^{-2Ls}\,\|\partial_x u_\varepsilon(s,\cdot)\|^2_{H^1(\mathbb{R})}\,ds \leq e^{2Lt}\,\|u_0\|^2_{H^1(\mathbb{R})} \qquad \forall\ t \geq 0.$$

Oleĭnik type estimates: $\forall\ T>0,\ \exists\ C_T = C_T(\|u_0\|_{H^1},\gamma,g,L,T)$ s.t.

$$\partial_x u_\varepsilon(t,x) \leq \frac{2}{\gamma t} + C_T \qquad \forall\ t \in]0,T],\ x \in \mathbb{R}.$$

Higher integrability: $\forall\ a,b,T>0,\ \exists\ K_{a,b,T} = K_{a,b,T}(\|u_0\|_{H^1},a,b,T)$ s.t.

$$\left\|\partial_x u_\varepsilon\right\|_{L^p([0,T]\times[a,b])} \leq K_{a,b,T} \qquad \forall\ 1 \leqslant p < 3.$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$

$$|f(\cdot,\cdot,u)|,\ |\partial_t f(\cdot,\cdot,u)| \leq L|u|, \qquad |\partial_u f(\cdot,\cdot,u)| \leq L \qquad \forall\ u$$

Energy estimates:

$$\|u_\varepsilon(t,\cdot)\|^2_{H^1(\mathbb{R})} + 2\varepsilon e^{2Lt} \int_0^t e^{-2Ls} \|\partial_x u_\varepsilon(s,\cdot)\|^2_{H^1(\mathbb{R})}\, ds \leq e^{2Lt} \|u_0\|^2_{H^1(\mathbb{R})} \qquad \forall\ t \geq 0.$$

Oleĭnik type estimates: $\forall\ T > 0,\ \exists\ C_T = C_T(\|u_0\|_{H^1}, \gamma, g, L, T)$ s.t.

$$\partial_x u_\varepsilon(t,x) \leq \frac{2}{\gamma t} + C_T \qquad \forall\ t \in]0,T],\ x \in \mathbb{R}.$$

Higher integrability: $\forall\ a, b, T > 0,\ \exists\ K_{a,b,T} = K_{a,b,T}(\|u_0\|_{H^1}, a, b, T)$ s.t.

$$\left\|\partial_x u_\varepsilon\right\|_{L^p([0,T]\times[a,b])} \leq K_{a,b,T} \qquad \forall\ 1 \leqslant p < 3.$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ ... continued

Equation for first derivative $q_\varepsilon = \partial_x u_\varepsilon$:

$$\partial_t q_\varepsilon + \gamma u_\varepsilon \partial_x q_\varepsilon - \varepsilon \partial^2_{xx} q_\varepsilon + \frac{\gamma}{2} q_\varepsilon^2 = \frac{g(u_\varepsilon) - \gamma {u_\varepsilon}^2}{2} - P_\varepsilon + \partial_x F_\varepsilon .$$

Thanks to higher integrability estimates $\exists \{\varepsilon_j\}_{\varepsilon>0}, \quad \varepsilon_j \to 0,$ s.t.:

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in} \quad H^1((0,T)\times\mathbb{R}) \quad \forall\, T \geq 0,$$

$$q_{\varepsilon_j} \rightharpoonup q = \partial_x u \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3,$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3/2,$$

Relying on eq'n for q_ε and using renormalization theory for transport equation of defect measure $\overline{q^2} - q^2$ we show:

$$q^2 = \overline{q^2},$$

$$q_{\varepsilon_j} \to q \quad \text{in} \quad L^2((0,\infty)\times\mathbb{R}).$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ ... continued

Equation for first derivative $q_\varepsilon = \partial_x u_\varepsilon$:

$$\partial_t q_\varepsilon + \gamma u_\varepsilon \partial_x q_\varepsilon - \varepsilon \partial_{xx}^2 q_\varepsilon + \frac{\gamma}{2} q_\varepsilon^2 = \frac{g(u_\varepsilon) - \gamma {u_\varepsilon}^2}{2} - P_\varepsilon + \partial_x F_\varepsilon .$$

Thanks to higher integrability estimates $\exists \{\varepsilon_j\}_{\varepsilon>0}, \quad \varepsilon_j \to 0$, s.t.:

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in} \quad H^1((0,T)\times\mathbb{R}) \quad \forall\, T \geq 0,$$

$$q_{\varepsilon_j} \rightharpoonup q = \partial_x u \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1<p<3,$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1<p<3/2,$$

Relying on eq'n for q_ε and using renormalization theory for transport equation of defect measure $\overline{q^2} - q^2$ we show:

$$q^2 = \overline{q^2},$$

$$q_{\varepsilon_j} \to q \quad \text{in} \quad L^2((0,\infty)\times\mathbb{R}).$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ ... continued

Equation for first derivative $q_\varepsilon = \partial_x u_\varepsilon$:

$$\partial_t q_\varepsilon + \gamma u_\varepsilon \partial_x q_\varepsilon - \varepsilon \partial_{xx}^2 q_\varepsilon + \frac{\gamma}{2} q_\varepsilon^2 = \frac{g(u_\varepsilon) - \gamma {u_\varepsilon}^2}{2} - P_\varepsilon + \partial_x F_\varepsilon .$$

Thanks to higher integrability estimates $\exists \{\varepsilon_j\}_{\varepsilon>0}, \quad \varepsilon_j \to 0$, s.t.:

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in} \quad H^1((0,T)\times\mathbb{R}) \quad \forall\, T \geq 0,$$

$$q_{\varepsilon_j} \rightharpoonup q = \partial_x u \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3,$$

$$q_{\varepsilon_j}^2 \rightharpoonup \overline{q^2} \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3/2,$$

Relying on eq'n for q_ε and using renormalization theory for transport equation of defect measure $\overline{q^2} - q^2$ we show:

$$q^2 = \overline{q^2},$$

$$q_{\varepsilon_j} \to q \quad \text{in} \quad L^2((0,\infty)\times\mathbb{R}).$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ ... continued

Equation for first derivative $q_\varepsilon = \partial_x u_\varepsilon$:

$$\partial_t q_\varepsilon + \gamma u_\varepsilon \partial_x q_\varepsilon - \varepsilon \partial_{xx}^2 q_\varepsilon + \frac{\gamma}{2} q_\varepsilon^2 = \frac{g(u_\varepsilon) - \gamma {u_\varepsilon}^2}{2} - P_\varepsilon + \partial_x F_\varepsilon .$$

Thanks to higher integrability estimates $\exists \{\varepsilon_j\}_{\varepsilon>0}, \quad \varepsilon_j \to 0,$ s.t.:

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in} \quad H^1((0,T)\times\mathbb{R}) \quad \forall\, T \geq 0,$$

$$q_{\varepsilon_j} \rightharpoonup q = \partial_x u \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3,$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3/2,$$

Relying on eq'n for q_ε and using renormalization theory for transport equation of defect measure $\overline{q^2} - q^2$ we show:

$$q^2 = \overline{q^2},$$

$$q_{\varepsilon_j} \to q \quad \text{in} \quad L^2((0,\infty)\times\mathbb{R}).$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ ... continued

Equation for first derivative $q_\varepsilon = \partial_x u_\varepsilon$:

$$\partial_t q_\varepsilon + \gamma u_\varepsilon \partial_x q_\varepsilon - \varepsilon \partial_{xx}^2 q_\varepsilon + \frac{\gamma}{2} q_\varepsilon^2 = \frac{g(u_\varepsilon) - \gamma {u_\varepsilon}^2}{2} - P_\varepsilon + \partial_x F_\varepsilon .$$

Thanks to higher integrability estimates $\exists\ \{\varepsilon_j\}_{\varepsilon>0}, \quad \varepsilon_j \to 0$, s.t.:

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in} \quad H^1((0,T)\times\mathbb{R}) \quad \forall\ T \geq 0,$$

$$q_{\varepsilon_j} \rightharpoonup q = \partial_x u \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\ 1 < p < 3,$$

$$q^2_{\varepsilon_j} \rightharpoonup \overline{q^2} \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\ 1 < p < 3/2,$$

Relying on eq'n for q_ε and using renormalization theory for transport equation of defect measure $\overline{q^2} - q^2$ we show:

$$q^2 = \overline{q^2},$$

$$q_{\varepsilon_j} \to q \quad \text{in} \quad L^2((0,\infty)\times\mathbb{R}).$$

Compactness of $\{u_\varepsilon\}_{\varepsilon>0}$ ... continued

Equation for first derivative $q_\varepsilon = \partial_x u_\varepsilon$:

$$\partial_t q_\varepsilon + \gamma u_\varepsilon \partial_x q_\varepsilon - \varepsilon \partial_{xx}^2 q_\varepsilon + \frac{\gamma}{2} q_\varepsilon^2 = \frac{g(u_\varepsilon) - \gamma {u_\varepsilon}^2}{2} - P_\varepsilon + \partial_x F_\varepsilon .$$

Thanks to higher integrability estimates $\exists \, \{\varepsilon_j\}_{\varepsilon>0}, \quad \varepsilon_j \to 0$, s.t.:

$$u_{\varepsilon_j} \rightharpoonup u \quad \text{in} \quad H^1((0,T)\times\mathbb{R}) \quad \forall\, T \geq 0,$$

$$q_{\varepsilon_j} \rightharpoonup q = \partial_x u \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3,$$

$$q_{\varepsilon_j}^2 \rightharpoonup \overline{q^2} \quad \text{in} \quad L^p_{loc}((0,\infty)\times\mathbb{R}) \quad \forall\, 1 < p < 3/2,$$

Relying on eq'n for q_ε and using renormalization theory for transport equation of defect measure $\overline{q^2} - q^2$ we show:

$$q^2 = \overline{q^2},$$

$$q_{\varepsilon_j} \to q \quad \text{in} \quad L^2((0,\infty)\times\mathbb{R}).$$

Thank you for your attention!