Semi-group theory for the Stokes and Navier-Stokes equations with Navier-type boundary conditions in L^p-spaces

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Introduction and motivation

This thesis is devoted to the mathematical theoretical study of the Stokes

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0, T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega, \end{cases}$$

and the Navier-Stokes

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} + \nabla \pi = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{u} = 0 & \text{in} \quad \Omega \times (0, T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} \quad \Omega \end{cases}$$

problems in a bounded domain Ω of $\mathbf{R}^3$ of class $C^{2,1}$.
The domain Ω is not necessarily simply-connected and the boundary Γ is not necessarily connected.

- Since the pioneer work of **Leray (1934)** and **Hopf (1951)**, these two problems have been often studied with Dirichlet boundary condition $\boldsymbol{u} = \mathbf{0}$ on Γ formulated by **Stokes (1845)**.

- **Navier (1827)** suggested an alternative boundary condition which is a type of slip boundary condition with friction on the wall based on a proportionality between the normal dynamical tensor $\mathbb{D}\boldsymbol{u} = \frac{1}{2}(\nabla \boldsymbol{u} + \nabla \boldsymbol{u}^T)$ and the velocity.

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad [\mathbb{D}\boldsymbol{u} \cdot \boldsymbol{n}]_{\boldsymbol{\tau}} + \alpha\, \boldsymbol{u}_{\boldsymbol{\tau}} = 0 \qquad \text{on } \Gamma \times (0, T),$$

where $\alpha \geq 0$ is the coefficient of friction.

In this thesis we consider a Navier-slip boundary condition but without friction

$$\boldsymbol{u} \cdot \boldsymbol{n} = 0, \qquad [\mathbb{D}\boldsymbol{u} \cdot \boldsymbol{n}]_{\boldsymbol{\tau}} = 0 \qquad \text{on } \Gamma \times (0, T). \tag{1}$$

The boundary condition (1) is used to simulate the flows near rough walls as in the work of **Feireisl et al. (2008, 2010)** for instance and in perforated walls as in the work of **Beavers (1967)** for instance. **Sueur (2014)** study the motion of a rigid body in a viscous incompressible fluid when some Navier slip conditions are prescribed on the body's boundary.

Another type of Navier-slip boundary condition without friction is the one involving the tangential component of the vorticity

$$\boldsymbol{u}\cdot\boldsymbol{n} = 0, \qquad \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} \qquad \text{on} \quad \Gamma\times(0,T), \qquad (2)$$

we call them Navier-type boundary conditions. In the case of flat boundary the two conditions (1) and (2) are equivalent.

The condition (2) has been recently studied by **Da Veiga (2005)**, **Berselli (2010)** and **Amrouche-Seloula (2011-2013)**.

In some real-life situations it is also natural to prescribe the value of the pressure at least on some part of the boundary.

There are at lest two types of boundary conditions involving the pressure, one of them is the one involving the tangential component of the velocity

$$\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0}, \qquad \pi = \pi_0 \qquad \text{on} \quad \Gamma \times (0, T). \tag{3}$$

The boundary conditions involving the pressure has been studied by **Conca et al. (1994-1995)**, **Łukaszewicz (1997)**, **Marušić (2007)** and **Amrouche-Seloula (2011-2013)**.

My thesis was devoted to the **mathematical study** of the **Stokes** and **Navier-Stokes** problems with the boundary conditions (1), (2) and (3) respectively.

My goal was to obtain a good **semi-group theory** and a good **mathematical framework** for the Stokes operator with these boundary conditions respectively leading to a good mathematical study of the non-linear problem.

In what follows we will consider only the **Navier-type boundary condition** (2). The same results holds for the boundary conditions (1) and (3) with some modifications in the proofs.

Analyticity of the Stokes semi-group

The Stokes operator $A_p \,:\, \mathbf{D}(A_p) \subset \boldsymbol{L}^p_{\sigma,\tau}(\Omega) \longmapsto \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, is defined by

$$\forall\, \boldsymbol{u} \in \mathbf{D}(A_p), \qquad A_p\, \boldsymbol{u} \,=\, -P\Delta \boldsymbol{u}. \tag{4}$$

The space

$$\boldsymbol{L}^p_{\sigma,\tau}(\Omega) \,=\, \Big\{ \boldsymbol{f} \in \boldsymbol{L}^p(\Omega);\; \mathrm{div}\, \boldsymbol{f} = 0 \text{ in } \Omega,\;\; \boldsymbol{f} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma \Big\}. \tag{5}$$

Thanks to the work of **Amrouche-Seloula (2011)** and the regularity of Ω we can characterize the domain of A_p by

$$\mathbf{D}(A_p) = \{ \boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega);,\;\; \mathrm{div}\, \boldsymbol{u} = 0 \text{ in } \Omega, \\ \boldsymbol{u} \cdot \boldsymbol{n} \,=\, 0,\;\; \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} \,=\, \mathbf{0} \text{ on } \Gamma \}. \tag{6}$$

A key observation

Proposition (H. Al Baba, C. Amrouche, M. Escobedo)

For all $\boldsymbol{u} \in \mathbf{D}(A_p)$, $A_p\,\boldsymbol{u} = -\Delta\boldsymbol{u}$.

In fact, because $\Delta\boldsymbol{u} = \mathbf{grad}(\mathrm{div}\,\boldsymbol{u}) - \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u}$ in Ω,

$$\mathrm{div}\,\boldsymbol{u} = 0 \quad \text{in } \Omega \quad \text{and} \quad \mathbf{curl}\,\boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \quad \text{on } \Gamma,$$

$$\text{then} \quad \mathrm{div}(\Delta\boldsymbol{u}) = 0 \ \text{ in } \Omega, \qquad \mathbf{curl}\,\mathbf{curl}\,\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma$$

and

$$\Delta\boldsymbol{u} \cdot \boldsymbol{n} = 0 \quad \text{on } \Gamma.$$

We consider the problem:

$$\begin{cases} \lambda\,\boldsymbol{u} - \Delta \boldsymbol{u} = \boldsymbol{f}, & \operatorname{div} \boldsymbol{u} = 0 \qquad \text{in} \quad \Omega, \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \mathbf{0} \qquad \text{on} \quad \Gamma, \end{cases} \tag{7}$$

where $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and $\lambda \in \Sigma_\varepsilon$, with $\varepsilon \in]0, \pi[$ be fixed.

$$\Sigma_\varepsilon = \{\lambda \in \mathbb{C}^*;\ |\arg \lambda| \leq \pi - \varepsilon\}.$$

Existence and Resolvent estimate

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

Let $\lambda \in \Sigma_\varepsilon$ *and let* $\boldsymbol{f} \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, *with* $1 < p < \infty$.

- *The Problem (7) has a unique solution* $\boldsymbol{u} \in \boldsymbol{W}^{2,p}(\Omega)$.
- *In addition for* $\mathrm{Re}\,\lambda \geq 0$, *i.e. for* $\varepsilon = \pi/2$, *the solution* $\boldsymbol{u}$ *satisfies the estimate*

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{|\lambda|} \|\boldsymbol{f}\|_{\boldsymbol{L}^p(\Omega)}, \tag{8}$$

where the constant $C(\Omega, p)$ *is independent of* λ *and* $\boldsymbol{f}$.

Corollary (H. Al Baba, C. Amrouche, M. Escobedo)

The Stokes operator with Navier-type boundary conditions generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ for all $1 < p < \infty$.

Al Baba Amrouche Escobedo, To appear-Contemporary Mathematics.

Complex and fractional powers

• The existence of complex and fractional powers of the Stokes operator A_p is guaranteed since the Stokes operator generates a bounded analytic semi-group on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ (See **Triebel (1978)**, **Komatsu (1966)** for instance).

• Since the Stokes operator with Navier-type boundary conditions doesn't have bounded inverse (**Amrouche-Seloula (2011)**), the complex and fractional powers of the Stokes operator may not be expressed by an integral formula and it is not easy to compute calculus inequalities involving these powers.

Pure imaginary powers

Our main object in this subsection is to prove the boundedness of the pure imaginary powers of the operator A_p on $\boldsymbol{L}^p_{\sigma,\tau}(\Omega)$. This is fundamental and has an important consequence in the associated parabolic problem. Our main result is the following :

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

There exists an angle $0 < \theta_0 < \pi/2$ such that for all $s \in \mathbb{R}$ we have

$$\|A_p^{i\,s}\|_{\mathcal{L}(\boldsymbol{L}^p_{\sigma,\tau}(\Omega))} \leq M\, C(\Omega, p)\, e^{|s|\,\theta_0}, \tag{9}$$

for some constant $M > 0$.

(Al Baba-Amrouche-Escobedo) in Revision in ARMA

$\mathbf{D}(A_p^{1/2})$

Consider the space

$$\boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega) = \{\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\Omega);\ \operatorname{div} \boldsymbol{v} = 0 \text{ in } \Omega,\ \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \text{ on } \Gamma\}. \tag{10}$$

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

- *For all* $1 < p < \infty$, $\mathbf{D}(A_p^{1/2}) = \boldsymbol{W}^{1,p}_{\sigma,\tau}(\Omega)$ *with equivalent norms.*
- *For all* $\boldsymbol{u} \in \mathbf{D}(A_p^{1/2})$

$$\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \simeq \|\mathbf{curl}\,\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)} \tag{11}$$

▷ For all $\boldsymbol{u} \in \mathbf{D}(A_p^{1/2})$, the norm $\|A_p^{1/2}\boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}$ is not a norm on $\mathbf{D}(A_p^{1/2})$ and (11) is not immediate.

$\mathbf{D}(A_p^\alpha)$

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

For all $1 < p < \infty$ and for all $\alpha \in \mathbb{R}$ such that $0 < \alpha < 3/2p$ the following Sobolev embedding holds

$$\mathbf{D}(A_p^\alpha) \hookrightarrow \boldsymbol{L}^q(\Omega), \qquad \frac{1}{q} = \frac{1}{p} - \frac{2\alpha}{3}. \tag{12}$$

Moreover for all $\boldsymbol{u} \in \mathbf{D}(A_p^\alpha)$ the following estimate holds

$$\|\boldsymbol{u}\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, \|A_p^\alpha \boldsymbol{u}\|_{\boldsymbol{L}^p(\Omega)}. \tag{13}$$

The homogeneous problem

• Thanks to our boundary condition the gradient of pressure disappears from the Stokes problem and our work is reduced to study the following homogeneous problem

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} = \boldsymbol{0}, & \operatorname{div} \boldsymbol{u} = 0 \quad \text{in} \quad \Omega \times (0, T), \\ \boldsymbol{u} \cdot \boldsymbol{n} = 0, & \mathbf{curl}\, \boldsymbol{u} \times \boldsymbol{n} = \boldsymbol{0} \quad \text{on} \quad \Gamma \times (0, T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 \quad \text{in} \quad \Omega. \end{cases} \tag{14}$$

As for the heat equation we have the following theorem :

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$*, then Problem (14) has a unique solution* $\boldsymbol{u}(t)$ *satisfying*

$$\boldsymbol{u} \in C([0, +\infty[, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap C(]0, +\infty[, \mathbf{D}_p(A)), \tag{15}$$

$$\boldsymbol{u} \in C^k(]0, +\infty[, \mathbf{D}_p(A^\ell)), \qquad \forall\, k \in \mathbb{N},\ \forall\, \ell \in \mathbb{N}^*. \tag{16}$$

Moreover we have the estimates

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^p(\Omega)} \leq\ C(\Omega, p)\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{17}$$

$$\left\|\frac{\partial \boldsymbol{u}(t)}{\partial t}\right\|_{\boldsymbol{L}^p(\Omega)} \leq \frac{C(\Omega, p)}{t}\,\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{18}$$

$L^p - L^q$ estimates

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

Let $1 < p \leq q < \infty$, $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ and let $\boldsymbol{u}(t)$ be the unique solution of Problem (14). The following estimates holds:

$$\|\boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}, \tag{19}$$

$$\|\mathbf{curl}\, \boldsymbol{u}(t)\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, t^{-1/2}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)} \tag{20}$$

and for all $m, n \in \mathbb{N}$ one has

$$\left\|\frac{\partial^m}{\partial t^m} \Delta^n \boldsymbol{u}(t)\right\|_{\boldsymbol{L}^q(\Omega)} \leq C(\Omega, p)\, t^{-(m+n)}\, t^{-3/2(1/p-1/q)} \|\boldsymbol{u}_0\|_{\boldsymbol{L}^p(\Omega)}. \tag{21}$$

The inhomogeneous problem

We consider the abstract Cauchy-Problem on a complex Banach space X:

$$\begin{cases} \frac{\partial u}{\partial t} + \mathcal{A}\, u(t) = f(t) & 0 \leq t \leq T \\ u(0) = 0, \end{cases} \tag{22}$$

where $-\mathcal{A}$ is the infinitesimal generator of an analytic semi-group in X.

• For $f \in L^p(0, T;\, X)$, the analyticity of the semi-group generated by $-\mathcal{A}$ is not enough to obtain a solution to Problem (22) satisfying

$$u \in W^{1,p}(0, T;\, X) \cap L^p(0, T;\, D(\mathcal{A})), \tag{23}$$

unless X is a Hilbert space (**Kato**). Usually we need to impose further regularity assumptions on f such that f is locally Hølder continuous (see **Pazy** for instance).

• Using the concept of ζ-convexity and a perturbation argument, **Giga-Sohr** have proved the existence of solution to Problem (22) satisfying (23), when the pure imaginary powers of $\mathcal{A}$ are bounded and satisfy estimate

$$\|\mathcal{A}^{is}\|_{\mathcal{L}(X)} \leq K\, e^{\theta\,|s|} \tag{24}$$

for some $0 < \theta < \pi/2$ and $K > 0$.

Strong solutions for the inhomogeneous problem

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

Let $1 < p, q < \infty$, $\boldsymbol{u}_0 = \boldsymbol{0}$ *and* $\boldsymbol{f} \in \boldsymbol{L}^q(0, T; \boldsymbol{L}^p(\Omega))$, $0 < T \leq \infty$. *The Problem (3) has a unique solution* $(\boldsymbol{u}, \pi)$ *such that*

$$\boldsymbol{u} \in L^q(0, T_0;\ \boldsymbol{W}^{2,p}(\Omega)), \qquad \frac{\partial \boldsymbol{u}}{\partial t} \in L^q(0, T;\ \boldsymbol{L}^p(\Omega)), \tag{25}$$

for all T_0 *satisfying* $T_0 \leq T$ *if* $T < \infty$ *and* $T_0 < T$ *if* $T = \infty$

$$and \qquad \pi \in L^q(0, T;\ W^{1,p}(\Omega)/\mathbb{R}) \tag{26}$$

Theorem

Moreover we have the estimate

$$\int_0^T \Big\| \frac{\partial \boldsymbol{u}}{\partial t} \Big\|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t + \int_0^T \| \Delta \boldsymbol{u}(t) \|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t + \int_0^T \| \pi(t) \|^q_{W^{1,p}(\Omega)/\mathbb{R}} \,\mathrm{d}\, t \leq C(p, q, \Omega) \int_0^T \| \boldsymbol{f}(t) \|^q_{\boldsymbol{L}^p(\Omega)} \,\mathrm{d}\, t. \quad (27)$$

Navier-Stokes problem

- In this section we consider the time dependent Navier-Stokes problem with Navier-type boundary conditions

$$\begin{cases} \frac{\partial \boldsymbol{u}}{\partial t} - \Delta \boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla\pi = \mathbf{0}, & \mathrm{div}\,\boldsymbol{u} = 0 & \text{in} & \Omega\times(0,T), \\ \boldsymbol{u}\cdot\boldsymbol{n} = 0, & \mathbf{curl}\,\boldsymbol{u}\times\boldsymbol{n} = \mathbf{0} & \text{on} & \Gamma\times(0,T), \\ & \boldsymbol{u}(0) = \boldsymbol{u}_0 & \text{in} & \Omega, \end{cases} \tag{28}$$

where $(\boldsymbol{u}\cdot\nabla) = \sum_{j=1}^{3} u_j \frac{\partial}{\partial x_j}$. For simplicity the external force is assumed to be zero.

• We showed that some informations on the linear problem can be used to obtain local mild and classical solutions to Problem (28).

• First, using the $L^p - L^q$ estimates for the Stokes semi-group and proceeding as **Giga** we will prove the existence of a local in time mild solution to the Problem (28).

• Next, using the fractional powers of the Stokes operator we will estimate the non-linear term $\boldsymbol{u} \cdot \nabla \boldsymbol{u}$. Then, proceeding as **Giga-Miyakawa** we prove that the solution $\boldsymbol{u} \in \mathbf{D}(A_p)$ for all $t \in (0, T_*]$ for certain $T_* < T$.

Local mild solution

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$, $p \geq 3$. *There is a* $T_0 > 0$ *and a unique mild solution of* (28) *on* $[0, T_0)$ *such that*

$$\boldsymbol{u} \in BC([0, T_0);\, \boldsymbol{L}^p_{\sigma,\tau}(\Omega)) \cap L^q(0, T_0;\, \boldsymbol{L}^r_{\sigma,\tau}(\Omega)) \qquad (29)$$

with

$$q > p,\ r > p, \qquad \frac{2}{q} + \frac{3}{r} = \frac{3}{p}. \qquad (30)$$

Moreover there is a positive constant ε *such that if* $\|\boldsymbol{u}_0\|_{\boldsymbol{L}^p_{\sigma,\tau}(\Omega)} \leq \varepsilon$ *then* T_0 *can be taken as infinity for* $p = 3$.

Regularity result

Theorem (H. Al Baba, C. Amrouche, M. Escobedo)

Let $\boldsymbol{u}_0 \in \boldsymbol{L}^p_{\sigma,\tau}(\Omega)$ *with* $p \geq 3$. *There exists a maximal interval of time* $T_* \in]0, T[$ *such that the unique solution* $\boldsymbol{u}(t)$ *of Problem* (28) *satisfies*

$$\boldsymbol{u} \in C((0, T_*], \mathbf{D}(A_p)) \cap C^1((0, T_*]; \boldsymbol{L}^p_{\sigma,\tau}(\Omega)). \tag{31}$$

Al Baba-Amrouche-Escobedo Jaca 2014

- Weak, strong and very weak solutions to the Navier-Stokes problem with the boundary conditions treated in this thesis.
- The effect of the Navier-type boundary conditions on the global existence of strong solution for the Navier-Stokes problem.
- The Navier-slip boundary condition with friction on the wall.
- The geometry of the domain Ω (half space, exterior domain, ...).

Further readings

- ***H. Al Baba, C. Amrouche and M. Escobedo***. Analyticity of the semi-group generated by the Stokes operator with Navier-type boundary conditions on L^p-spaces. *Contemporary Mathematics-To appear*.

- ***H. Al Baba, C. Amrouche, A. Rejaiba***. The time dependent Stokes problem with Navier boundary conditions on L^p-spaces. *Analysis-To appear*.

- ***H. Al Baba, C. Amrouche, N. Seloula***. Time dependent Stokes problem with Normal and pressure boundary conditions on L^p-spaces. *Analysis-To appear*.

- ***H. Al Baba, C. Amrouche and M. Escobedo***. Semi-group theory for the Stokes operator with Navier-type boundary conditions on L^p-spaces. *Arch. Ration. Mech. Anal-in Revision.*

- ***H. Al Baba, C. Amrouche and M. Escobedo***. L^p- theory for the time dependent Navier-Stokes Problem with Navier-type boundary conditions. *Thirteenth International Conference Zaragoza-Pau on Mathematics.*

Thank you for your attention