# 3D-2D analysis for the optimal elastic compliance problem .

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Motivation: Optimality conditions and mechanical justification for optimization pbs of the kind

$$\sup \left\{ < F, u > : \ \nabla^2 u \in K \right\}$$

$\leftrightsquigarrow$ optimal measure $\mu$ (Lagrange multiplier).

I.Fragalà, GB : JFA (2003), ARMA **184** (2), 257–284 (2007)], SICON (to appear)

I.Fragalà, GB, P. Seppecher : CRAS and in preparation

# Classical Kirchoff model for thin plates

- $\Omega \subset \mathbb{R}^2$ midplane of the plate
- $F$ smooth load
- Unknown: optimal thickness $h$

$$\left\{ h \in L^\infty(\Omega) \ : \ a \leq h(x) \leq b \ , \ \int_\Omega h \, dx = m \right\}$$

- Criterion: miminization of compliance

$$\boxed{\mathcal{E}(h) = \inf \left\{ \int_\Omega F u \, dx \right\}}$$

- $u$ deflection

$$\mathcal{E}(h) = - \inf \left\{ \int_\Omega \left( M(h) \nabla^2 u \cdot \nabla^2 u - F u \right) dx \ : \ u \in H^2(\Omega) + \text{b.c.} \right\}$$

- Cubic dependence: $M(h) \sim M_0 \, h^3$

# Appearance of concentrated micro-structures

• '80s: nonesistence of solutions [Banichuk, Bendsoe, Caillerie, Cheng-Olhoff, Gibiansky-Cherkaev, Lurie, Kohn-Vogelius]

• Same homogenization phenomena for conductors/ elastic materials [Allaire-Kohn, Francfort-Murat, Kohn-Strang, Murat-Tartar]

•'90s: relaxations [Bonnetier-Conca, Bonnetier-Vogelius, Munoz-Pedregal]

• Still: – thickness $h$ depending on only *one* variable

– If upperbound $b$ increases and $m \ll 1$, $h$ becomes maximal on thin perforated 1D-layers (stiffeners).

– no efficient (better concentrate material on top and buttom)

$\rightsquigarrow$ adopt a different point of view

# Mass optimization problem (MOP)

- $\Omega \subset \mathbb{R}^2$ design region
- Unknown: optimal distribution of mass $\mu$

$$\mathcal{K} := \left\{ \mu \in \mathcal{M}^+ \ : \ \mathrm{spt}(\mu) \subseteq \overline{\Omega} \, , \ \int d\mu = m \right\}$$

- Criterion: minimize the plate compliance under a given load $F \in \mathcal{D}'$

$$\mathcal{C}(\mu, j, F) = -\inf \left\{ \int j(\nabla^2 u) \, d\mu - \langle F, u \rangle_{\mathbb{R}} \ : \ u \in \mathcal{D} \right\}$$

$$(\mathrm{MOP}) \qquad \boxed{\mathcal{I} = \inf \left\{ \mathcal{C}(\mu, j, F) \ : \ \mu \in \mathcal{K} \right\}} \qquad \text{is attained}$$

# (MOP) $\leftrightsquigarrow$ linear constrained problem (LCP)

**Theorem.** There holds $\mathcal{I} = \mathcal{S}^2/2$ , where

$$(\text{LCP}) \qquad \boxed{\mathcal{S} = \sup \Big\{ \langle f, u\rangle \ : \ j(\nabla^2 u) \leq 1/2 \text{ on } \Omega \Big\}}$$

*Proof.*

$$\begin{aligned}
\mathcal{I} &= \inf_{\mu\in\mathcal{K}} \Big\{ -\inf_{u\in\mathcal{D}} \Big[ \int j(\nabla^2 u)\, d\mu - \langle f, u\rangle \Big] \Big\} \\
&= \inf_{\mu\in\mathcal{K}} \Big\{ \sup_{u\in\mathcal{D}} \Big[ -\int j(\nabla^2 u)\, d\mu + \langle f, u\rangle \Big] \Big\} \\
&= \sup_{u\in\mathcal{D}} \Big\{ -\sup_{\mu\in\mathcal{K}} \Big[ \int j(\nabla^2 u)\, d\mu \Big] + \langle f, u\rangle \Big\} \\
&= \sup_{u\in\mathcal{D}} \Big\{ \langle f, u\rangle - \| j(\nabla^2 u)\|_{L^\infty(\Omega)} \Big\} = \frac{\mathcal{S}^2}{2}
\end{aligned}$$

□

REMARK: The unknown $\mu$ disappeared !

# Goals

• MODELLING

Does (MOP) (or (LCP)) admit any mechanical justification ?

Can it be derived from $3D$ elasticity, and which is the link (if any) with the Kirchoff model ?

• OPTIMIZATION

How to compute $\mathcal{I}$ (or $\mathcal{S}$) ?

Is it possible to give optimality conditions useful to find out explicit solutions ?

# 3D- shape optimization problem

- $m > 0$ given amount of mass
- $Q = \Omega \times [-h, h] \subset \mathbb{R}^3$ design region
- Unknown : $\{A \text{ open } \subset Q \; : \; |A| = m\}$
- Criterion: minimize over admissible $A$ the *elastic compliance*

$$\mathcal{C}^{el}(A) := - \inf_{u \in \mathcal{D}} \left\{ \int_A j(e(u))\, dx - \langle F, u \rangle \right\}$$

(under a given system of forces $F \in H^{-1}(Q; \mathbb{R}^3)$)

- Elastic potentials:

*Strain potential*: $j : \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}$ positive, quadratic form

*Stress potential*: $j^* : \mathbb{R}^{3\times 3}_{sym} \to \mathbb{R}$ the Moreau-Fenchel conjugate of $j$.

- It will be useful writing: $j(z) = \frac{1}{2}\big(\rho(z)\big)^2 \quad , \quad j^*(z) = \frac{1}{2}\big(\rho^0(z^*)\big)^2$

where $\rho^0(z^*) := \sup\{z \cdot z^* \; : \; \rho(z) \leq 1\}$ .

The shape optmization pb is ill-posed (minimizing sequences may oscillate). To avoid the heavy relaxation procedure (through composite micro-structures), engineers often adopt the

Convexification procedure: $A \rightsquigarrow \theta \in L^\infty(Q, [0,1])$

The class of admissible sets is enlarged to measures $\mu = \theta\, dx$ such that $\theta \in [0,1], \int \theta\, dx = m$

Definition: Let $\mu$ be a positive measure supported on $Q$. We associate the elastic compliance ($F$ is now a vector distribution )

$$\mathcal{C}^{el}(\mu, j, F) := -\inf_{u \in \mathcal{D}} \left\{ \int j(e(u))\, d\mu - \langle F, u \rangle \right\}$$

Scaling property: $\mathcal{C}^{el}(\varepsilon\mu, j, tF) \;=\; \dfrac{t^2}{\varepsilon}\, \mathcal{C}^{el}(\mu, j, F) \quad (v = \dfrac{\varepsilon}{t} u)$ .

# 3D-2D Analysis

Asymptotic analysis: $\begin{cases} m \rightsquigarrow \varepsilon \\ h \rightsquigarrow \delta \quad , \quad Q \rightsquigarrow Q_\delta \\ \varepsilon \to 0 \quad , \quad \delta \to 0 \end{cases}$

Different strategies: $\boxed{\tau := \frac{\varepsilon}{\delta}}$

**A** $\varepsilon \to 0$ , then $\delta \to 0$ : vanishing filling ratio $\tau$

**B** $\varepsilon \to 0$ with $\varepsilon = \tau\,\delta$: fixed filling ratio $\tau$

(and after possibly $\tau \to 0$)

**C** Additional topological constraint: $A = \{|x_3| \leq \varepsilon\, f(x_1, x_2)\}$ , $\int_\Omega f = 1$

**GOAL :**

Characterize the related rescaled limits as $(\varepsilon, \delta) \to (0,0)$ of

$$\mathcal{I}_{\varepsilon,\delta} := \inf \left\{ C^{el}(\theta, j, \sqrt{\varepsilon} F^{\delta}) \ : \ \theta \in \{0,1\} \ , \ \int_{Q_\delta} \theta = \varepsilon \right\}$$

$$\tilde{\mathcal{I}}_{\varepsilon,\delta} := \inf \left\{ C^{el}(\theta, j, \sqrt{\varepsilon} F^{\delta}) \ : \ \theta \in L^{\infty}(Q, [0,1])\} \ , \ \int_{Q_\delta} \theta = \varepsilon \right\}$$

# Plan

1. Limit as $\varepsilon \to 0$ (first step in strategy A)

2. Duality and linear constraint problem.

3. Compliance model from stategy A

4. Compliance model from strategy B (fixed filling rato $\tau$))

5. Example (mixed flexion and membrane regime)

6. Some explicit solutions of (LCP)

# 1- Limit as $\varepsilon \to 0$

It falls in the theory of light structures (truss-like Michell's structures).

**THEOREM 1** For fixed $\delta > 0$, one has

i) $\lim_{\varepsilon\to 0} \tilde{\tau}_{\varepsilon,\delta} = \inf\{\mathcal{C}^{el}(\mu, j, F^\delta) \ : \ \int \mu = 1, \ \mathrm{spt}(\mu) \subset Q_\delta\}$

ii) $\lim_{\varepsilon\to 0} \tau_{\varepsilon,\delta} = \inf\{\mathcal{C}^{el}(\mu, j_0, F^\delta) \ : \ \int \mu = 1, \ \mathrm{spt}(\mu) \subset Q_\delta\}$

where $j_0$ is given by (still 2-homogeneous, non quadratic)

$$j_0(e) \ := \ \sup\{e \cdot \xi - j^*(\xi) \ : \ det\, \xi = 0\} \ .$$

**Comments:** - We are reduced to a max-min problem in $(u, \mu)$ for which existence of solutions holds.

NB: $\mu$ might be a concentrated measure (surely if $F$ is a discrete force)

- Assertion ii) is a reformulation of a result by [G. Allaire and R. Kohn]. Note that $j^0(z) < j(z)$ except for degenerate tensors.

Ex.: $j = |z|^2 \Rightarrow j_0(z) = |\lambda_3(z)|^2 + |\lambda_2(z)|^2$

# 2- Duality and linear constraint problem

- Commutation argument for $\sup\inf = \inf\sup$ applies. Thus

$$\mathcal{I}_{0,\delta}(\tilde{\mathcal{I}}_{0,\delta}) := \inf_{\int \mu = 1} \sup_{u \in \mathcal{D}} \left\{ \langle F^\delta, u \rangle - \int j_0(e(u))\, d\mu \right\}$$

$$= \sup_{u \in \mathcal{D}} \left\{ \langle F^\delta, u \rangle - \| j_0(e(u)) \|_{L^\infty(Q_\delta)} \right\} = \frac{S_\delta^2}{2} \left( \frac{\tilde{S}_\delta^2}{2} \right) ,$$

$$(LCP)_\delta \qquad S_\delta(\tilde{S}_\delta) := \sup_{u \in \mathcal{D}} \left\{ \langle F^\delta, u \rangle \ : \ \rho_0^0(e(u)) (\rho^0(e(u))) \leq 1 \text{ in } Q_\delta \right\} .$$

- Recovering measure $\mu$ trough dual problem

$$(MOP)_\delta \qquad \inf \left\{ \int \rho_0^0(\lambda) \ : \ \mathrm{spt}\lambda \subset (Q_\delta) \ , \ -\mathrm{div}\lambda = F^\delta \text{ in } \mathbb{R}^3 \right\} ,$$

- Optimality of a triple $(u, \mu, \sigma)$ (where $\lambda = \sigma\mu$ , $\rho_0^0(\sigma) = 1$)
- Link with Monge-Kantorovich mass transport theory (only in scalar case) [G. Buttazzo, GB, JEMS(2001)])

# Michell's bridge example

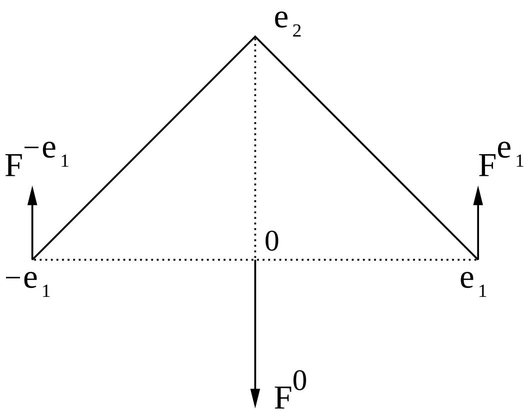



Figure 1: An admissible truss for the "bridge".

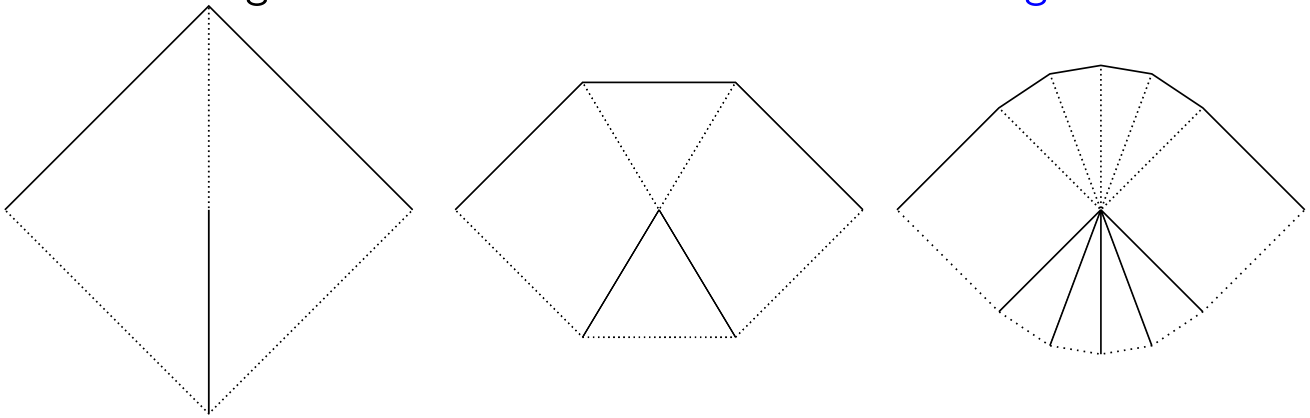


Figure 2: Construction of the optimal measure

[W. Gangbo, P. Seppecher, GB, preprint]

# 3- Compliance model from stategy A

• Rescaling of the load: We need to pass to the limit in $\delta$ in $(LCP)_\delta$ (or $(MOP)_\delta$). This limit blows up if $F_3 \neq 0$, due to Korn constant of order $\delta^{-1}$ As usual in flexion regime, we start with $F \in H^{-1}(Q;\mathbb{R}^3)$ and set (Important: the vertical component is multiplied by $\delta$)

$$F^\delta = \left(\frac{1}{\delta}F_1(x_1,x_2,\frac{x_3}{\delta}),\frac{1}{\delta}F_2(x_1,x_2,\frac{x_3}{\delta}),F_3(x_1,x_2,\frac{x_3}{\delta})\right)$$

• Averaging the load in $x_3$:

$$\begin{aligned}\overline{F_\alpha}(x_1,x_2) &= [F_\alpha](x_1,x_2) := \int_{-h}^{h} F_\alpha(x_1,x_2,s)ds \\ \overline{F_3}(x_1,x_2) &= [F_3] - \left[x_3(\frac{\partial F_1}{\partial x_1}+\frac{\partial F_2}{\partial x_2}\right] \text{ (moments résultants ) .}\end{aligned}$$

# 3D-2D-reduction of stress and strain potentials

In the limit as $\delta \to 0$, all 3D-stress tensors take the form

$$\begin{pmatrix} \xi_{1,1} & \xi_{1,2} & 0 \\ \xi_{1,2} & \xi_{2,2} & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Accordingly effective 2D strain potentials are obtained by infimal convolution:

$$\overline{j}(e) \;=\; \inf \left\{ j \begin{pmatrix} e_{1,1} & e_{1,2} & a \\ e_{1,2} & e_{2,2} & b \\ a & b & c \end{pmatrix} \; : \; a, b, c \in \mathbb{R} \right\} \quad \text{(real materials )}$$

$$\overline{j_0}(e) \;=\; \inf \{ \; j_0 \, (\ldots) \quad : \quad a, b, c \in \mathbb{R} \} \quad \text{(fictitious materials )}$$

LEMMA: $\overline{j}(e) = \overline{j_0}(e)$ $\Rightarrow$ **Fictitious approach and relaxation approach are equivalent in the limit as $\delta \to 0$ !!**

**Theorem 2** The limit $S_0$ of $S_\delta$ (resp $\tilde{S}_\delta$) is given by

$$(1) \qquad \sup \left\{ < \overline{F}_\alpha, v_\alpha > + < \overline{F}_3, v_3 > : \ \bar{j}\big(e_{\alpha,\beta}(v) \pm \nabla^2 v_3\big) \leq \frac{1}{2} \text{ in } \overline{\Omega} \right\} ,$$

or alternatively (dual problem)

$$(2) \qquad \min \left\{ \int \overline{\rho^0}(\lambda^+) + \int \overline{\rho^0}(\lambda^-) \ : \ (\lambda^+, \lambda^-) \in \mathcal{M}(\overline{\Omega}, \mathbb{R}^{2\times 2}_{sym}) \right\} .$$

subject to the differential constraints

$$(3) \qquad -\mathrm{div}(\lambda^+ + \lambda^-) = (\overline{F_1}, \overline{F_2}) \quad , \quad \mathrm{div}^2(\lambda^+ - \lambda^-) = \overline{F_3}.$$

Furthermore, an admissible triple $(v, \lambda^+, \lambda^-)$ is optimal iff:

$$(4) \qquad \overline{\rho^0}(\lambda^+) =< \lambda^+, e_{\alpha,\beta}(v) + \nabla^2 v_3 \ , \ \overline{\rho^0}(\lambda^-) =< \lambda^-, e_{\alpha,\beta}(v) - \nabla^2 v_3 >$$

( $\rightsquigarrow$ inequalities in (1) are saturated resp $\lambda^\pm$ a.e. ( eikonal eq) )

• Notice that in our limit model, due to the $L^\infty$ constraint, the membrane energy and the flexion energy cannot be decoupled (in contrast with usual linear elasticity).

• If $\overline{F}_1 = \overline{F}_2 = 0$, then $\lambda^+ = -\lambda^- := \lambda$ in the dual problem (2) and we are reduced to $\min\left\{\int \overline{\rho^0}(\lambda) \ : \ \mathrm{div}^2(\lambda) = \overline{F_3}\right\}$ , which is exactly the dual problem of the (LCP) pb in introduction.

**Proof** Using the rescaled displacement on $Q$:

$$U_\delta(x_\alpha, x_3) = \left(u_\alpha(x', \frac{x_3}{\delta}), \delta^{-1} u_3(x', \frac{x_3}{\delta})\right) \ ,$$

we know that the matrix $e_\delta := e(U_\delta)$ is bounded in $L^\infty$ and deduce that the limit of $U_\delta$ satisfies $e_{3,\alpha}(U) = e_{3,3}(U) = 0$ . Thus (Kirchoff-Love)

$$U = \left(v_1(x') - x_3 \frac{\partial v_3}{\partial x_1} \ , \ v_2(x') - x_3 \frac{\partial v_3}{\partial x_2} \ , \ v_3(x')\right) \quad , \quad -1 \leq x_3 \leq 1 \ .$$

Recall that now $\varepsilon$ and $\delta$ go contemporarily to zero but with fixed $\tau = \frac{\varepsilon}{\delta} \in (0,1]$. Let

$$\mathcal{I}^\delta(\tau) := \mathcal{I}_{\tau\delta,\delta}$$

Denote by $H^1_{KL}(Q;\mathbb{R}^3)$ the space of *Kirchoff-Love displacements*:

$$H^1_{KL}(Q;\mathbb{R}^3) := \left\{ u \in H^1(Q;\mathbb{R}^3) \text{ such that } e_{i3}(u) = 0 \text{ for } i = 1,2,3 \right\} .$$

We introduce the limit compliance on the reference 3D subset $Q$:

$$(4) \qquad \begin{aligned} \mathcal{C}(\theta) \quad &:= \sup \left\{ \langle F, u \rangle_{\mathbb{R}^3} - \textstyle\int_Q \overline{j}(e_{\alpha\beta}(u))\, \theta \, dx \; : \; u \in H^1_{KL} \right\} \\ &= \inf \left\{ \textstyle\int_Q \theta^{-1} \overline{j}^*(\sigma)\, dx \; : \; \sigma \in L^2(Q;\mathbb{R}^{2\times 2}_{\text{sym}}) \right., \\ &\qquad \left. -\operatorname{div}[\sigma] = (\overline{F}_1, \overline{F}_2) \;,\; -\operatorname{div}^2[x_3\sigma] = \overline{F}_3 \right\} . \end{aligned}$$

and for every positive value of Lagrange parameter $k$:

$$(5) \qquad \phi(k) := \inf \left\{ \mathcal{C}(\theta) + k \int_Q \theta \, dx \; : \; \theta \in L^\infty(Q;[0,1]) \right\}$$

**Theorem 3** There holds

(*i*)
$$\lim_{\delta \to 0} \mathcal{I}^\delta(\tau) = \mathcal{I}(\tau) := \sup_{k \in \mathbb{R}^+} \left\{ \Phi(k) - k\tau \right\}$$

(ii) Up to subsequences $(\theta^d, \sigma^\delta))$ converges weakly star to an optimal pair $(\overline{\theta}, \overline{\sigma})$ for $\phi(k)$ (see (4) and (5)).

(iii) Vanishing filling ratio ( link with strategy A):

$$\lim_{\tau \to 0} \tau \mathcal{I}(\tau) = \frac{S_0^2}{2}$$

where

$$S_0 := \sup \left\{ < \overline{F}_\alpha, v_\alpha > + < \overline{F}_3, v_3 > : \ \overline{j}\big(e_{\alpha,\beta}(v) \pm \nabla^2 v_3\big) \leq \frac{1}{2} \text{ in } \overline{\Omega} \right\} .$$

and alternatively to (5)

$$\begin{aligned} \phi(k) \quad &:= \sup \left\{ \langle F, u \rangle_{\mathbb{R}^3} - \int_Q [\overline{j}(e_{\alpha\beta}(u)) - k]_+ \, dx \ : \ u \in H^1_{KL}(Q; \mathbb{R}^3) \right\} \\ &= \sup \Big\{ \langle \overline{F}, v \rangle_{\mathbb{R}^2} - \int_\Omega W_k(e(v_1, v_2), \nabla^2 v_3) \, dx' \ : \\ &\qquad v_1, v_2 \in H^1(\Omega) \ , \ v_3 \in H^2(\Omega) \Big\} . \end{aligned}$$

**Corollary 4** Let $(\overline{\theta}, \overline{u}, \overline{\sigma})$ be an optimal triple for $\phi(k)$ (see (4) and (5)). Then:

$\overline{\theta} = 0$ and $\overline{\sigma} = 0$ on $\{\overline{j}(e_{\alpha\beta}(\overline{u})) < k\}$ ; $\overline{\theta} = 1$ and $\overline{\sigma}(x', \cdot)$ is affine on $\{\overline{j}(e_{\alpha\beta}(\overline{u})) > k\}$ .

In particular , if the set $\{\overline{j}(e_{\alpha\beta}(\overline{u})) = k\}$ has null measure ( THIS HAPPENS in particular if $\nabla^2 \overline{u}_3 \neq 0$), then $\overline{\theta}$ is unique and it is **the characteristic function** of the set $\overline{\omega} := \{\overline{j}(e_{\alpha\beta}(\overline{u})) > k\}$:

For all $x' \in D$, each fiber $\{x_3 : (x', x_3) \in \overline{\omega}\}$ is the complement of a subinterval of $I$.

Consider the following axially symmetric system of forces supported on the design region $Q = \overline{\Omega} \times [-h, h]$ :

$$F_1 := \alpha(\delta_B - \delta_A) \ , \qquad F_2 = \delta_C - \frac{1}{2}(\delta_A + \delta_B) \ .$$

where $O := (0,0)$, $A := \big(-\frac{l}{2}, 0\big)$ ,$B := \big(\frac{l}{2}, 0\big)$ and $C := (0, h_0)$ . (we need that $1 \geq h_0$ and $\Omega$ contains $O$ and $A$ )


Figure 3: *loads yielding a membrane/flexion regime in the clear/dark part of* $\overline{AB}$

# 5.2 Example (continued)

We apply Theorem 1 to compute $\mathcal{S}_0$ given by

$$\sup \quad \Big\{ \alpha\big[v_1(B) - v_1(A)\big] + v_2(O) - \frac{1}{2}\big[v_2(A) + v_2(B)\big]$$

$$v \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^2) \text{ such that } \; |(v_1)' \pm \mathbf{h}(v_2)''| \leq 1 \text{ on } \Omega \Big\} \ .$$

If $\lambda^+, \lambda^-$ solutions of dual problem (2)(3), then by (4):

$$\begin{cases} -(\lambda^+ + \lambda^-)' = \alpha\big(\delta_A - \delta_B\big) \\ h(\lambda^+ - \lambda^-)'' = \delta_O - \dfrac{1}{2}\big(\delta_A + \delta_B\big) \\ |(v_1)' \pm \mathbf{h}(v_2)''| \leq 1 \\ |\lambda^\pm| = \langle \lambda^\pm, (v_1)' \pm (v_2)'' \rangle_{\mathbb{R}} \ . \end{cases}$$

# 5.3 Example (continued)

The first two equations determine $\lambda^{\pm}$ as follows:

$$(5) \qquad \lambda^+ + \lambda^- = \alpha \mathcal{L}^1 \llcorner \overline{AB} \,, \qquad \lambda^+ - \lambda^- = \frac{1}{2}\Big(|x_1| - \frac{l}{2}\Big)\mathcal{L}^1 \llcorner \overline{AB} \,,$$

and the last two conditions are satisfied provided

$$(6) \qquad (v_1)' \pm (v_2)'' = \mathrm{sign}\,(\lambda^{\pm}) \;.$$

From (5), we see in particular that $\lambda^-$ remains always nonnegative, whereas for $\lambda^+$ two cases may occur:

*case 1)*: if $1 \geq l/(4\alpha)$, then $\lambda^+$ remains nonnegative;

*case 2)*: if $1 < l/(4\alpha)$, then

$$\begin{cases} \lambda^+ \geq 0 & \text{if } |x_1| \geq (l/2) - 2h\alpha \\ \lambda^+ < 0 & \text{if } |x_1| < (l/2) - 2\alpha \;. \end{cases}$$

Accordingly, solution $v$ and the value of $\mathcal{S}_0$ can be easily computed:

*Case* $1 \geq l/(4\alpha)$: we have $(v_1)' = 1$, $(v_2)'' = 0$ (membrane regime), and

$$\mathcal{S}_0 = \int \lambda^+ + \int \lambda^- = \alpha l \ ;$$

*Case* $1 \leq l/(4\alpha)$: we have

$$\begin{cases} (v_1)' = 1 \text{ and } (v_2)'' = 0 \quad (membrane) & \text{if } |x_1| \geq (l/2) - 2\alpha \\ (v_1)' = 0 \text{ and } (v_2)'' = 1 \quad (flexion) & \text{if } |x_1| < (l/2) - 2\alpha \ , \end{cases}$$

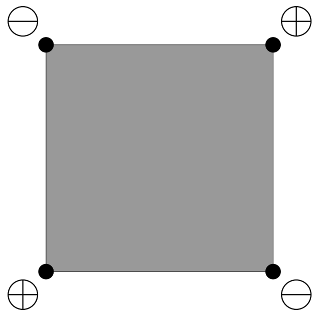
and

$$\mathcal{S}_0 = \int |\lambda^+| + \int \lambda^- = 2\big[\alpha^2 + l^2/(16)\big] \ .$$

# 6- Solving (LCP): ex. 1/4

$$\sup \left\{ u(0,0) + u(1,1) - u(1,0) - u(0,1) \ : \ |\nabla^2 u| \leq 1 \right\}$$

$$\mu = \frac{1}{\sqrt{2}} \, \mathcal{L}^2 \llcorner (0,1)^2 \quad , \quad u(x_1, x_2) = \frac{x_1 x_2}{\sqrt{2}}$$



# 6- Solving (LCP): ex. 2/4

$$\sup\left\{\alpha\big(u(1,0)-u(0,0)\big)+g_1\cdot\nabla u(1,0)-g_0\cdot\nabla u(0,0)\ :\ |\nabla^2 u|\le 1\right\}$$

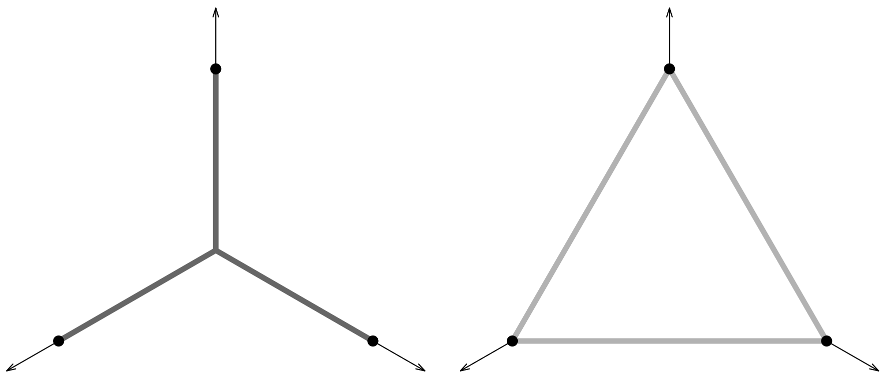
(with $\alpha+g_1\cdot e_1-g_0\cdot e_1=0, \quad g_1\cdot e_2=g_0\cdot e_2$)

$$\mu=\sqrt{(g_0\cdot e_1-\alpha s)^2+\frac{1}{2}(g_0\cdot e_2)^2}\ \mathcal{H}^1\llcorner S$$

# 6- Solving (LCP): ex. 3/4

$$\sup \left\{ \sum_{i=1}^{3} \nabla u(P_i) \cdot v_i \ : \ |\lambda_1(\nabla^2 u)| \leq 1 \right\}$$

$$\mu_1 = \mathcal{H}^1 \llcorner T \ , \qquad \mu_2 = \frac{1}{\sqrt{3}} \mathcal{H}^1 \llcorner \Delta \quad , \quad u(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2)$$



# 6.Solving (MOP): ex. 4/4

$$\sup\left\{\sum_{i=1}^{3}\nabla u(P_i)\cdot v_i \ : \ |\lambda_1(\nabla^2 u)|^2+|\lambda_2(\nabla^2 u)|^2\leq 1\right\}$$

Optimal $\mu$ is unique 2D (and no hole)

[GOLAY- SEPPECHER], Eur. J. Mech. A Solids (2001)

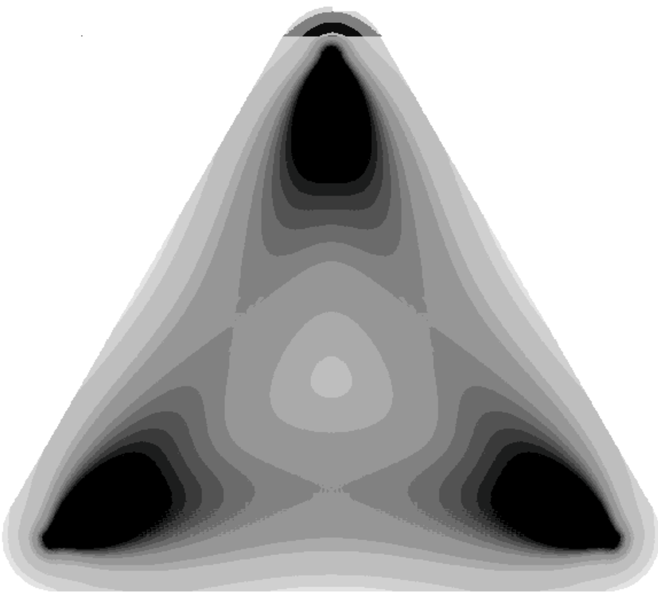


Figure 3 — the two-dimensional optimal mass distribution.