The PML Method: Continuous and Semidiscrete Waves.

Sylvain Ervedoza[1] Enrique Zuazua[2]

[1]Laboratoire de Mathématiques de Versailles.

[2]Universidad Autónoma, Madrid.

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Outline of the talk.

1. Introduction to the PML method.
2. In the continuous level:
   Exponential decay of the energy.
3. In the semi-discrete level:
   1. Dynamics of the semi-discrete waves.
   2. Spectral theory: Existence of localized eigenvectors.
4. How to recover the exponential decay in the semi-discrete level ?
   - Numerical Viscosity.
   - Other numerical schemes.

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Problem.

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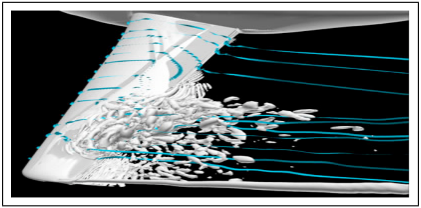
To solve *numerically* a wave type equation in exterior domain.

- Bound the domain ? With which boundary conditions ?

Ex: CEM, Aerodynamics, Sismology ...



On the 1d wave equation.

$$\begin{cases} \partial_{tt}^2 u - \partial_{xx}^2 u = 0, & t > 0,\ x \in (-1, \infty) \\ u(t=0) = u_0, & \partial_t u(t=0) = u_1 \\ u(x=-1) = 0 \end{cases} \tag{1}$$

with $\mathrm{Supp}(u_0, u_1) \subset (-1, 0)$.

Absorbing Boundary Conditions (Engquist-Majda)*:

$$\partial_t u(x=0,t) + \partial_x u(x=0,t) = 0, \quad t > 0.$$

Problem:

In higher dimension, **non local** operators, hard to compute **numerically**.

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Perfectly Matched Layer.

Perfectly Matched Layer (PML, Bérenger†): Adding a layer $\mathcal{C}$ around the domain we are interesting in such that:

- No reflexion at the interface.
- Any wave coming inside $\mathcal{C}$ does **not** (or almost not) come back.

In **1d**, in the case where the interesting domain is $(-1, 0)$.

▷ Step 1: Put the system in the hyperbolic form:

$$\begin{cases} \partial_t P + \partial_x V = 0 & t > 0,\ x \in (-1, \infty) \\ \partial_t V + \partial_x P = 0 & t > 0,\ x \in (-1, \infty) \\ P(x = -1, t) = 0 & \\ P(t = 0) = P_0, \quad V(t = 0) = V_0, & \end{cases} \tag{2}$$

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The PML system.

▷ Step 2: Putting an absorbing coefficient in the added layer $\mathcal{C} = (0, r)$:

$$\begin{cases} \partial_t P + \partial_x V + \chi_{(0,r)}\sigma P = 0 & t > 0,\ x \in (-1, r) \\ \partial_t V + \partial_x P + \chi_{(0,r)}\sigma V = 0 & t > 0,\ x \in (-1, r) \\ P(x = -1) = P(x = r) = 0 & \\ P(t = 0) = P_0, \quad V(t = 0) = V_0, & \end{cases} \tag{3}$$

σ nonnegative function of our choice, $\mathrm{Supp}(\sigma) \subset (0, r)$.

To simplify, we choose $r = 1$ in the sequel.

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Advantages and inconvenients.

Inconvenients:

- System is only weakly well-posed !
  $\rightarrow$ Loss of regularity on the data[‡].
- Multiplication of the numbers of variables: The PML system for the 2d wave equation has 4 unknows !

Advantages:

- Excellent numerical results.
- Robust and adaptable in higher dimension and on more complex systems (for instance advective acoustics.

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Some references:

- Abarbanel and Gottlieb, 1997, 1998, 1999.
- Collino-Monk, 1998.
- Lassas-Sommersalo, 1998.
- Petropoulos, 1998.
- Lions-Metral-Vacus, 2002.
- Bécache and Joly, 2002.
- Bécache, Fauqueux and Joly, 2003.
- Bermùdez, Hervella-Nieto, Prieto and Rodriguez, 2006.
- etc...

Survey: Tsynkov, *Numerical solution of problems on unbounded domains. A review*, Appl. Num. Analysis, 27:533-557, 1998.

Energy of the continuous system (3).

The PML system (3) has an energy

$$E(t) = \frac{1}{2}\int_{-1}^{1} \left(|P(t,x)|^2 + |V(t,x)|^2\right)\, dx$$

dissipated according to

$$\frac{dE}{dt}(t) = -\int_{0}^{1} \sigma(x)\left(|P(t,x)|^2 + |V(t,x)|^2\right)\, dx.$$

Hence the system (3) is well-posed.

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Decay rate.

For the free system (2),

$$P(t,x) = V(t,x) = 0, \quad t > 2, \; x \in (-1,0).$$

$\Longrightarrow$ The energy of the solution of (3) should be small when $t > 2$.

More precisely, the energy is exponentially decreasing according to the rate

$$\omega(\sigma) = \sup \{\omega : \; \exists C(\omega), \; \forall (P_0, V_0) \in (L^2(-1,1))^2, \\ \forall t, \; E(t) \leq C(\omega) E(P_0, V_0) \exp(-\omega t)\}$$

Measure of the efficiency of the PML system.

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Measure of the efficiency of the PML system.

Study of the spatial operator L.

$$L(P,V) = (\partial_x V + \chi_{(0,1)}\sigma P, \partial_x P + \chi_{(0,1)}\sigma V)$$
$$D(L) = H_0^1(-1,1) \times H^1(-1,1).$$

Theorem

1. L^{-1} is a compact operator.
2. The eigenvalues of L are given by

$$\lambda_k = \frac{1}{2}\Big(\int_0^1 \sigma(x)\, dx + ik\pi\Big), \ k \in \mathbb{Z}.$$

3. The eigenvectors constitute a Riesz basis.

Sketch of the proof.

The system becomes diagonal by setting

$$Q = P + V, \quad R = P - V.$$

$\Longrightarrow$ *Explicit* construction of an isomorphism $\mathcal{I} : L^2(-1,1)^2 \to L^2(-3,1)$ mapping the eigenvectors to the canonical Fourier basis of $L^2(-3,1)$.

If

$$\theta(x) = \int_{-1}^{x} (\sigma(z) - \frac{I}{2})\, dz, \quad I = \int_0^1 \sigma. \tag{4}$$

then $W = \mathcal{I}(f,g)$ is defined by

$$W(x) = \begin{cases} (f+g)(x)e^{\theta(x)}, & -1 < x < 1 \\ (g-f)(-2-x)e^{-\theta(-2-x)}, & -3 < x < -1. \end{cases}$$

Consequences.

Theorem

The energy of system (3) is exponentially decreasing:

$$\forall t > 0, \ E(t) \leq E(0)\exp(4\|\theta\|_\infty - It).$$

Besides, $\|\theta\|_\infty \leq I = \int_0^1 \sigma(x)\, dx$.

Very precise estimate of the conditioning number thanks to the *explicit* isomorphism $\mathcal{I}$

$$\kappa(\mathcal{I}) = \|\mathcal{I}\| \|\mathcal{I}^{-1}\| = \exp(2\|\theta\|_\infty).$$

Better ?

Let us define respectively the left and right energies:

$$E_l(P,V) = \frac{1}{2}\int_{-1}^{0} \left(|P(x)|^2 + |V(x)|^2\right)\, dx,$$
$$E_r(P,V) = \frac{1}{2}\int_{0}^{1} \left(|P(x)|^2 + |V(x)|^2\right)\, dx.$$

Theorem

Let P_0, V_0 initial data with support in $(-1,0)$. Then

$$E_l(P(t),V(t)) \leq \exp(l(2-t))E_0$$
$$E_r(P(t),V(t)) \leq \exp(l + 2\|\theta\|_\infty - lt)E_0.$$

Remarks.

- The spectrum is totally explicit: On the contrary, for the damped wave equation[§] it is not. However the spectrums are close in high frequencies.
- Contrary to the damped wave equation, there is no overdamping phenomenon.
- The spectral theory provides optimal results, cf caracteristics.
- We recover the critical time $t = 2$, corresponding to the time needed by the waves to get out of $(-1, 0)$.

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- The decay rate depends only on $I = \int_0^1 \sigma(x)\, dx$.
- The function θ is explicit,

$$\theta(x) = \int_{-1}^{x} (\sigma(z) - \frac{I}{2})\, dz$$

and satisfies the following properties:

$$\theta(-1) = \theta(1) = 0, \quad \theta(0) = -\frac{I}{2},$$
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Semi-discrete system.

$$\begin{cases} \partial_t P_j + \frac{V_{j+1}-V_j}{h} + \sigma_j P_j = 0, \ |j| \leq N-1 \\ \partial_t V_j + \frac{P_j - P_{j-1}}{h} + \sigma_{j-1/2} V_j = 0, \ -N+1 \leq j \leq N \\ P_{-N} = P_N = 0, \end{cases} \tag{5}$$

The energy

$$E_h(t) = \frac{h}{2} \sum_{j=-N+1}^{N} \left(|P_j(t)|^2 + |V_j(t)|^2 \right)$$

is dissipated:

$$\frac{dE_h}{dt}(t) = -h \sum_{j=-N+1}^{N} \left(\sigma_j |P_j|^2 + \sigma_{j-1/2} |V_j|^2 \right).$$

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What the discretization changes...

Theorem

There are no positive constants C and μ such that for all h small enough,

$$E_h(t) \leq C\, E_h(0)\exp(-\mu t),$$

Two proofs:

1. Existence of localized spurious waves.
2. Existence of localized eigenvectors.

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Propagation of the rays.

Bicaracteristics associated to the principal symbol[¶]:

$$\tau^2 - \omega_h(\xi)^2, \quad \omega_h(\xi) = \frac{2}{h}\sin\left(\frac{\xi h}{2}\right).$$

The rays are the solution of

$$\begin{cases} \frac{dx}{ds} = 2\omega_h(\xi)\frac{d\omega_h}{d\xi}(\xi), & \frac{d\xi}{ds} = 0, \\ \frac{dt}{ds} = -2\tau, & \frac{d\tau}{ds} = 0. \end{cases}$$

with initial data τ_0, ξ_0 such that

$$\tau_0^2 - \omega_h(\xi_0)^2 = 0.$$

Especially,

$$\frac{dx}{dt} = \pm\frac{d\omega_h}{d\xi}(\xi_0) = \pm\cos\left(\frac{\xi_0 h}{2}\right).$$

[¶] Burq-Gérard, Cours à l'X

Setting $\zeta_0 = \xi_0 h$, the rays are straight lines

$$X_{\pm}^{\zeta_0} : (x_0, t) \to x_0 \pm t\cos(\zeta_0/2). \quad (6)$$

To justify this approach, one can construct a solution of (5) localized around the rays‖ (6) which do not propagate.

‖See also Maciá, phd thesis.

Spectral analysis.

In the special case $\sigma(x) = \sigma\chi_{(0,1)}(x)$.

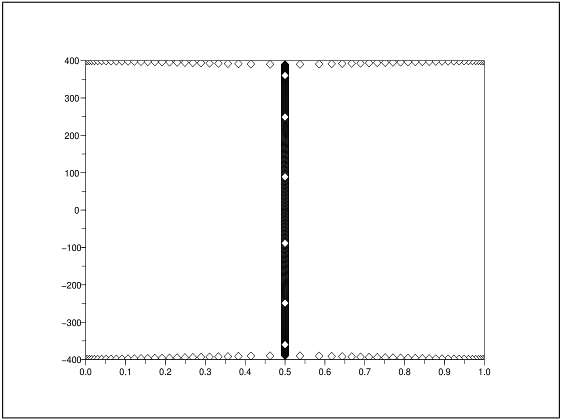
In this case, the eigenvalues λ of the discrete space operator L_h satisfy:

$$\sinh\left(\frac{\alpha h}{2}\right) = \frac{\lambda h}{2} \quad ; \quad \sinh\left(\frac{\beta h}{2}\right) = \frac{(\lambda - \sigma)h}{2}.$$

$$\sinh(\alpha)\cosh(\beta)\cosh\left(\frac{\beta h}{2}\right) + \cosh(\alpha)\sinh(\beta)\cosh\left(\frac{\alpha h}{2}\right) = 0.$$

$\Rightarrow$ Complex Analysis, especially Rouché's theorem.

Numerical Simulation $\sigma = 1$, $N = 200$.



The high frequencies eigenvectors are not dissipated !

Spectral analysis: Low frequency.

Theorem

Let $\delta < 1$. Then there exists a constant C such that the set of the eigenvalues λ_h of L_h such that $|\mathcal{I}m(\lambda_h)h| < 2\delta$ has exactly one point in each disk $D_k^h(\hat{\lambda}_k^h, Ch)$ of center

$$\hat{\lambda}_k^h = \frac{2i}{h}\sin\left(\frac{k\pi h}{4}\right) + \frac{\sigma}{2},$$

k satisfying $|\sin\left(\frac{k\pi h}{4}\right)| \leq \delta$.

Spectral analysis: High Frequency.

Theorem

One can find a sequence of eigenvalues λ_h of L_h such that

- $\mathcal{I}m(\lambda_h)h \to 2$.
- $\mathcal{R}e(\lambda_h) \to 0$.

Remark: These two theorems describe entirely the behaviour of the spectrum numerically computed.

Repartition of the eigenvectors.

If (P_h, V_h) is an eigenvector corresponding to the eigenvalue $\lambda_h = a_h + ib_h$, setting

$$\begin{cases} E_h^l = \frac{h}{4}|P_0|^2 + \frac{h}{2}\sum_{j=1}^N(|P_j|^2 + |V_j|^2), \\ E_h^r = \frac{h}{4}|P_0|^2 + \frac{h}{2}\sum_{j=-N+1}^0(|V_j|^2 + |P_{j-1}|^2), \end{cases}$$

the left and right energies, then

$$\frac{E_h^r(P_h, V_h)}{E_h^l(P_h, V_h)} = \frac{a_h}{\sigma - a_h}.$$

$\Longrightarrow$ Existence of localized eigenvectors.

On the damped wave equation.

Finite difference semi-discrete damped wave equation:

$$\begin{cases} \partial_{tt}^2 u_j - \Delta_h u_j + 2a_j \partial_t u_j = 0, & |j| < N-1, \\ u_{-N} = u_N = 0 \end{cases} \tag{7}$$

Energy:

$$E_h(t) = \frac{h}{2} \sum_{j=-N}^{N-1} |\partial_t u_j|^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2$$

Known results**: No uniform exponential decay of the energy!

**Infante-Zuazua, Boundary Observability for the space semi-discretizations of the 1d wave equation, 1999

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Remedies.

- Damp the high frequencies: Adding a numerical viscosity.
  Ramdani-Takahashi-Tucsnak, Tcheugoué-Tébou-Zuazua.
- Modify the propagation of the waves: Looking for other discretizations.
  Banks-Ito-Wang, Castro-Micu, Mixed finite element method.
- Other methods: Bi-grid method, ...
  Glowinski, Negreanu-Zuazua, Ignat-Zuazua, etc.

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Viscous System.

$$\begin{cases} \partial_t P_j + \frac{V_{j+1}-V_j}{h} + \sigma_j P_j - \alpha h^2 (\Delta_h P)_j = 0, \\ \partial_t V_j + \frac{P_j - P_{j-1}}{h} + \sigma_{j-1/2} V_j - \alpha h^2 (\Delta_h V)_j = 0, \\ P_{-N} = P_N = 0, \quad V_{-N} = V_{-N+1}, \quad V_{N+1} = V_N, \end{cases} \tag{8}$$

with $\alpha > 0$ and

$$(\Delta_h A)_j = \frac{1}{h^2}(A_{j+1} + A_{j-1} - 2A_j).$$

In the sequel, we fix $\alpha > 0$, for instance $\alpha = 1$.

Dissipation Law.

The energy

$$E_h(t) = \frac{h}{2} \sum_{j=-N+1}^{N} \left(|P_j(t)|^2 + |V_j(t)|^2 \right)$$

is dissipated according to

$$\frac{dE_h}{dt}(t) = -h \sum_{j=-N+1}^{N} \sigma_j |P_j|^2 - h \sum_{j=-N+1}^{N} \sigma_{j-1/2} |V_j|^2$$
$$- \alpha h^3 \sum_{j=-N}^{N-1} \left(\left(\frac{P_{j+1} - P_j}{h} \right)^2 + \left(\frac{V_{j+1} - V_j}{h} \right)^2 \right).$$

Uniform decay.

Theorem

If α is positive and σ is a non-trivial non-negative function, then there exist two positive constants C and μ such that for all $h > 0$, for any initial datum (P_0^h, V_0^h), the energy of the solution (P, V) of (8) satisfies

$$E_h(t) \leq CE_h(0)\exp(-\mu t), \ t > 0.$$

Sketch of the proof.

1. Show the equivalence with the following observability inequality (HUM)

$$E_h(0) \leq C\Big(h\sum_j \int_0^T (\sigma_j |P_j|^2 + \sigma_{j-1/2}|V_j|^2)\,dt$$
$$+\alpha h^3 \sum_j \int_0^T \Big[\Big(\frac{P_{j+1}-P_j}{h}\Big)^2 + \Big(\frac{V_{j+1}-V_j}{h}\Big)^2\Big]\,dt \Big).$$

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Can we estimate μ ?

We proved

$$E_h(t) \leq C E_h(0) \exp(-\mu t), \; t > 0.$$

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Open problem.

- Can we choose the viscous parameter such that the spectral abscissa of the spatial viscous operator L_h^{visc} coincides with the one of the continuous spatial operator L ?
- Do the eigenvectors constitute a uniform Riesz basis ?

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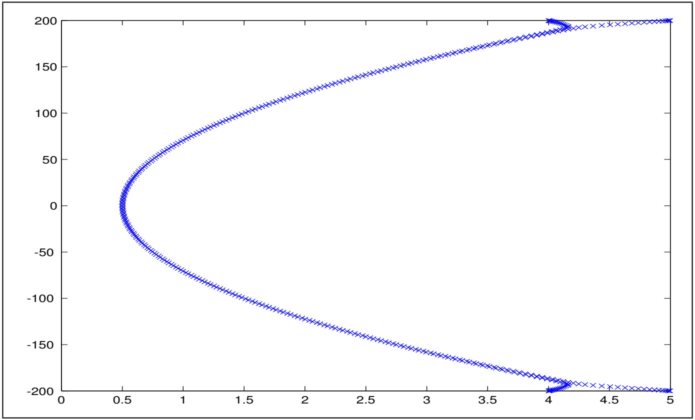
Spectrum of L_h^{visc}, $N = 200$, $\sigma = 1$.

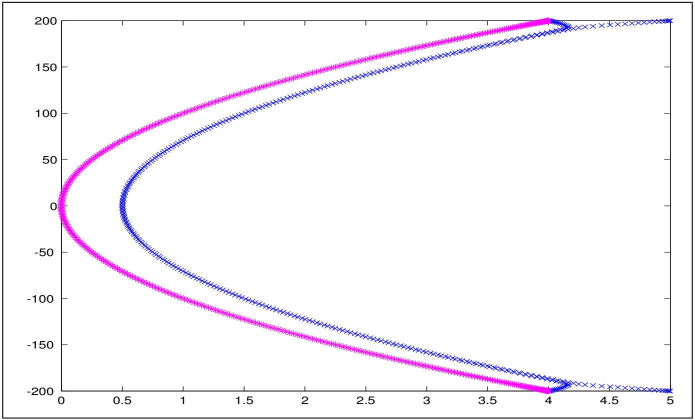


The high frequencies are damped !

Comments.

- Parabolic shape $\mathcal{C}$ similar to the one we would obtain for $\sigma = 0$.
- The curve is very close to the one we would obtain by adding at a given frequency the abscissas of both curves $\mathcal{C}$ and the one given by the spectrum without viscosity.
- In high frequencies, the spectrum splits up into two branches.

Graphics.



Parabolic shape.

What we can do...

Hypothesis:

The spectrum of L_h is close to the spectrum of L untill an order ϵ/h.

Theorem

Setting $\alpha = l/\epsilon$, the spectral abscissa of L_h^{visc} converges to $l/2$ when $h \to 0$.

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Sketch of the proof.

We write system (8) under the form

$$\partial_t(P, V) + (A_h + B_h)(P, V) = \alpha h^2 A_h^2 (P, V),$$

with $A_h + B_h = L_h$,

$$A_h = \begin{pmatrix} 0 & \partial_x^h \\ \partial_x^h & 0 \end{pmatrix}, \quad B_h = \begin{pmatrix} \sigma^h & 0 \\ 0 & \sigma^h \end{pmatrix}.$$

We study the perturbed system

$$\partial_t(P, V) + (A_h + B_h)(P, V) = \alpha h^2 (A_h + B_h)^2 (P, V),$$

whose spectrum Λ can be deduced easily from the spectrum of $\Lambda(L_h)$:

$$\lambda = a + ib \in \Lambda(L_h) \Leftrightarrow \mu(\alpha) = \lambda - \alpha h^2 \lambda^2 \in \Lambda.$$

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Another discretization.

Another discretization (still finite difference approximation):

$$\begin{cases} \partial_t \left(\frac{P_j + P_{j+1}}{2} \right) + \frac{V_{j+1} - V_j}{h} + \sigma_j \frac{P_j + P_{j+1}}{2} = 0, \; j \leq N, \\ \partial_t \left(\frac{V_j + V_{j+1}}{2} \right) + \frac{P_{j+1} - P_j}{h} + \sigma_j \frac{V_j + V_{j+1}}{2} = 0, \; -N \leq j, \\ P_{-N} = V_N = 0. \end{cases} \tag{9}$$

Remark: This also corresponds to a *Mixed Finite Element* discretization.

New dynamics ?

The principal symbol becomes

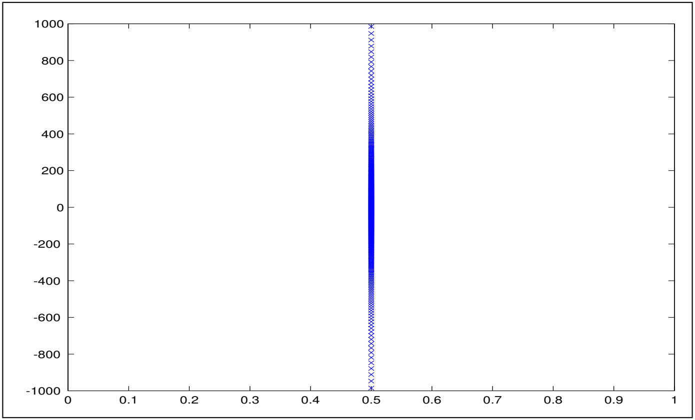
$$\tau^2 - \omega_h(\xi)^2, \quad \omega_h(\xi) = \frac{2}{h}\tan\left(\frac{\xi h}{2}\right)$$

In particular, the velocity of the waves is

$$\frac{d\omega_h}{d\xi}(\xi) = 1 + \tan\left(\frac{\xi h}{2}\right)^2 \geq 1.$$

$\Longrightarrow$ The waves propagate.

Spectrum of the spatial operator in (9), $N = 100$, $\sigma = 1$.



Blow up and localization of the abscissa !

Remarks.

- The spectral abscissa of the spatial operator in (9) is concentrated around $l/2 = \int_0^1 \sigma(x)\, dx/2$ as in the continuous case.
- The ordinates of the high frequency eigenvectors blow up. This can create numerical instabilities.
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Energies.

System (9) has a whole family of energies:

$$E_h^{\alpha}(t) = \frac{h}{2}\sum_{j=-N}^{N-1}\Big(\frac{P_j(t)+P_{j+1}(t)}{2}\Big)^2 + \Big(\frac{V_j(t)+V_{j+1}(t)}{2}\Big)^2$$

$$+\,\alpha\left(\frac{h}{2}\sum_{j=-N}^{N-1}\Big(\frac{P_j'(t)+P_{j+1}'(t)}{2}\Big)^2 + \Big(\frac{V_j'(t)+V_{j+1}'(t)}{2}\Big)^2\right).$$

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Dissipation law.

These energies are dissipated:

$$\frac{d}{dt}E_h^{\alpha} = -h\sum_{j=-N}^{N-1}\sigma_j\Big(\Big(\frac{P_j(t)+P_{j+1}(t)}{2}\Big)^2+\Big(\frac{V_j(t)+V_{j+1}(t)}{2}\Big)^2\Big)$$

$$-\alpha h\sum_{j=-N}^{N-1}\sigma_j\Big(\Big(\frac{P_j'(t)+P_{j+1}'(t)}{2}\Big)^2+\Big(\frac{V_j'(t)+V_{j+1}'(t)}{2}\Big)^2\Big).$$

Uniform exponential decay.

Theorem

There exist two positive constants C and μ such that for all h small enough, for any initial data (P^0, V^0), the solution (P, V) of (9) satisfies :

$$\forall t, \; E_h^{\frac{h^2}{4}}(t) \leq C E_h^{\frac{h^2}{4}}(0) \exp(-\mu t)$$

Idea: Using HUM to restrict ourselves to the case $\sigma = 0$.

Sketch of the proof.

HUM Method: We have to prove that the observability inequality

$$e_h^{h^2/4}(0) \leq$$

$$Ch \int_0^T \sum_{j=-N}^{N-1} \sigma_j \Big(\Big(\frac{p_j(t)+p_{j+1}(t)}{2}\Big)^2 + \Big(\frac{v_j(t)+v_{j+1}(t)}{2}\Big)^2 \Big)\, dt$$

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holds for any (p, v) solution of the corresponding conservative system ($\sigma = 0$ in (9)).

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- Can we estimate the decay rate μ ? Open problem.
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Conclusion.

- Discretizing a wave equation change the dynamics, especially in the high frequencies. This high frequency dynamic is very sensitive to the scheme we use.
- It is difficult to estimate precisely the uniform decay rate μ.
- In higher dimension ? With other discretizations ? Many open questions.
- What happens on non-uniform mesh ? *Work in progress...*

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Thanks.

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New Materials, adaptative systems and and their nonlinearities: modeling, control and numerical simulation.

Thank you for your attention.

For more details, see *Perfectly Matched Layers in 1d: Energy decay for continuous and semi-discrete waves*, to be published.