Obstacle Problems for the p-Laplacian via Tug-of-War games

Partial differential equations, optimal design and numerics
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Description of the problem

This is joint work with my colleague Marta Lewicka. The cartoons in the next slides are drawn by our colleague Kiumars Kaveh.

- $\Omega \subset \mathbb{R}^N$ open, bounded set with Lipschitz boundary
- $\Psi : \mathbb{R}^n \to \mathbb{R}$ bounded Lipschitz function (the obstacle)
- $F : \partial\Omega \to \mathbb{R}$ bounded Lipschtiz function (boundary data) compatibility condition : $F(x) \geq \Psi(x)$ for $x \in \partial\Omega$.

$$\begin{cases} -\Delta_p u & \geq & 0 & \text{in } \Omega, \\ u & \geq & \Psi & \text{in } \Omega, \\ -\Delta_p u & = & 0 & \text{in } \{x \in \Omega;\ u(x) > \Psi(x)\}, \\ u & = & F & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Notions of weak solutions I

For the p-Laplace operator:

$$-\Delta_p u = -\text{div}\big(|\nabla u|^{p-2}\nabla u\big)$$

we can consider FOUR notions of supersolution:

1. Weak (or Sobolev) supersolutions: These are functions $v \in W^{1,p}_{\mathrm{loc}(\Omega)}$ such that:

$$\int_\Omega \langle |\nabla v|^{p-2}\nabla v, \nabla\phi\rangle \, \mathrm{d}x \geq 0$$

for all test functions $\phi \in C_0^\infty(\Omega)$ that are non-negative in Ω.

2. Potential theoretic supersolutions or p-superhamonic functions: A lower-semicontinuous function $v : \Omega \to \mathbb{R} \cup \{\infty\}$ is p-superharmonic if is not identically ∞ on any connected component of Ω and it satisfies the comparison principle with respect to p-harmonic functions; that is: if $D \Subset \Omega$, and $w \in C(\bar{D})$ is p-harmonic in D satisfying $w \leq v$ on ∂D, then we must have: $w \leq v$ on D.

Notions of weak solutions II

3. Viscosity supersolutions: A lower-semicontinuous function $v : \Omega \to \mathbb{R} \cup \{\infty\}$ is a viscosity p-supersolution if it is not identically ∞ on any connected component of Ω, and if whenever $\phi \in C_0^\infty(\Omega)$ is such that $\phi(x) \leq v(x)$ for all $x \in \Omega$ with equality at one point $\phi(x_0) = v(x_0)$ (ϕ touches v from below at x_0), and $\nabla\phi(x_0) \neq 0$, then we have:

$$- \Delta_p \phi(x_0) \geq 0. \tag{2}$$

▷ weak supersolutions ⇒ potential theoretic and viscosity supersolutions (easy)
▷ bounded p-superharmonic functions are weak supersolutions (not easy even for $p = 2$), Lindqvist (1986)
▷ viscosity supersolutions ⇔ p-superharmonic Juutinen, Lindqvist, M (1999).
Therefore, these three notions of supersolution agree on the class of bounded functions.

4. Supersolutions in the sense of means (M,Parvianinen,Rossi, 2010): Choose α and β as follows:

$$\alpha = \frac{p-2}{N+p}, \qquad \beta = \frac{2+N}{N+p}.$$

A continuous function $v : \Omega \to \mathbb{R} \cup \{\infty\}$ is a supersolution in the sense of means if whenever $\phi \in C_0^\infty(\Omega)$ is such that $\phi(x) \leq v(x)$ for all $x \in \Omega$, with equality at one point $\phi(x_0) = v(x_0)$ (ϕ touches v from below at x_0), then we have:

$$0 \leq -\phi(x_0) + \frac{\alpha}{2} \sup_{B_\epsilon(x_0)} \phi + \frac{\alpha}{2} \inf_{B_\epsilon(x_0)} \phi + \beta \fint_{B_\epsilon(x_0)} \phi + o(\epsilon^2) \qquad \text{as } \epsilon \to 0^+. \tag{3}$$

By $0 \leq h(\epsilon) + o(\epsilon^2)$ as $\epsilon \to 0^+$ we mean that:

$$\lim_{\epsilon \to 0^+} \frac{[h(\epsilon)]^-}{\epsilon^2} = 0.$$

p-harmonious functions

Let $0 < \epsilon_0 \ll 1$. Define

$$\Gamma = \{x \in \mathbb{R}^N \setminus \Omega;\ \operatorname{dist}(x, \Omega) < \epsilon_0\}, \qquad X = \Omega \cup \Gamma.$$

Let now $0 < \epsilon \leq \epsilon_0$ be a fixed scale.
ϵ-p-harmonious functions: A bounded function $u : X \to \mathbb{R}$ is ϵ-p-harmonious with boundary values given by a (Borel) function $F : \bar{\Gamma} \to \mathbb{R}$ if:

$$u_\epsilon(x) = \begin{cases} \dfrac{\alpha}{2} \sup\limits_{B_\epsilon(x)} u_\epsilon + \dfrac{\alpha}{2} \inf\limits_{B_\epsilon(x)} u_\epsilon + \beta \fint_{B_\epsilon(x)} u_\epsilon & \text{for } x \in \Omega \\ F(x) & \text{for } x \in \Gamma. \end{cases} \tag{4}$$

p-harmonious functions II

M-Parvianen-Rossi (2012) proved that $u = \lim_{\epsilon \to 0} u_\epsilon$ is a solution to the Dirichlet problem:

$$\begin{cases} -\Delta_p u = 0 & \text{in } \Omega, \\ u = F & \text{on } \partial\Omega. \end{cases} \tag{5}$$

The proof consist of two parts:
(1) show that the family $\{u_\epsilon\}$ is equicontinuous in a certain sense so that we can extract limits that are continuous functions (here where the probability is used in the form of Tug-of-War games), and
(2) prove that any such limit is a weak solution in the means (and therefore viscosity and weak) of the Dirichlet problem (5).This follows from general stability theory of viscosity solutions.

Obstacle problems in the linear case

Consider a second order differential operator

$$\mathfrak{L}(v(x)) = \frac{1}{2}\,\mathrm{trace}(\sigma(x)\sigma'(x)D^2 v),$$

where the matrix function σ is Lipschitz continuous. Consider the obstacle problem in $\mathbb{R}^n$

$$\min\left(-\mathfrak{L}v, v - g\right) = 0, \tag{6}$$

where g are appropriately regular. To solve this problem probabilistically we first solve the stochastic differential equation

$$d\mathbf{X}_t = \sigma(\mathbf{X}_t)\, d\mathbf{W}_t \tag{7}$$

starting from x at time t=0. Denoting by $\{\mathbf{X}_t^x, t \geq 0\}$ its solution, we write the value function

$$v(x) = \sup_{\tau \in \mathfrak{T}} \mathbb{E}\left[g(\mathbf{X}_\tau^x)\right], \tag{8}$$

where $\mathfrak{T}$ denotes the set of all stopping times valued in $[0, \infty]$.

Obstacle problems in the non-linear case

We are looking for a probabilistic approach to the obstacle problem whtn second order linear differential operator $\mathfrak{L}$ is replaced by the p-Laplace operator

$$-\Delta_p u = -\text{div}\big(|\nabla u|^{p-2}\nabla u\big).$$

Since our operator is non-linear, we don't have a suitable variant of the linear stochastic differential equation that we could use to write a formula similar to (8). Instead we will use tug-of-war games with noise as our basic stochastic process. Tug-of-war games run on discrete time, so we will show that the solutions to the obstacle problem for the p-Laplacian for $p \in [2, \infty)$, can be interpreted as limits of values of a specific obstacle tug-of-war game with noise, when the step-size ϵ determining the allowed length of move of a token, at each step of the game, converges to 0.

ϵ-approximate obstacle problem

We fix ϵ fixed in this section and write u instead of u_ϵ

Theorem (Existence and uniqueness)

There exists a unique bounded Borel function $u : X \to \mathbb{R}$ which satisifes:

$$u(x) = \begin{cases} \max\left\{\Psi(x), \frac{\alpha}{2}\sup_{B_\epsilon(x)} u + \frac{\alpha}{2}\inf_{B_\epsilon(x)} u + \beta \fint_{B_\epsilon(x)} u\right\} & \text{for } x \in \Omega \\ F(x) & \text{for } x \in \Gamma. \end{cases}$$

This formula is similar to the Wald-Bellman equation of optimal stopping.

Playing tug-of-war games

Our board is a domain Ω, which we assume bounded and Lipschitz for simplicity. We fix a step-size $\epsilon > 0$ small. Start with a token at a point $x_0 \in \Omega$. Two players take turns to move (following a specific rule) token to another point $x_1 \in \Omega$ at most at distance ϵ from x_0. We keep applying the rules and go from x_1 to x_2, from x_2 to x_3,... such that

$$x_n \in B(x_{n-1}, \epsilon).$$

We need to specify the game rules and a stopping criterium.

The game will stop once the token reaches the boundary strip Γ (or is within ϵ from the boundary of Ω).
On the boundary the rules will be determined by two positive numbers α and β,

$$\alpha + \beta = 1$$

and two players I and II.

Playing tug-of-war games, II

With probability α the players flip an unbiased coin and whoever wins makes a move; that is, each player gets to move the taken with probability $\alpha/2$.
With probability β the token is moved at random by a distance at most ϵ.
Finally, we have a pay off-function

$$F\colon \Gamma \to \mathbb{R},$$

which we assume Lipschitz and bounded.
When the token reaches the boundary at $x_\tau \in \Gamma$, player II pays player I the amount $F(x_\tau)$ euros.
A smart player I would steer the token towards the maximum values of F, while a smart player II will steer the token towards the minimum values of F.

Tug of War Games with Noise

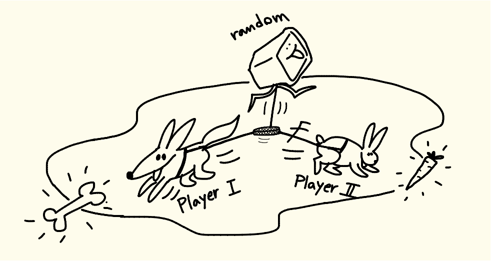

Figure: Player I and Player II compete in a Tug-of-War with random noise

Tug of War Games with Noise

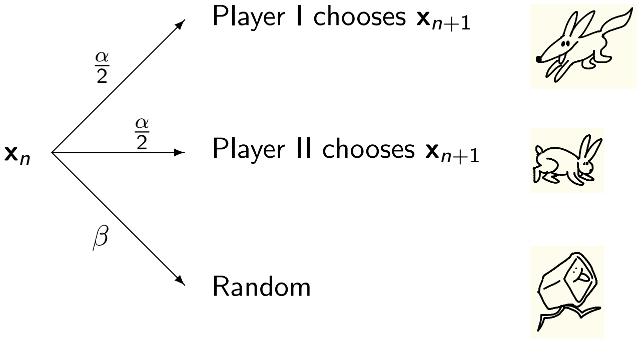

Figure: Player I, Player II and random noise with their probabilities

The measure spaces $(X^{\infty,x_0}, \mathcal{F}_n^{x_0})$ and $(X^{\infty,x_0}, \mathcal{F}^{x_0})$.

Fix any $x_0 \in X$ and consider the space of infinite sequences ω (recording positions of token during the game), starting at x_0:

$$X^{\infty,x_0} = \{\omega = (x_0, x_1, x_2 \ldots);\ x_n \in X \text{ for all } n \geq 1\}.$$

For each $n \geq 1$, let $\mathcal{F}_n^{x_0}$ be the σ-algebra of subsets of X^{∞,x_0}, containing sets of the form:

$$A_1 \times \ldots \times A_n := \{\omega \in X^{\infty,x_0};\ x_i \in A_i \text{ for } i : 1 \ldots n\},$$

for all n-tuples of Borel sets $A_1, \ldots, A_n \subset X$.
Let $\mathcal{F}^{x_0}$ be now defined as the smallest σ-algebra of subsets of X^{∞,x_0}, contaning $\bigcup_{n=1}^{\infty} \mathcal{F}_n^{x_0}$. Clearly, the increasing sequence $\{\mathcal{F}_n^{x_0}\}_{n\geq 1}$ is a filtration of $\mathcal{F}^{x_0}$, and the coordinate projections $x_n(\omega) = x_n$ are $\mathcal{F}^{x_0}$ measurable (random variables) on X^{∞,x_0}.

Stopping times

Define the exit time from the set Ω:

$$\tau_0(\omega) = \min\{n \geq 0;\ x_n \in \Gamma\}$$

$\tau_0 : X^{\infty,x_0} \to \mathbb{N} \cup \{+\infty\}$ is $\mathcal{F}^{x_0}$ measurable and, in fact, it is a stopping time with respect to the filtration $\{\mathcal{F}_n^{x_0}\}$, that is:

$$\forall n \geq 0 \qquad \{\omega \in X^{\infty,x_0};\ \tau_0(\omega) \leq n\} \in \mathcal{F}_n^{x_0}.$$

Let now $\tau : X^{\infty,x_0} \to \mathbb{N} \cup \{+\infty\}$ be any stopping time such that $\tau \leq \tau_0$. For $n \geq 1$ we define the Borel sets:

$$A_n^\tau = \{(x_0, x_1, \ldots, x_n) : \exists\, \omega = (x_0, x_1, \ldots, x_n, x_{n+1}, \ldots), \tau(\omega) \leq n\}.$$

Note that $(x_0, \ldots, x_n) \in A_n^\tau$ whenever $x_n \in \Gamma$.

Strategies

For every $n \geq 1$, let $\sigma_I^n, \sigma_{II}^n : X^{n+1} \to X$ be Borel measurable functions with the property that:

$$\sigma_I^n(x_0, x_1, \ldots, x_n),\ \sigma_{II}^n(x_0, x_1, \ldots, x_n) \in B_\epsilon(x_n) \cap X.$$

We call $\sigma_I = \{\sigma_I^n\}_{n\geq 1}$ and $\sigma_{II} = \{\sigma_{II}^n\}_{n\geq 1}$ the strategies of Players I and II, respectively.

The probability measure $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}$

Given $\tau, \sigma_I, \sigma_{II}$ as above, we define now a family of probabilistic (Borel) measures on X, parametrised by the finite histories $(x_0, \dots, x_n)$:

$$\gamma_n[x_0, x_1, \dots, x_n] = \begin{cases} \frac{\alpha}{2}\delta_{\sigma_I^n(x_0,x_1,\dots,x_n)} + \frac{\alpha}{2}\delta_{\sigma_{II}^n(x_0,x_1,\dots,x_n)} + \beta\frac{\mathcal{L}_N\lfloor B_\epsilon(x_n)}{|B_\epsilon(x_n)|} \\ \delta_{x_n} \end{cases}$$

where δ_y denotes the Dirac delta at a given $y \in X$. Note that since $\tau \leq \tau_0$, then $\gamma_n[x_0, x_1, \dots, x_n] = \delta_{x_n}$ whenever $x_n \in \Gamma$

The probability measure $\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}$, II

For every $n \geq 1$ we now define the probability measure $\mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}$ on $(X^{\infty,x_0}, \mathcal{F}^{x_0}_n)$ by setting:

$$\mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}(A_1 \times \ldots \times A_n) = \int_{A_1} \ldots \int_{A_n} 1 \, \mathrm{d}\gamma_{n-1}[x_0, x_1, \ldots, x_{n-1}] \ldots \mathrm{d}\gamma_0[x_0]$$

for every n-tuple of Borel sets $A_1, \ldots, A_n \subset X$. Here, A_1 is interpreted as the set of possible successors x_1 of the initial position x_0, which we integrate $\mathrm{d}\gamma_0[x_0]$, while $x_n \in A_n$ is a possible successor of x_{n-1} which we integrate $\mathrm{d}\gamma_{n-1}[x_0, x_1, \ldots, x_{n-1}]$, etc. For every $n \geq 1$ and every Borel set $A \subset X$, the function:

$$X^{n+1} \ni (x_0, x_1, \ldots, x_n) \mapsto \gamma_n[x_0, x_1, \ldots, x_n](A) \in \mathbb{R}$$

is Borel measurable.

From Kolmogorov's construction it follows that

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}} = \lim_{n\to\infty} \mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}$$

on $(X^{\infty,x_0}, \mathcal{F}_n)$ so that:

$$A_1 \times \ldots \times A_n \in \mathcal{F}^{x_0}_n \qquad \mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}(A_1 \times \ldots \times A_n) = \mathbb{P}^{n,x_0}_{\tau,\sigma_I,\sigma_{II}}(A_1 \times \ldots \times A_n).$$

Lemma

Let $v : X \to \mathbb{R}$ be a bounded Borel function. For any $n \geq 1$, the conditional expectation $\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}\{v \circ x_n \mid \mathcal{F}^{x_0}_{n-1}\}$ of the random variable $v \circ x_n$ is a $\mathcal{F}^{x_0}_{n-1}$ measurable function on X^{∞,x_0} (and hence it depends only on the initial n positions in the history $\omega = (x_0, x_1, \ldots, x_{n-1})$, given by:

$$\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}\{v \circ x_n \mid \mathcal{F}^{x_0}_{n-1}\}(x_0, \ldots, x_{n-1}) = \int_X v \; d\gamma_{n-1}[x_0, \ldots, x_{n-1}].$$

The game stops almost surely and the game has a value

Lemma

Assume that $\beta > 0$. *Then each game stops almost surely, i.e.:*

$$\mathbb{P}^{x_0}_{\tau,\sigma_I,\sigma_{II}}\big(\{\tau < \infty\}\big) = 1.$$

Theorem

Define:

$$G : X \to \mathbb{R} \qquad G = \chi_\Gamma F + \chi_\Omega \Psi,$$

Define the two value functions:

$$u_I(x_0) = \sup_{\tau,\sigma_I} \inf_{\sigma_{II}} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}[G \circ x_\tau], \quad u_{II}(x_0) = \inf_{\sigma_{II}} \sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{II}}[G \circ x_\tau],$$

where sup *and* inf *are taken over all strategies* σ_I, σ_{II} *and stopping times* $\tau \leq \tau_0$. *Then:*

$$u_I = u = u_{II} \quad \textit{in } \Omega,$$

The heart of the matter: Strategies $\Rightarrow$ Estimates

To see that $u_{II} \leq u$ in Ω fix $\eta > 0$ and consider an arbitrary strategy σ_I and an arbitrary stopping time $\tau \leq \tau_0$. Choose a strategy $\sigma_{0,II}$ for Player II, such that

$$u(\sigma_{0,II}^n(x_n)) \leq \inf_{B_\epsilon(x_n)} u + \frac{\eta}{2^{n+1}}$$

Then, the sequence of random variables

$$\left\{ u \circ x_n + \frac{\eta}{2^n} \right\}_{n \geq 0}$$

is a **supermartingale** with respect to the filtration $\{\mathcal{F}_n^{x_0}\}$. It follows that:

$$\begin{aligned} u_{II}(x_0) &\leq \sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[G \circ x_\tau + \frac{\eta}{2^\tau}] \leq \sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[u \circ x_\tau + \frac{\eta}{2^\tau}] \\ &\leq \sup_{\tau,\sigma_I} \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[u \circ x_0 + \frac{\eta}{2^0}] = u(x_0) + \eta. \end{aligned}$$

Main Result

Theorem

Let $p \in [2, \infty)$. Let $F : \partial\Omega \to \mathbb{R}$, $\Psi : \bar{\Omega} \to \mathbb{R}$ be two Lipschitz continuous functions, satisfying $\Psi \leq F$ on $\partial\Omega$.

Let $u_\epsilon : \Omega \cup \Gamma \to \mathbb{R}$ be the unique ϵ-p-superharmonious function with boundary values F and obstacle Ψ.

Then u_ϵ converge as $\epsilon \to 0$, uniformly in $\bar{\Omega}$, to a continuous function u which is the unique viscosity solution to the obstacle problem for the p-Laplacian with boundary values F and obstacle Ψ.

The key is to prove the uniform convergence of u_ϵ, as $\epsilon \to 0$, in $\bar{\Omega}$. This follows form aversion of the Ascoli-Arzelá theorem, valid for equibounded (possibly discontinous) functions with "uniformly vanishing oscillation":

Lemma (M-Parviainen-Rossi, 2012)

Let $u_\epsilon : \bar{\Omega} \to \mathbb{R}$ be a set of functions such that:
(i) $\exists C > 0 \quad \forall \epsilon > 0 \qquad \|u_\epsilon\|_{L^\infty(\bar{\Omega})} \leq C,$
(ii) $\forall \eta > 0 \quad \exists r_0, \epsilon_0 > 0 \quad \forall \epsilon < \epsilon_0 \quad \forall x_0, y_0 \in \bar{\Omega} \qquad |x_0 - y_0| < r_0 \implies |u_\epsilon(x_0) - u_\epsilon(y_0)| < \eta$
Then, a subsequence of u_ϵ converges uniformly in $\bar{\Omega}$, to a continuous function u.

Lemma (KEY LEMMA)

Let $u_\epsilon : X \to \mathbb{R}$ be the ϵ-p-superharmonious in our main theorem. Then, for every $\eta > 0$ there exist $r_0, \epsilon_0 > 0$ such that $\forall \epsilon < \epsilon_0, \forall y_0 \in \partial\Omega, \forall x_0 \in \bar{\Omega}$ we have

$$|x_0 - y_0| < r_0 \implies |u_\epsilon(x_0) - u_\epsilon(y_0)| < \eta.$$

Strategies ⇒ Estimates, II

Let $\delta > 0$ and $z_0 \in \mathbb{R}^N \setminus \Omega$ satisfy: $B_\delta(z_0) \cap \bar{\Omega} = \{y_0\}$. Define strategy $\sigma_{0,II}$ for Player II:

$$\sigma_{0,II}^n(x_0, \dots x_n) = \sigma_{0,II}^n(x_n) = \begin{cases} x_n + (\epsilon - \epsilon^3)\frac{z_0 - x_n}{|z_0 - x_n|} & \text{if } x_n \in \Omega \\ x_n & \text{if } x_n \in \Gamma. \end{cases}$$

Let σ_I be an arbitrary strategy for Player I and let $\tau \leq \tau_0$ be any admissible stopping time. We then have

$$\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[|x_\tau - y_0|] \leq |x_0 - y_0| + 2\delta + C_\delta \epsilon^2 \mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[\tau].$$

Lemma

$$\mathbb{E}^{x_0}_{\tau,\sigma_I,\sigma_{0,II}}[|x_\tau - y_0|] \leq C\delta + C_\delta(|x_0 - x_0| + \epsilon)$$

for all ϵ sufficiently small

.

The double obstacle problem (Codenotti-Lewicka-M)

Let $\Omega \subset \mathbb{R}^N$ and $F : \partial\Omega \to \mathbb{R}$ as before, and bounded and Lipschitz functions $\Psi_1, \Psi_2 : \mathbb{R}^N \to \mathbb{R}$ such that $\Psi_1 \leq \Psi_2$ in $\bar{\Omega}$ and $\Psi_1 \leq F \leq \Psi_2$ on $\partial\Omega$. Consider the following double-obstacle problem:

$$\begin{cases} -\Delta_p u \geq 0 & \text{in } \{x \in \Omega;\ u(x) < \Psi_2(x)\} \\ -\Delta_p u \leq 0 & \text{in } \{x \in \Omega;\ u(x) > \Psi_1(x)\} \\ \Psi_1 \leq u \leq \Psi_2 & \text{in } \Omega \\ u = F & \text{on } \partial\Omega. \end{cases} \tag{9}$$

Note that under the third condition in (9), the first two conditions are jointly equivalent to:

$$\max\Big\{u - \Psi_2, \min\big\{ -\Delta_p u, u - \Psi_1\big\}\Big\} = 0.$$

Double Obstacle Problem

Theorem

Let $\Psi_1, \Psi_2 : \mathbb{R}^N \to \mathbb{R}$ and $F : \Gamma \to \mathbb{R}$ be bounded Borel functions such that $\Psi_1 \leq \Psi_2$ in X and $\Psi_1 \leq F \leq \Psi_2$ in Γ. Then, for every $\epsilon < \bar{\epsilon}_0$, there exists a unique Borel function $u : X \to \mathbb{R}$ which satisfies:

$$u(x) = \max \left\{ \Psi_1(x), \min \left\{ \Psi_2(x), \frac{\alpha}{2} \sup_{B_\epsilon(x)} u + \frac{\alpha}{2} \inf_{B_\epsilon(x)} u + \beta \fint_{B_\epsilon(x)} u \right\} \right\}$$

for $x \in \Omega$ and

$$u(x) = F(x)$$

for $x \in \Gamma$.

Double Obstacle Problem

Theorem

Let $p \in [2, \infty)$ *and define:*

$$\alpha = \frac{p-2}{p+N}, \qquad \beta = \frac{2+N}{p+N}.$$

Let $F, \Psi_1, \Psi_2 : \mathbb{R}^N \to \mathbb{R}$ *be bounded Lipschitz continuous functions such that:*

$$\Psi_1 \leq \Psi_2 \quad \text{in } \bar{\Omega} \qquad \text{and} \qquad \Psi_1 \leq F \leq \Psi_2 \quad \text{in } \mathbb{R}^N \setminus \Omega.$$

Let u_ϵ *be the unique solution from the previous theorem. Then* $\{u_\epsilon\}$ *converge, as* $\epsilon \to 0$*, uniformly in* $\bar{\Omega}$*, to a continuous function* $u : \bar{\Omega} \to \mathbb{R}$ *which is a viscosity solution to the double-obstacle problem (9).*

Thank you very much

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