Modelisation of a flute

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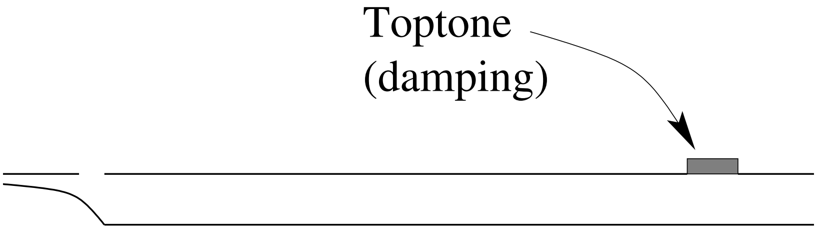
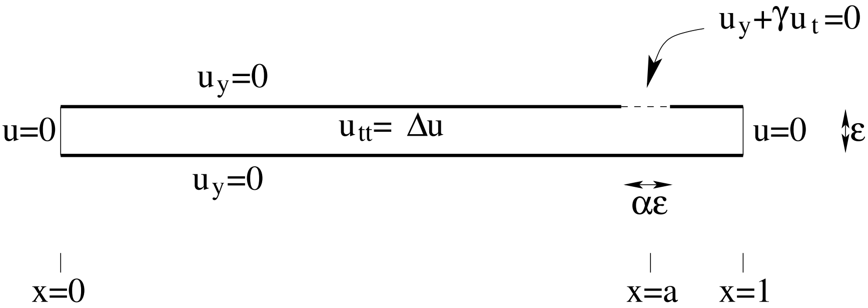
The soundless flute



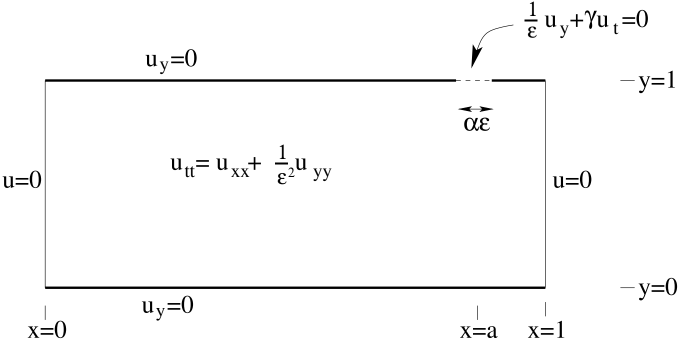
Mathematical modelisation

Let u be the pressure in the tube of the flute.



Thin domain technics : Arrieta, Ciarlet, Ciuperca, Hale, Jimbo, Lions, Morita, Raugel, Sell...

Change of variable : $y \longmapsto \frac{1}{\varepsilon} y$



New space norm : $\|u\|^2_{\mathbb{H}^1_\varepsilon} = \|\partial_x u\|^2_{\mathbb{L}^2} + \frac{1}{\varepsilon^2}\|\partial_y u\|^2_{\mathbb{L}^2}$.

Linear operator

The associated linear stationnary operator A_ε is defined by

$$A_\varepsilon \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in \mathbb{H}^1_\varepsilon \times \mathbb{L}^2$$

is equivalent to

$$\begin{cases} v &= f \\ \partial^2_{xx} u + \frac{1}{\varepsilon^2} \partial^2_{yy} u &= g \\ \frac{1}{\varepsilon} \partial_y u + \gamma v &= 0 \quad \text{on } \Gamma_\varepsilon \\ \text{other B.C.} & \end{cases}$$

NB : $\gamma > 0$ so the operator is invertible.

Finding the limit equation

Assume that $\varepsilon_n \longrightarrow 0$ and $\|(f_n, g_n)\|_{\mathbb{H}^1_{\varepsilon_n} \times \mathbb{L}^2} = 1$.
Set

$$A_{\varepsilon_n} \begin{pmatrix} u_n \\ v_n \end{pmatrix} = \begin{pmatrix} f_n \\ g_n \end{pmatrix} .$$

We can assume :

- $f_n \longrightarrow f$ strongly in $\mathbb{L}^2$, weakly in $\mathbb{H}^1$. Moreover $f(x, y) = f(x)$ does not depend on y.
- $g_n \longrightarrow g$ weakly in $\mathbb{L}^2$.

The central lemma

Proposition :

Let φ, ψ be functions in $\mathbb{H}^1(\Omega)$.

$$\left| \frac{1}{\varepsilon} \int_{a-\frac{\alpha\varepsilon}{2}}^{a+\frac{\alpha\varepsilon}{2}} \varphi(x,1)\psi(x,1)dx \; - \; \alpha \int_0^1 \varphi(a,y)\psi(a,y)dy \right|$$

$$\leq \sqrt{\varepsilon} \|\varphi\|_{\mathbb{H}^1_\varepsilon} \|\psi\|_{\mathbb{H}^1_\varepsilon} \; .$$

In particular, there exists $C > 0$ such that for any $\varepsilon < 1$,

$$\left| \frac{1}{\varepsilon} \int_{\Gamma_\varepsilon} \varphi\psi \right| \leq C \|\varphi\|_{\mathbb{H}^1_\varepsilon} \|\psi\|_{\mathbb{H}^1_\varepsilon} \; .$$

Proof of the lemma

To prove the lemma, we write

$$\left| \frac{1}{\varepsilon} \int_{a-\frac{\alpha\varepsilon}{2}}^{a+\frac{\alpha\varepsilon}{2}} \varphi(x,1)\psi(x,1)dx \;-\; \alpha \int_0^1 \varphi(a,y)\psi(a,y)dy \right|$$

$$\leq \frac{1}{\varepsilon} \int_{a-\frac{\alpha\varepsilon}{2}}^{a+\frac{\alpha\varepsilon}{2}} \int_0^1 |\varphi(x,1)\psi(x,1) - \varphi(x,y)\psi(x,y)| \;\; dxdy$$

$$+\frac{1}{\varepsilon} \int_{a-\frac{\alpha\varepsilon}{2}}^{a+\frac{\alpha\varepsilon}{2}} \int_0^1 |\varphi(x,y)\psi(x,y) - \varphi(a,y)\psi(a,y)| \;\; dxdy \;\; .$$

Proof of the lemma

Then, we use estimates similar to the following ones.

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Gamma_\varepsilon}\int_0^1 |\psi(x,1)-\psi(x,y)|^2\,dydx &= \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon}\int_0^1 \left(\int_y^1 \partial_y\psi(x,\zeta)d\zeta\right)^2 dydx \\ &\le \frac{1}{\varepsilon}\|\partial_y\psi\|^2_{\mathbb{L}^2} \le \varepsilon\|\psi\|^2_{\mathbb{H}^1_\varepsilon}\end{aligned}$$

$$\begin{aligned}\frac{1}{\varepsilon}\int_{\Gamma_\varepsilon}\int_0^1 |\psi(a,y)-\psi(x,y)|^2\,dydx &= \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon}\int_0^1 \left(\int_x^a \partial_x\psi(\xi,y)d\xi\right)^2 dydx \\ &\le \frac{1}{\varepsilon}\int_{\Gamma_\varepsilon}\int_0^1 |x-a|\int_0^1 |\partial_x\psi(\xi,y)|^2\,d\xi dydx \\ &\le \varepsilon\|\partial_x\psi\|^2_{\mathbb{L}^2} \le \varepsilon\|\psi\|^2_{\mathbb{H}^1_\varepsilon}\end{aligned}$$

Finding the limit equation

We have $\begin{cases} \partial^2_{xx} u_n + \frac{1}{{\varepsilon_n}^2}\partial^2_{yy} u_n = g_n \\ v_n = f_n \\ \frac{1}{\varepsilon_n}\partial_y u_n + \gamma v_n = 0 \text{ on } \Gamma_{\varepsilon_n} \end{cases}$

Therefore,

$$-\int_\Omega |\partial_x u_n|^2 - \frac{1}{{\varepsilon_n}^2}\int_\Omega |\partial_y u_n|^2 - \frac{\gamma}{\varepsilon_n}\int_{\Gamma_{\varepsilon_n}} f_n u_n = \int_\Omega g_n u_n \ ,$$

and so $\|u_n\|_{\mathbb{H}^1_\varepsilon} \leq C\|(f_n, g_n)\|_{\mathbb{H}^1_\varepsilon \times \mathbb{L}^2}$.

Thus $\begin{cases} u_n \longrightarrow u(x,y) = u(x) \\ v_n \longrightarrow f(x,y) = f(x) = v(x) \end{cases}$ weakly in $\mathbb{H}^1$, strongly in $\mathbb{L}^2$.

Finding the limit equation

Let $\varphi(x)$ be a test function independent of y. We have

$$- \int_\Omega \partial_x u_n \partial_x \varphi - \frac{\gamma}{\varepsilon_n} \int_{\Gamma_{\varepsilon_n}} v_n \varphi = \int_\Omega g_n \varphi \ .$$

Thus, the variational formulation of the limit equation is

$$- \int_0^1 \partial_x u \partial_x \varphi - \alpha\gamma v(a)\varphi(a) = \int_0^1 g\varphi \ .$$

Finding the limit equation

$$A_0 = \begin{pmatrix} 0 & Id \\ \partial_{xx}^2 & -\alpha\gamma\delta_{x=a} \end{pmatrix}$$

Proposition

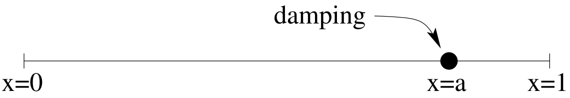
M : vertical mean operator.
J : injection onto vertically constant functions.

$$\|A_\varepsilon^{-1} - JA_0^{-1}M\|_{\mathcal{L}(\mathbb{H}_\varepsilon^1 \times \mathbb{L}^2)} \leq C\sqrt{\varepsilon} \ .$$

The limit equation

The limit equation is the one-dimensional wave equation with ponctual damping :

$$\begin{cases} u_{tt} + \alpha\gamma\delta_{x=a}u_t(a) = u_{xx} \\ u(0) = u(1) = 0 \end{cases}$$

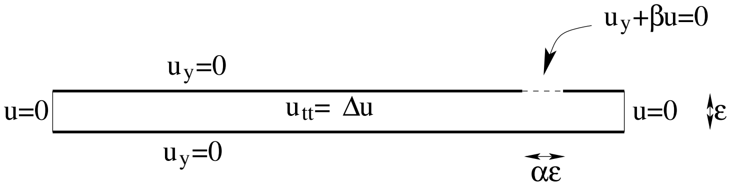


When a is close to the boundary, the low frequencies are efficiently damped.

References : Fabre, Jaffard, Puel, Tucsnak, Zuazua...

When the hole is open

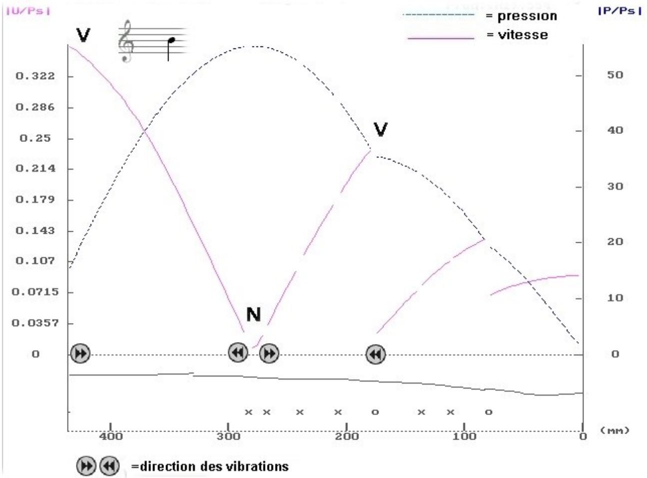
Let us now open the lateral hole.

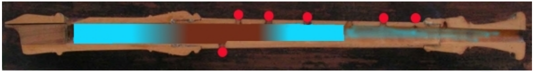


The limit equation is $\begin{cases} u_{tt} = u_{xx} - \alpha\beta\delta_{x=a}u(a) \\ u(0) = u(1) = 0 \end{cases}$.

Real measure of the pressure

The measure of pressure in the tube of a recorder (flûte à bec)



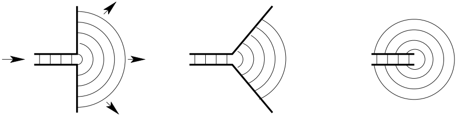


The choice of the BC

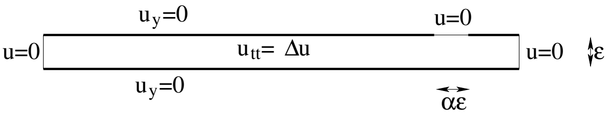
At the exit of a tube we put :

- Dirichlet B.C. $u = 0$
  Rigourously obtained by mathematicians
  (*dumbbell domain*)
  Arrieta, Jimbo, Morita...

- Robin B.C. $\frac{\partial u}{\partial \nu} + \beta u = 0$ (or better $\frac{\partial u}{\partial \nu} + \gamma u_t = 0$?)
  Obtained by acousticians.

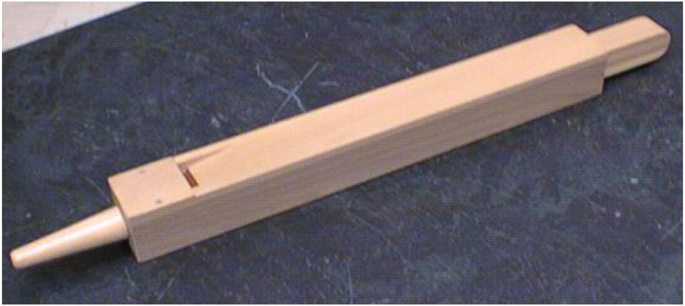


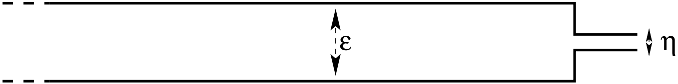
Lateral Dirichlet BC



The limit equation is $\begin{cases} u_{tt} = u_{xx} & \text{on }]0,a[\cup]a,1[\\ u(0) = u(a) = u(1) = 0 \end{cases}$.

Organ pipe





$\eta = \varepsilon \implies$ Dirichlet BC

$\eta = \varepsilon^p \ (p > 1) \implies$ Neumann BC

References : Arrieta, Ciuperca