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On the motion of a rigid body immersed in a bidimensional incompressible perfect fluid

Sur le mouvement d'un corps rigide immergé dans un fluide parfait incompressible bidimensionnel

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Abstract

We consider the motion of a rigid body immersed in a bidimensional incompressible perfect fluid. The motion of the fluid is governed by the Euler equations and the conservation laws of linear and angular momentum rule the dynamics of the rigid body. We prove the existence and uniqueness of a global classical solution for this fluid–structure interaction problem. The proof relies mainly on weighted estimates for the vorticity associated with the strong solution of a fluid–structure interaction problem obtained by incorporating some viscosity.

Résumé

Nous étudions le mouvement d'un corps rigide immergé dans un fluide parfait incompressible bidimensionnel. Le mouvement du fluide est modélisé par les équations d'Euler, et la dynamique du corps rigide est régie par les lois de conservation des moments linéaires et angulaires. Nous prouvons l'existence et l'unicité d'une solution globale classique pour ce problème d'interaction fluide–structure. La preuve repose essentiellement sur des estimées à poids pour la vorticité associée à la solution forte d'un problème d'interaction fluide–structure obtenu en incorporant de la viscosité.

Keywords: Euler equations; Fluid–rigid body interaction; Exterior domain; Classical solutions

Mots-clés : Equations d'Euler ; Interaction fluide–corps rigide ; Domaine extérieur ; Solutions classiques

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1. Introduction

In this paper we continue our investigation of the Cauchy problem for the system describing the motion of a rigid body immersed in an incompressible perfect fluid. In [27], the global existence and uniqueness of a *classical* solution were established when the rigid body was a *ball*. Here, the rigid body may take an *arbitrary* form. To be more precise, we assume that the rigid body fills a bounded, simply connected domain $S(t) \subset \mathbb{R}^2$ of class C^1 and piecewise C^2 and which is different from a ball, and that it is surrounded by a perfect incompressible fluid. For the sake of simplicity, both the fluid and the solid are assumed to be homogeneous. The domain occupied by the fluid is denoted by $\Omega(t) = \mathbb{R}^2 \setminus \overline{S(t)}$. The dynamics of the fluid is described by the Euler equations, whereas the motion of the rigid body is governed by the balance equations for linear and angular momentum (Newton's laws). The equations modelling the dynamics of the system read then

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + \nabla p = 0 \quad \text{in } \Omega(t) \times [0, T], \tag{1.1}$$

$$\operatorname{div} u = 0 \quad \text{in } \Omega(t) \times [0, T], \tag{1.2}$$

$$u \cdot n = \big(h' + r(x - h)^{\perp}\big) \cdot n \quad \text{on } \partial S(t) \times [0, T], \tag{1.3}$$

$$mh'' = \int\limits_{\partial S(t)} pn \, \mathrm{d}\Gamma \quad \text{in } [0, T], \tag{1.4}$$

$$Jr' = \int\limits_{\partial S(t)} (x - h(t))^{\perp} \cdot pn \, \mathrm{d}\Gamma \quad \text{in } [0, T], \tag{1.5}$$

$$u(x, 0) = a(x) \quad \forall x \in \Omega, \tag{1.6}$$

$$h(0) = 0 \in \mathbb{R}^2, \quad h'(0) = b \in \mathbb{R}^2, \quad r(0) = c \in \mathbb{R}. \tag{1.7}$$

In the above equations u (resp., p) is the velocity field (resp., the pressure) of the fluid, and h (resp., r) denotes the position of the center of mass (resp., the angular velocity) of the rigid body, $y^{\perp} = (-y_2, y_1)$ if $y = (y_1, y_2)$, and $\partial S(t) = \partial\Omega(t)$. Note that we have assumed the center of mass of the solid to be located at the origin at time $t = 0$. We have denoted by n the unit outward normal to $\partial\Omega(t)$. The continuity equation for the velocity (1.3) means that the normal component of the velocity is the same for the fluid and the rigid body on $\partial S(t)$. In other words, the fluid does not enter into the rigid body. The (positive) constants m and J are respectively the *mass* and the *moment of inertia* of the rigid body. They are defined by

$$m = \int\limits_{S} \gamma \, \mathrm{d}x, \qquad J = \int\limits_{S} \gamma |x|^2 \, \mathrm{d}x,$$

where γ denotes the (uniform) density of the rigid body. In Newton's law (1.4) (resp., (1.5)), we notice that the only exterior force (resp., torque) applied to the rigid body is the one resulting from the fluid pressure integrated along the boundary $\partial S(t)$. For a derivation of (1.1)–(1.5), we refer e.g. to [13].

As for many fluid–structure interaction problems, the main difficulties come from the fact that the system (1.1)–(1.7) is nonlinear, strongly coupled and that the domain of the fluid is an unknown function of time. Several papers devoted to the study of this kind of systems have been published in the last decade. More precisely, when the dynamics of the fluid is modelled by the Navier–Stokes equations, the existence of solutions has been studied in [5,6,2,16,19, 20,15,28,7,8,33] when the fluid fills a bounded domain, and in [29,21,30,34,12] when the fluid fills the whole space. The stationary problem was studied in [29] and in [9]. The asymptotic behavior of the solutions has been investigated (with simplified models) in [37] and in [26].

When the fluid is perfect, the *only* available result is the one by the authors [27] when the solid is a ball and the fluid fills $\mathbb{R}^2$. Notice, however, that a theory providing *classical* solutions to this kind of problems seems desirable for control purposes, as most of the control results for the Euler flows involve classical solutions. (See e.g. Coron [3,4], and Glass [14].)

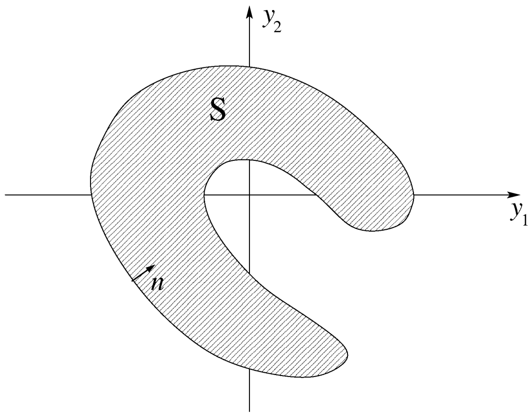

Fig. 1. Configuration at $t = 0$.

In order to write the equations of the fluid in a *fixed* domain, we perform a change of variables. Denoting by S the set occupied by the solid at $t = 0$ (see Fig. 1) and by $\Omega = \mathbb{R}^2 \setminus \overline{S}$ the initial domain occupied by the fluid, we set

$$\theta(t) = \int_0^t r(s)\,\mathrm{d}s, \qquad Q(t) = \begin{pmatrix} \cos\theta(t) & -\sin\theta(t) \\ \sin\theta(t) & \cos\theta(t) \end{pmatrix}, \tag{1.8}$$

and

$$\begin{cases} v(y,t) = Q(t)^* u\big(Q(t)y + h(t), t\big), \\ q(y,t) = p\big(Q(t)y + h(t), t\big), \\ l(t) = Q(t)^* h'(t). \end{cases} \tag{1.9}$$

Then, the functions (v, q, l, r) satisfy the following system

$$\frac{\partial v}{\partial t} + \big[\big(v - l - ry^\perp\big)\cdot\nabla\big]v + rv^\perp + \nabla q = 0 \quad \text{in } \Omega \times [0,T], \tag{1.10}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega \times [0,T], \tag{1.11}$$

$$v \cdot n = \big(l + ry^\perp\big)\cdot n \quad \text{on } \partial S \times [0,T], \tag{1.12}$$

$$ml' = \int_{\partial S} qn\,\mathrm{d}\Gamma - mrl^\perp \quad \text{in } [0,T], \tag{1.13}$$

$$Jr' = \int_{\partial S} qn \cdot y^\perp\,\mathrm{d}\Gamma \quad \text{in } [0,T], \tag{1.14}$$

$$v(y,0) = a(y) \quad \forall y \in \Omega, \tag{1.15}$$

$$l(0) = b, \quad r(0) = c. \tag{1.16}$$

The study of the Cauchy problem for the system (1.10)–(1.16) is more tricky than for the system considered in [27] (rigid ball). When comparing both systems, we first notice the presence of the additional terms $-r(y^\perp \cdot \nabla)v$ and $rv^\perp$ in (1.10), $ry^\perp \cdot n$ in (1.12) and $-mrl^\perp$ in (1.13). Moreover, the angular velocity r fails to be constant here, and its dynamics, which is governed by (1.14), has to be taken into account. Besides some modifications in the computations and in the analysis (see below Lemmas 2.1, 6.1 and the section devoted to the uniqueness of the solution), the main difficulty comes from the presence of the term $-r(y^\perp \cdot \nabla)v$ in (1.10), which looks difficult to control as $|y| \to \infty$. The idea is to first replace $y^\perp$ by a truncated vector $y_R^\perp$ in (1.10), and next to derive appropriate estimates to pass to the limit in the modified equation. As a matter of fact, the theory of weighted estimates for singular integrals (see e.g. [32]) does not provide any estimate of the form

$$\big\| |y|\nabla v \big\|_{L^2(\mathbb{R}^2)} \leqslant C \big\| f\big(|y|\big) \operatorname{curl} v \big\|_{L^2(\mathbb{R}^2)},$$

for any choice of the weight function f. The key observation thanks to which we shall be able to control the term $-r(y^\perp \cdot \nabla)v$ is that $|y|\nabla v \in L^2(\Omega)$ whenever $\operatorname{curl} v \in L^\infty(\Omega) \cap L^1_\theta(\Omega)$ (with $\theta > 2$) *and* $v \in L^2(\Omega)$ (see below Proposition 2.2).

A Navier–Stokes based system similar to (1.10)–(1.16) has been recently studied in [18,11,10], but it should be noticed that the global existence of strong solutions in the 2D case has not been proved because of the term $r(y^\perp \cdot \nabla)v$.

Before stating the main result of the paper, we introduce some notations borrowed from Kikuchi [23]. If V denotes any scalar-valued function space and $u = (u_1, u_2)$ is any vector-valued function, we shall say that $u \in V$ if $u_i \in V$ for all i, for the sake of simplicity. Let T be any positive number, and let $Q_T = \Omega \times (0, T)$. $\mathcal{B}(\overline{\Omega})$ (resp., $\mathcal{B}(\overline{Q_T})$) is the Banach space of all real-valued, continuous and bounded functions defined on $\overline{\Omega}$ (resp. $\overline{Q_T}$), endowed with the L^∞ norm. For any $\theta > 0$, $L^1_\theta(\Omega)$ denotes the space of (class of) measurable functions ω on Ω such that

$$\|\omega\|_{L^1_\theta(\Omega)} := \int_\Omega \big|\omega(y)\big||y|^\theta \, \mathrm{d}y < \infty.$$

Finally, for any $\lambda \in (0, 1)$, $C^\lambda(\overline{\Omega})$ (resp., $C^{\lambda,0}(\overline{Q_T})$) is the space of all the functions $\omega \in \mathcal{B}(\overline{\Omega})$ (resp., $\omega \in \mathcal{B}(\overline{Q_T})$) which are uniformly Hölder continuous in y with exponent λ on $\overline{\Omega}$ (resp., on $\overline{Q_T}$). $B_r(y)$ will denote the open ball in $\mathbb{R}^2$ with center y and radius r. At any point $y \in \partial\Omega$ $(= \partial S)$, $n = (n_1, n_2)$ will denote the unit outer normal vector to $\partial\Omega$ and $\tau = (\tau_1, \tau_2)$ will denote the unit tangent vector $\tau = -n^\perp$. For any scalar-valued function ω, we set $\operatorname{curl}\omega = (\partial\omega/\partial y_2, -\partial\omega/\partial y_1)$ and $\nabla\omega = (\partial\omega/\partial y_1, \partial\omega/\partial y_2)$, while for any vector-valued function $v = (v_1, v_2)$, we set $\operatorname{curl} v = \partial v_2/\partial y_1 - \partial v_1/\partial y_2$, $\operatorname{div} v = \partial v_1/\partial y_1 + \partial v_2/\partial y_2$ and $\nabla v = (\partial v_i/\partial y_j)_{1\leqslant i,j\leqslant 2}$. The main result in this paper is the following one.

Theorem 1.1. *Let* $\theta > 2$, $0 < \lambda < 1$, $a \in \mathcal{B}(\overline{\Omega}) \cap H^1(\Omega)$, $b \in \mathbb{R}^2$, *and* $c \in \mathbb{R}$. *Assume that* $\operatorname{div} a = 0$, $(a - b - cy^\perp) \cdot n|_{\partial S} = 0$, $\lim_{|y|\to+\infty} a(y) = 0$, *and* $\operatorname{curl} a \in L^1_\theta(\Omega) \cap C^\lambda(\overline{\Omega})$. *Then there exists a solution* (v, q, l, r) *of* (1.10)–(1.16) *such that*

$$v, \frac{\partial v}{\partial t}, \nabla v \in \mathcal{B}\big(\overline{Q_T}\big), \quad \nabla q \in C\big(\overline{Q_T}\big), \quad v \in C^1\big([0,T], L^2(\Omega)\big) \cap C\big([0,T], H^1(\Omega)\big),$$
$$y^\perp \cdot \nabla v \in C\big([0,T], L^2(\Omega)\big), \quad q \in C\big([0,T], \widehat{H}^1(\Omega)\big), \quad l \in C^1\big([0,T]\big) \quad \text{and} \quad r \in C^1\big([0,T]\big).$$

Such a solution is unique up to an arbitrary function of t which may be added to q.

In the above theorem, we have denoted by $\widehat{H}^1(\Omega)$ the homogeneous Sobolev space

$$\widehat{H}^1(\Omega) = \big\{q \in L^2_{\mathrm{loc}}\big(\overline{\Omega}\big) \bigm| \nabla q \in L^2(\Omega)\big\},$$

where $q \in L^2_{\mathrm{loc}}(\overline{\Omega})$ means that $q \in L^2(\Omega \cap B_0)$ for any open ball $B_0 \subset \mathbb{R}^2$ with $B_0 \cap \Omega \neq \emptyset$.

Notice that, with the above regularity, the solution v satisfies the following property

$$\lim_{|y|\to\infty} v(y, t) = 0 \tag{1.17}$$

uniformly with respect to $t \in [0, T]$. Indeed, $v \in W^{1,\infty}(Q_T) \cap C([0, T]; L^2(\Omega))$, which implies (1.17) thanks to a simple modification of Barbalat's lemma.

The *kinetic energy* of the system is given by

$$E(t) = \frac{1}{2}m\big|l(t)\big|^2 + \frac{1}{2}J\big|r(t)\big|^2 + \frac{1}{2}\int_\Omega \big|v(y,t)\big|^2 \, \mathrm{d}y.$$

A great role will be played in the sequel by the scalar vorticity $\omega := \operatorname{curl} v$, which will be proved to be bounded in $L^1_\theta(\Omega) \cap L^\infty(\Omega)$. (The initial vorticity $\omega_0 := \operatorname{curl} a \in L^1_\theta(\Omega) \cap L^\infty(\Omega)$ by assumption.) An integral term of the form $\int_\Omega f(\omega(y,t))\,\mathrm{d}y$, where $f : \mathbb{R} \to \mathbb{R}$ is any continuous function such that $f(\omega)$ is integrable, is termed a *generalized enstrophy*.

Using the regularity of the solution provided by Theorem 1.1 and the incompressibility of the flow associated with $v - l - ry^\perp$ (see below), we readily obtain the following result.

Corollary 1.2. *Let (a, b, c) be as in Theorem* 1.1. *Then the kinetic energy and all the generalized enstrophies of the solution given in Theorem* 1.1 *remain constant.*

In particular, any L^p-norm of the vorticity is conserved along the flow.

A large part of the proof of Theorem 1.1 rests on the machinery developed in [23] to prove the existence of classical solutions to the Euler system in an exterior domain. However, unlike [23], a fixed-point argument cannot be applied directly to the Euler system, due to a lack of pressure estimate. On the other hand, when we compare the assumptions of our main result to those required in [23], we note that

(1) no additional assumption has to be made here in order to insure the uniqueness of the solution;
(2) the initial velocity a has to belong to $H^1(\Omega)$.

The intrusion of an L^2-estimate in a classical theory, which may look awkward at first sight, is nevertheless necessary. Indeed, the boundedness of the speed of the rigid body cannot be proved without the aid of the conservation of the kinetic energy of the system solid+fluid. Thus, a feature of the problem investigated here is that we need estimates both in $L^\infty(\Omega)$ and in $L^2(\Omega)$.

To prove Theorem 1.1 we proceed in three steps. In the first step, we construct a strong solution of an approximated system in which the Euler equations have been replaced by the Navier–Stokes equations (with suitable boundary conditions and with $y^\perp$ replaced by a truncated vector $y_R^\perp$ depending on some parameter R). In the second step, we demonstrate that the vorticity associated with the strong solution of the Navier–Stokes system is bounded in $L^\infty(\Omega) \cap L^1_\theta(\Omega)$, uniformly with respect to the viscosity coefficient ν and to the parameter R. These estimates, combined with a standard energy estimate, provide the velocity estimates needed to pass to the limit as $R \nearrow \infty$ and $\nu \to 0$. In the final step, we prove that the solution to (1.10)–(1.16) has the regularity depicted in Theorem 1.1.

The paper is outlined as follows. Section 2 contains some preliminary results. Section 3 is devoted to the existence of strong solutions to the approximated Navier–Stokes system. In Sections 4 and 5, we prove some energy and vorticity estimates needed to pass to the limit as $R \to \infty$ and $\nu \to 0$. Finally, the proof of Theorem 1.1 is given in Section 6.

2. Preliminaries

2.1. Extension of the velocity field to the plane

In the system (1.10)–(1.16), we can extend v to $\mathbb{R}^2$ by setting $v(y,t) = l(t) + r(t)y^\perp$ for all $y \in S$ and all $t \geqslant 0$. Then $\operatorname{div} v = 0$ in $\mathbb{R}^2 \times [0, T]$ and $D(v) = 0$ in $S \times [0, T]$, where

$$D(v)_{k,l} = \frac{1}{2}\left(\frac{\partial v_k}{\partial y_l} + \frac{\partial v_l}{\partial y_k}\right).$$

We are led to introduce the following spaces

$$\mathcal{H} = \{\phi \in L^2(\mathbb{R}^2) \mid \operatorname{div}(\phi) = 0 \text{ in } \mathbb{R}^2,\ D(\phi) = 0 \text{ in } S\} \tag{2.1}$$

and

$$\mathcal{V} = \{\phi \in \mathcal{H} \mid \phi_{|\Omega} \in H^1(\Omega)\}. \tag{2.2}$$

We define a scalar product in $L^2(\mathbb{R}^2)$ which is equivalent to the usual one

$$(u, v)_\gamma := \int_\Omega u \cdot v \,\mathrm{d}x + \gamma \int_S u \cdot v \,\mathrm{d}x.$$

The spaces $L^2(\mathbb{R}^2)$ and $\mathcal{H}$ are clearly Hilbert spaces for the scalar product $(\cdot,\cdot)_\gamma$. Notice that for every $u \in \mathcal{H}$ there exists a unique $(l_u, r_u) \in \mathbb{R}^2 \times \mathbb{R}$ such that $u = l_u + r_u y^\perp$ in S (see e.g. [35, Lemma 1.1, pp. 18]). It follows that for all $u, v \in \mathcal{H}$

$$(u, v)_\gamma = \int_\Omega u \cdot v \,\mathrm{d}x + m l_u \cdot l_v + J r_u r_v.$$

The space $\mathcal{V}$ is also a Hilbert space for the scalar product

$$(u,v)_{\mathcal{V}} := (u,v)_{\mathcal{Y}} + \int_{\Omega} \nabla u : \nabla v \, \mathrm{d}x.$$

A first technical result is the following

Lemma 2.1. *Let $u, v \in \mathcal{V}$ and suppose that $u_{|\Omega} \in H^2(\Omega)$ and that $\operatorname{curl} u = 0$ on ∂S. Then we have the following identity*

$$\int_{\partial S} v \cdot \frac{\partial u}{\partial n} \, \mathrm{d}\Gamma = \int_{\partial S} \kappa \big(v - l_v - r_v y^{\perp}\big) \cdot \big(u - l_u - r_u y^{\perp}\big) \, \mathrm{d}\Gamma + \int_{\partial S} (r_v u \cdot \tau + r_u v \cdot \tau + r_u r_v y \cdot n) \, \mathrm{d}\Gamma, \tag{2.3}$$

where κ denotes the curvature of ∂S.

Proof. For the sake of simplicity we assume that the domain S is of class C^2, the extension to the general framework being straightforward. We may extend n as a vector field of class C^1 on a neighborhood of ∂S. Since $\operatorname{div} u = 0$ and $\operatorname{div} v = 0$ in $\mathcal{D}'(\mathbb{R}^2)$, we have that

$$\big(u - l_u - r_u y^{\perp}\big) \cdot n = \big(v - l_v - r_v y^{\perp}\big) \cdot n = 0 \quad \text{on } \partial S.$$

By using the above equations, we deduce that

$$\big(v - l_v - r_v y^{\perp}\big) \cdot \nabla\big[\big(u - l_u - r_u y^{\perp}\big) \cdot n\big] = 0 \quad \text{on } \partial S. \tag{2.4}$$

Since $\operatorname{curl} u = 0$ on ∂S, we infer that

$$\big(v - l_v - r_v y^{\perp}\big) \cdot (\nabla n)^* \big(u - l_u - r_u y^{\perp}\big) + \big(v - l_v - r_v y^{\perp}\big) \cdot \left(\frac{\partial u}{\partial n} + r_u n^{\perp}\right) = 0 \quad \text{on } \partial S$$

hence

$$v \cdot \frac{\partial u}{\partial n} = \kappa \big(v - l_v - r_v y^{\perp}\big) \cdot \big(u - l_u - r_u y^{\perp}\big) + \big(l_v + r_v y^{\perp}\big) \cdot \frac{\partial u}{\partial n} - r_u \big(v - l_v - r_v y^{\perp}\big) \cdot n^{\perp}. \tag{2.5}$$

On the other hand, since $\operatorname{div} u = 0$ and $\operatorname{curl} u = 0$ on ∂S, we have that

$$\frac{\partial u}{\partial n} = -\left(\frac{\partial u}{\partial \tau}\right)^{\perp}$$

where $\tau = -n^{\perp}$. The above equation implies that

$$\int_{\partial S} \big(l_v + r_v y^{\perp}\big) \cdot \frac{\partial u}{\partial n} \, \mathrm{d}\Gamma = \int_{\partial S} r_v u \cdot \tau \, \mathrm{d}\Gamma. \tag{2.6}$$

Gathering (2.5) and (2.6), we obtain the result. □

2.2. Velocity versus vorticity

The following result, which relates the velocity of the fluid to the vorticity, the velocity of the rigid body and the circulation of the flow along ∂S, will play a great role later.

Proposition 2.2. *Let $l \in \mathbb{R}^2$, $r \in \mathbb{R}$, $C \in \mathbb{R}$ and $\omega \in L^1(\Omega) \cap L^\infty(\Omega)$. Then there exists a unique vector field $v \in \mathcal{B}(\overline{\Omega})$ fulfilling*

$$\operatorname{curl} v = \omega \quad \text{in } \Omega, \tag{2.7}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega, \tag{2.8}$$

$$v \cdot n = \big(l + r y^{\perp}\big) \cdot n \quad \text{on } \partial S, \tag{2.9}$$

$$\int_{\partial S} v \cdot \tau \, d\Gamma = C \quad \textit{and} \tag{2.10}$$

$$\lim_{|y|\to+\infty} v(y) = 0. \tag{2.11}$$

Furthermore, $v \in L^p(\Omega)$ $\forall p \in (2,+\infty]$, $\nabla v \in L^p(\Omega)$ $\forall p \in (1,+\infty)$ *and there exist some positive constants* K_p, K'_p *such that*

$$\|v\|_{L^p(\Omega)} \leqslant K_p\big(|l| + |r| + |C| + \|\omega\|_{L^1(\Omega)} + \|\omega\|_{L^\infty(\Omega)}\big) \quad \forall p \in (2,+\infty], \tag{2.12}$$

$$\|\nabla v\|_{L^p(\Omega)} \leqslant K'_p\big(|l| + |r| + |C| + \|\omega\|_{L^p(\Omega)}\big) \quad \forall p \in (1,+\infty). \tag{2.13}$$

If in addition $v \in L^2(\Omega)$ *and* $\omega \in L^1_\theta(\Omega)$ *with* $\theta > 2$, *then* $\int_\Omega \omega \, dy = -C$, $|y|\nabla v \in L^2(\Omega)$ *and there exists some positive constant* K'' *such that*

$$\big\| |y|\nabla v\big\|_{L^2(\Omega)} \leqslant K''\big(|l| + |r| + \|\omega\|_{L^\infty(\Omega)} + \|\omega\|_{L^1_\theta(\Omega)}\big). \tag{2.14}$$

Proof. As the proof is very similar to the one of [27, Proposition 2.3], we limit ourselves to pointing out the main differences.

First Step: Reduction to the case $l = 0$, $r = 0$, *and* $C = 0$.

Let us introduce

$$R_0 := \sup_{y\in\partial S} |y|.$$

(i) *Reduction to the case* $l = 0$ *and* $r = 0$.

We need the following lemma.

Lemma 2.3. *Let* $l \in \mathbb{R}^2$ *and* $r \in \mathbb{R}$. *Then there exists a vector field* $d_1 \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ *such that* $\operatorname{div} d_1 = 0$ *on* $\mathbb{R}^2$ *and*

$$d_1(y) = \begin{cases} l + ry^\perp & \textit{if } |y| \leqslant R_0, \\ 0 & \textit{if } |y| \geqslant R_0 + 1. \end{cases}$$

Proof of Lemma 2.3. It is sufficient to pick any function $\theta \in C^\infty(\mathbb{R}^+, \mathbb{R}^+)$ such that

$$\theta(s) = \begin{cases} 1 & \text{if } s \leqslant R_0, \\ 0 & \text{if } s \geqslant R_0 + 1, \end{cases}$$

and to set $d_1(y) := \operatorname{curl}(\theta(|y|)\, y \cdot l^\perp) + r\theta(|y|)y^\perp$. □

Setting $v_1 := v - d_1$, we see that (2.7)–(2.11) is changed into

$$\operatorname{curl} v_1 = \omega_1 := \omega - \operatorname{curl} d_1 \quad \text{in } \Omega, \tag{2.15}$$

$$\operatorname{div} v_1 = 0 \quad \text{in } \Omega, \tag{2.16}$$

$$v_1 \cdot n = 0 \quad \text{on } \partial S, \tag{2.17}$$

$$\int_{\partial S} v_1 \cdot \tau \, d\Gamma = C_1 := C - r\int_{\partial S} y^\perp \cdot \tau \, d\Gamma \quad \text{and} \tag{2.18}$$

$$\lim_{|y|\to+\infty} v_1(y) = 0. \tag{2.19}$$

Notice that $\omega_1 \in L^1(\Omega) \cap L^\infty(\Omega)$, as $\operatorname{curl} d_1 \in C_0^\infty(\mathbb{R}^2)$.

(ii) *Reduction to the case* $C = 0$.

We need the following lemma.

Lemma 2.4. *There exists a vector field* $d_2 \in C^1(\mathbb{R}^2, \mathbb{R}^2)$ *such that* $\operatorname{div} d_2 = 0$, $d_2(y) = 0$ *for* $|y| \geqslant R_0 + 1$, $d_2 \cdot n = 0$ *on* ∂S *and* $\int_{\partial S} d_2 \cdot \tau \, d\Gamma = 1$.

Proof of Lemma 2.4. We pick a function $\psi \in C_0^2(\mathbb{R}^2, \mathbb{R}^+)$ such that $\overline{S} = \{y \in \mathbb{R}^2 \mid \psi(y) \geqslant 1\}$, $\psi(y) = 1$ and $\nabla\psi \not\equiv 0$ on ∂S, and $\psi(y) = 0$ for $|y| \geqslant R_0 + 1$. Then $d_2(y) = \operatorname{curl}\psi$ fulfills all the requirements of the lemma, except possibly $\int_{\partial S} d_2 \cdot \tau \,\mathrm{d}\Gamma = 1$. As $\int_{\partial S} d_2 \cdot \tau \,\mathrm{d}\Gamma \neq 0$, the last condition may be satisfied thanks to a normalization. □

The change of unknown function

$$v_2 := v_1 - C_1 d_2$$

transforms (2.15)–(2.19) into

$$\operatorname{curl} v_2 = \omega_2 := \omega_1 - C_1 \operatorname{curl} d_2 \quad \text{in } \Omega, \tag{2.20}$$

$$\operatorname{div} v_2 = 0 \quad \text{in } \Omega, \tag{2.21}$$

$$v_2 \cdot n = 0 \quad \text{on } \partial S, \tag{2.22}$$

$$\int_{\partial S} v_2 \cdot \tau \,\mathrm{d}\Gamma = 0 \quad \text{and} \tag{2.23}$$

$$\lim_{|y| \to +\infty} v_2(y) = 0. \tag{2.24}$$

Notice that $\omega_2 \in L^1(\Omega) \cap L^\infty(\Omega)$, as $d_2(y) = 0$ for $|y| \geqslant R_0 + 1$.

Second Step: Construction of a solution to (2.20)–(2.24).

Proceeding as in [27, Proposition 2.3], one obtains the existence and the uniqueness of the solution $v = d_1 + C_1 d_2 + v_2 \in \mathcal{B}(\overline{\Omega})$ of (2.7)–(2.11).

Third Step: L^p-estimates.

The estimates (2.12) and (2.13) may be proved as in [27, Proposition 2.3]. Assume now that $v \in L^2(\Omega)$ and that $\omega \in L^1_\theta(\Omega)$, with $\theta > 2$. For each $R > R_0$ let $\Omega_R := \Omega \cap B_R(0)$. The following result is needed.

Lemma 2.5. *Let $v : \Omega \to \mathbb{R}^2$ be a function such that $v \in H^1(\Omega_R)$ for any $R > R_0$ and* $\operatorname{curl} v \in L^1(\Omega)$. *Assume further that $v \in L^2(\Omega)$. Then the following Stokes' formula holds true*

$$\int_{\Omega} \operatorname{curl} v \,\mathrm{d}y = -\int_{\partial S} v \cdot \tau \,\mathrm{d}\Gamma. \tag{2.25}$$

Proof of Lemma 2.5. An application of the usual Stokes' formula in Ω_R yields

$$\int_{\Omega_R} \operatorname{curl} v \,\mathrm{d}y = -\int_{|y|=R} v \cdot \tau \,\mathrm{d}\Gamma - \int_{\partial S} v \cdot \tau \,\mathrm{d}\Gamma, \tag{2.26}$$

where $\tau := -n^\perp$ and n denotes the unit outward normal to $\partial\Omega_R$. As $v \in L^2(\Omega)$, there exists a sequence $R_n \nearrow \infty$ such that $\varepsilon_n := R_n \int_{|y|=R_n} |v|^2 \,\mathrm{d}\Gamma \to 0$ as $n \to \infty$. Hence, by Cauchy–Schwarz inequality,

$$\Bigg(\int_{|y|=R_n} v \cdot \tau \,\mathrm{d}\Gamma\Bigg)^2 \leqslant \int_{|y|=R_n} |v|^2 \,\mathrm{d}\Gamma \cdot \int_{|y|=R_n} |\tau|^2 \,\mathrm{d}\Gamma = 2\pi\varepsilon_n \to 0.$$

Letting $R_n \to \infty$ in (2.26) yields (2.25), since curl $v \in L^1(\Omega)$. □

It follows from Lemma 2.5 and (2.7), (2.10) that

$$\int_{\Omega} \omega \,\mathrm{d}y = -\int_{\partial S} v \cdot \tau \,\mathrm{d}\Gamma = -C,$$

hence $\int_\Omega \omega_2(y)\,\mathrm{d}y = 0$ and $|C| \leqslant \mathrm{Const}(\|\omega\|_{L^\infty(\Omega)} + \|\omega\|_{L^1_\theta(\Omega)})$.

We now turn to the estimate (2.14) for $v = d_1 + C_1 d_2 + v_2$. As $d_1(y) = d_2(y) = 0$ for $|y| \geqslant R_0 + 1$, we only have to prove the following result.

Lemma 2.6. *Let $\omega_2 \in L^\infty(\Omega) \cap L^1_\theta(\Omega)$ with $\int_\Omega \omega_2(y)\,\mathrm{d}y = 0$, and let v_2 denote the solution of* (2.20)–(2.24). *Then there exists a constant $K_2 > 0$ (independent of ω_2) such that*

$$\big\| |y|\nabla v_2 \big\|_{L^2(\Omega)} \leqslant K_2\big(\|\omega_2\|_{L^\infty(\Omega)} + \|\omega_2\|_{L^1_\theta(\Omega)}\big). \tag{2.27}$$

Proof of Lemma 2.6. Let us introduce the following weights on $\mathbb{R}^2$

$$\rho(y) = \big(1+|y|^2\big)^{1/2} \quad \text{and} \quad lg(y) = \ln\big(2+|y|^2\big). \tag{2.28}$$

Since $\omega_2 \in L^1_\theta(\Omega) \cap L^\infty(\Omega)$ (with $\theta > 2$), we obtain (with the notations of [1]) that

$$\omega_2 \in W_1^{0,p}(\Omega) := \big\{w \in \mathcal{D}'(\Omega) \mid \rho(y)w(y) \in L^p(\Omega)\big\} \quad \forall p \in [2,\theta]$$

and

$$\omega_2 \in W_0^{-1,2}(\Omega) = \big(W_0^{1,2}(\Omega)\big)'$$

where

$$W_0^{1,2}(\Omega) := \big\{w \in \mathcal{D}'(\Omega) \mid \big(\rho(y)lg(y)\big)^{-1}w \in L^2(\Omega) \text{ and } \nabla w \in L^2(\Omega)\big\}.$$

It follows then from [1, Remark 2.11] that there exists a function $\psi_2 \in W_1^{2,2}(\Omega)$, with $\psi_2 \in W_1^{2,p}(\Omega)$ for all $p \in (2,\theta]$, such that $-\Delta\psi_2 = \omega_2$ in Ω and $\psi_2 = 0$ on ∂S. Recall that, with the notations of [1], a function w belongs to $W_1^{2,p}(\Omega)$ with $p > 2$ (resp. $p = 2$) if $\rho(y)^{-1}w(y) \in L^p(\Omega)$ (resp., $(\rho(y)lg(y))^{-1}w \in L^2(\Omega)$), $\partial w/\partial y_i \in L^p(\Omega)$ and $\rho(y)\partial^2 w/\partial y_i\partial y_j \in L^p(\Omega)$ for all i, j. It follows that $\bar v_2 := \operatorname{curl}\psi_2$ belongs to $W^{1,p}(\Omega)$ ($\subset \mathcal{B}(\overline{\Omega})$) for all $p \in (2,\theta]$ and it fulfills (2.20)–(2.22) and (2.24). As $\bar v_2 \in L^2(\Omega)$ and $\int_\Omega \omega_2(y)\,\mathrm{d}y = 0$, we infer from Lemma 2.5 that (2.23) holds as well for $\bar v_2$, hence $v_2 = \bar v_2$ by [23, Lemma 2.14]. We conclude that $|y|\nabla v_2 \in L^2(\Omega)$, and that (2.27) holds true. This completes the proof of Lemma 2.6 and of Proposition 2.2. □

Remark 2.7. It may occur that $|y|\nabla v \notin L^2(\Omega)$ when $v \notin L^2(\Omega)$. Indeed, let $\psi(y) = -\frac{1}{2\pi}\ln|y|$ denote the classical fundamental solution of Laplace's equation in $\mathbb{R}^2$, and let $v(y) := \operatorname{curl}\psi(y) = \frac{1}{2\pi}\frac{y^\perp}{|y|^2}$ for $y \in \Omega := \mathbb{R}^2 \setminus \overline{B_1(0)}$. Then (2.7)–(2.11) are fulfilled with $\omega = 0$, $l = (0,0)$ $r = 0$ and $C = 1$. It is easy to see that $v \in L^p(\Omega)$ if and only if $p > 2$, and that $|y|\nabla v \in L^p(\Omega)$ if and only if $p > 2$. Note that (2.25) also fails to be true for v.

3. Navier–Stokes approximation for the fluid

To solve (1.10)–(1.16), we follow an idea of P.-L. Lions ([25]). Namely, we replace the Euler equations by the Navier–Stokes equations and we supplement the system with the boundary condition $\operatorname{rot} v = 0$ on ∂S. As the term $ry^\perp \cdot \nabla v$ may be unbounded with y, we first study an approximated system in which the (unbounded) vector $y^\perp$ is replaced by the (bounded) vector $y_R^\perp$, which is defined for each number $R > R_0$ by

$$y_R^\perp = \begin{cases} y^\perp & \text{if } |y| \leqslant R, \\ \frac{R}{|y|}y^\perp & \text{if } |y| \geqslant R. \end{cases}$$

We then consider the following system

$$\frac{\partial v}{\partial t} + \big[\big(v - l - ry_R^\perp\big)\cdot\nabla\big]v + rv^\perp - \nu\Delta v + \nabla q = 0 \quad \text{in } \Omega \times [0,T], \tag{3.1}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega \times [0,T], \tag{3.2}$$

$$v \cdot n = (l + ry^\perp)\cdot n \quad \text{on } \partial S \times [0,T], \tag{3.3}$$

$$\operatorname{curl} v = 0 \quad \text{on } \partial S \times [0,T], \tag{3.4}$$

$$ml' = \int_{\partial S} qn\,\mathrm{d}\Gamma - mrl^\perp \quad \text{in } [0,T], \tag{3.5}$$

$$Jr' = \int_{\partial S} qn \cdot y^\perp\,\mathrm{d}\Gamma \quad \text{in } [0,T], \tag{3.6}$$

$$v(y,0)=a(y) \quad y\in\Omega, \tag{3.7}$$

$$l(0)=b,\ r(0)=c. \tag{3.8}$$

Proceeding as in [27], we may prove the following result.

Proposition 3.1. *Let $a\in H^1(\Omega)$, $b\in\mathbb{R}^2$ and $c\in\mathbb{R}$ be such that*

$$\operatorname{div} a=0 \quad \text{in } \Omega,$$
$$a\cdot n=(b+cy^{\perp})\cdot n \quad \text{on } \partial S.$$

Then for any $T>0$ the system (3.1)–(3.8) *admits a unique solution* $(v_\nu^R,\ q_\nu^R,\ l_\nu^R,\ r_\nu^R)$ *with*

$$v_\nu^R\in L^2\big(0,T;H^2(\Omega)\big)\cap C\big([0,T];H^1(\Omega)\big)\cap H^1\big(0,T;L^2(\Omega)\big),$$
$$q_\nu^R\in L^2\big(0,T;\widehat{H}^1(\Omega)\big),\quad l_\nu^R\in H^1\big(0,T;\mathbb{R}^2\big),\quad r_\nu^R\in H^1(0,T;\mathbb{R}).$$

4. First passage to the limit

In this section, we pass to the limit as $R\to\infty$.

4.1. Some estimates

We first prove an energy estimate for the system (3.1)–(3.8).

Proposition 4.1. *Let $a\in H^1(\Omega)$ be a function satisfying*

$$\operatorname{div} a=0 \quad \text{in } \Omega \quad \text{and} \quad a\cdot n=(b+cy^{\perp})\cdot n \quad \text{on } \partial S. \tag{4.1}$$

Then there exists a positive constant $C=C(S,m,J,\|\kappa\|_{L^\infty(\partial S)})$ independent of R and ν *such that the unique strong solution* $(v_\nu^R,q_\nu^R,l_\nu^R,r_\nu^R)$ *of* (3.1)–(3.8) *satisfies*

$$\int_\Omega \big|v_\nu^R(y,t)\big|^2\,\mathrm{d}y+m\big|l_\nu^R(t)\big|^2+J\big|r_\nu^R(t)\big|^2+\nu\int_0^t\int_\Omega\big|\nabla v_\nu^R(y,s)\big|^2\,\mathrm{d}y\,\mathrm{d}s$$
$$\leqslant \mathrm{e}^{C\nu t}\bigg[\int_\Omega\big|a(y)\big|^2\,\mathrm{d}y+m|b|^2+J|c|^2\bigg]\quad \forall t\in[0,T]. \tag{4.2}$$

Proof. In this proof, we drop the sub and superscripts ($v=v_\nu^R$) for the sake of readability.

Multiplying (3.1) by v and integrating over $\Omega\times(0,t)$ for any $t<T$ we get

$$\int_0^t\int_\Omega\frac{\partial v}{\partial t}\cdot v\,\mathrm{d}y\,\mathrm{d}s+\int_0^t\int_\Omega\big[\big(\big(v-l-ry_R^{\perp}\big)\cdot\nabla\big)v\big]\cdot v\,\mathrm{d}y\,\mathrm{d}s-\int_0^t\nu\int_\Omega\Delta v\cdot v\,\mathrm{d}y\,\mathrm{d}s+\int_0^t\int_\Omega\nabla q\cdot v\,\mathrm{d}y\,\mathrm{d}s$$
$$=I_1+I_2-I_3+I_4=0.$$

After some integrations by parts we obtain $I_2=0$ and

$$I_4=\bigg[\frac{m}{2}|l|^2+\frac{J}{2}r^2\bigg]_0^t,$$

hence

$$\int_\Omega\big|v(y,t)\big|^2\,\mathrm{d}y+m\big|l(t)\big|^2+J\big|r(t)\big|^2+2\nu\int_0^t\int_\Omega|\nabla v|^2\,\mathrm{d}y\,\mathrm{d}s$$
$$\leqslant\int_\Omega\big|a(y)\big|^2\,\mathrm{d}y+m|b|^2+J|c|^2+2\nu\int_0^t\int_{\partial S}\frac{\partial v}{\partial n}\cdot v\,\mathrm{d}\Gamma. \tag{4.3}$$

According to Lemma 2.1 we have that

$$\int_{\partial S} v \cdot \frac{\partial v}{\partial n}\, \mathrm{d}\Gamma = \int_{\partial S} \kappa |v - l - r y^{\perp}|^2\, \mathrm{d}\Gamma + 2\int_{\partial S} r v \cdot \tau\, \mathrm{d}\Gamma + \int_{\partial S} r^2 y \cdot n\, \mathrm{d}\Gamma,$$

hence there exists a positive constant $C_1 = C_1(S, \|\kappa\|_{L^\infty(\partial S)})$ such that

$$\left|\int_{\partial S} v \cdot \frac{\partial v}{\partial n}\, \mathrm{d}\Gamma\right| \leqslant C_1\left(\int_{\partial S} |v|^2\, \mathrm{d}\Gamma + |l|^2 + r^2\right).$$

Using a trace inequality, we see that there exists a positive constant $C_2 = C_2(S)$ such that

$$\int_{\partial S} |v|^2\, \mathrm{d}\Gamma \leqslant C_2 \int_{\Omega} |v|^2\, \mathrm{d}y + \frac{1}{2C_1}\int_{\Omega} |\nabla v|^2\, \mathrm{d}y.$$

It follows that there exists a positive constant $C = C(S, m, J, \|\kappa\|_{L^\infty(\partial S)})$ such that

$$\int_0^t \left|\int_{\partial S} v \cdot \frac{\partial v}{\partial n}\, \mathrm{d}\Gamma\right| \mathrm{d}s \leqslant \frac{C}{2}\left\{\int_0^t\int_{\Omega} |v|^2\, \mathrm{d}y\, \mathrm{d}s + m\int_0^t |l|^2\, \mathrm{d}s + J\int_0^t r^2\, \mathrm{d}s\right\} + \frac{1}{2}\int_0^t\int_{\Omega} |\nabla v|^2\, \mathrm{d}y\, \mathrm{d}s$$

which, combined to (4.3), yields

$$\begin{aligned} &\int_{\Omega} \left|v(y,t)\right|^2 \mathrm{d}y + m\left|l(t)\right|^2 + J\left|r(t)\right|^2 + \nu\int_0^t\int_{\Omega} |\nabla v|^2\, \mathrm{d}y\, \mathrm{d}s \\ &\quad \leqslant \int_{\Omega} \left|a(y)\right|^2 \mathrm{d}y + m|b|^2 + J|c|^2 + \nu C\left\{\int_0^t\int_{\Omega} |v|^2\, \mathrm{d}y\, \mathrm{d}s + m\int_0^t |l|^2\, \mathrm{d}s + J\int_0^t r^2\, \mathrm{d}s\right\}. \end{aligned} \tag{4.4}$$

An application of Gronwall's lemma gives then

$$\int_{\Omega} \left|v(t)\right|^2 \mathrm{d}y + m\left|l(t)\right|^2 + J\left|r(t)\right|^2 + \nu\int_0^t\int_{\Omega} |\nabla v|^2\, \mathrm{d}y\, \mathrm{d}s \leqslant \mathrm{e}^{C\nu t}\left\{\int_{\Omega} \left|a(y)\right|^2 \mathrm{d}y + m|b|^2 + J|c|^2\right\}.$$

The proof is completed. □

Let us now introduce the vorticity $\omega_\nu^R := \operatorname{curl} v_\nu^R$. Then

$$\omega_\nu^R \in L^2\big(0,T; H_0^1(\Omega)\big) \cap C\big([0,T]; L^2(\Omega)\big) \cap H^1\big(0,T; H^{-1}(\Omega)\big).$$

Taking the "curl" in (3.1) results in

$$\frac{\partial \omega_\nu^R}{\partial t} - \nu\Delta\omega_\nu^R + \left(v_\nu^R - l_\nu^R - r_\nu^R y_R^{\perp}\right)\cdot\nabla\omega_\nu^R - r_\nu^R D_R(y) : \nabla v_\nu^R = 0 \quad \text{in } \Omega \times [0,T] \tag{4.5}$$

where

$$\left\{D_R(y)\right\}_{i,j} := R\, 1_{|y|>R} \frac{y_i y_j}{|y|^3}. \tag{4.6}$$

Eq. (4.5) has to be supplemented with the boundary condition $\omega_\nu^R = 0$ on $\partial S \times [0,T]$ and the initial condition $\omega_\nu^R(0) = \operatorname{curl} a$ in Ω.

The next result asserts that ω_ν^R remains bounded in $L^2(0,T; H_0^1(\Omega))$ *uniformly with respect to* R.

Proposition 4.2. *Let $a \in H^1(\Omega)$ be a function fulfilling* (4.1). *Then there exists a positive constant C* independent of *R such that*

$$\int_{\Omega} \left|\omega_\nu^R(y,t)\right|^2 \mathrm{d}y + 2\nu\int_0^t\int_{\Omega} \left|\nabla\omega_\nu^R(y,s)\right|^2 \mathrm{d}y\, \mathrm{d}s \leqslant C \quad \forall t \in [0,T]. \tag{4.7}$$

Proof. Scaling in (4.5) by ω, we obtain after some integrations by parts

$$\int_0^t \left\langle \frac{\partial \omega_\nu^R}{\partial t}, \omega_\nu^R \right\rangle_{H^{-1}\times H_0^1} \mathrm{d}s + \nu \int_0^t \int_\Omega \left|\nabla \omega_\nu^R\right|^2 \mathrm{d}y\,\mathrm{d}s - \int_0^t \int_\Omega r_\nu^R \big(D_R(y) : \nabla v_\nu^R\big)\omega_\nu^R(y,s)\,\mathrm{d}y\,\mathrm{d}s = 0.$$

Then (4.7) follows from (4.2) and (4.6). □

4.2. *Passage to the limit $R \to \infty$*

In what follows, we fix $\nu > 0$ and we let $R \to +\infty$. According to Propositions 4.1 and 4.2, the functions v_ν^R and ω_ν^R are bounded in $L^2(0,T;H^1(\Omega))$ (as $\nu > 0$ is kept constant) and the functions l_ν^R and r_ν^R are bounded in $L^\infty(0,T)$. Therefore, there exist a sequence $R_k \nearrow \infty$ and some functions $v_\nu \in L^2(0,T;H^1(\Omega))$, $\omega_\nu \in L^2(0,T;H_0^1(\Omega))$, $l_\nu \in L^\infty(0,T;\mathbb{R}^2)$ and $r_\nu \in L^\infty(0,T)$ such that

$$\begin{aligned}
&v_\nu^{R_k} \rightharpoonup v_\nu \quad \text{in } L^2\big(0,T;H^1(\Omega)\big),\\
&\omega_\nu^{R_k} \rightharpoonup \omega_\nu \quad \text{in } L^2\big(0,T;H_0^1(\Omega)\big),\\
&l_\nu^{R_k} \rightharpoonup l_\nu \quad \text{in } L^\infty\big(0,T;\mathbb{R}^2\big)\text{-weak}*,\\
&r_\nu^{R_k} \rightharpoonup r_\nu \quad \text{in } L^\infty(0,T)\text{-weak}*
\end{aligned}$$

as $k \to +\infty$. Clearly

$$\operatorname{div} v_\nu = 0 \quad \text{in } \Omega \times [0,T] \tag{4.8}$$

and

$$v_\nu \cdot n = \big(l_\nu + r_\nu y^\perp\big)\cdot n \quad \text{on } \partial S \times [0,T]. \tag{4.9}$$

We now aim to take the limit in (4.5). For any $f \in L^2(0,T;L^2(\Omega))$, we have that $fD_{R_k} \to 0$ in $L^2(0,T;L^2(\Omega))$, hence

$$r_\nu^{R_k} D_{R_k}(y) : \nabla v_\nu^{R_k} \rightharpoonup 0 \quad \text{in } L^2\big(0,T;L^2(\Omega)\big).$$

Since $\operatorname{div}(v_\nu^{R_k} - l_\nu^{R_k} - r_\nu^{R_k} y_{R_k}^\perp) = 0$, we obtain that

$$\big(v_\nu^{R_k} - l_\nu^{R_k} - r_\nu^{R_k} y_{R_k}^\perp\big)\cdot \nabla \omega_\nu^{R_k} = \operatorname{div}\big(\omega_\nu^{R_k}\big(v_\nu^{R_k} - l_\nu^{R_k} - r_\nu^{R_k} y_{R_k}^\perp\big)\big). \tag{4.10}$$

Pick any $R > 0$. It follows from (4.5) that the sequence $(\partial \omega_\nu^{R_k}/\partial t)$ is bounded in $L^2(0,T;H^{-2}(\Omega_R))$. An application of Aubin's lemma gives that (for a subsequence)

$$\omega_\nu^{R_k} \to \omega_\nu \quad \text{in } L^2\big(0,T;L^2(\Omega_R)\big),$$

hence

$$\omega_\nu^{R_k}\big(v_\nu^{R_k} - l_\nu^{R_k} - r_\nu^{R_k} y_{R_k}^\perp\big) \to \omega_\nu\big(v_\nu - l_\nu - r_\nu y^\perp\big) \quad \text{in } \mathcal{D}'\big(\Omega_R \times (0,T)\big)$$

and therefore, using (4.10),

$$\big(v_\nu^{R_k} - l_\nu^{R_k} - r_\nu^{R_k} y_{R_k}^\perp\big)\cdot \nabla \omega_\nu^{R_k} \to \big(v_\nu - l_\nu - r_\nu y^\perp\big)\cdot \nabla \omega_\nu \quad \text{in } \mathcal{D}'\big(\Omega_R \times (0,T)\big)$$

as $k \to +\infty$. It follows that ω_ν fulfills the equation

$$\frac{\partial \omega_\nu}{\partial t} - \nu \Delta \omega_\nu + \big[\big(v_\nu - l_\nu - r_\nu y^\perp\big)\cdot \nabla\big]\omega_\nu = 0 \quad \text{in } \mathcal{D}'\big(\Omega \times (0,T)\big).$$

Clearly, the equations $\omega_\nu = \operatorname{curl} v_\nu$ and $\omega_\nu|_{\partial S} = 0$ are satisfied. We now turn to the initial condition. Let us introduce the Hilbert space

$$H := \left\{\varphi \in H_0^1(\Omega)^2 \mid \operatorname{div}\varphi = 0 \text{ and } \int_\Omega \left|\varphi(y)\right|^2 \rho(y)^2\,\mathrm{d}y < \infty\right\}$$

endowed with the norm $\|\varphi\|_H^2 := \int_\Omega (|\nabla\varphi(y)|^2 + \rho(y)^2|\varphi(y)|^2)\,dy$. It may be seen that the sequence $(v_\nu^{R_k})_t$ is bounded in $L^{3/2}(0,T;H')$. (Scaling in (3.1) by a test function $\varphi \in L^3(0,T;H)$, the result follows at once in integrating by parts in the integral term $\int_0^T\int_\Omega (v\cdot\nabla v)\cdot\varphi\,dy\,dt$ and in using the boundedness of v_ν^R in $L^3(0,T;L^4(\Omega))$.)

Observing that the first embedding in

$$H^1(\Omega) \subset L^2_{\rho(y)^{-2}\,dy}(\Omega) \subset H'$$

is compact and that $L^2(\Omega) \subset H'$ compactly, we deduce from [31, Corollary 4] that the sequence $(v_\nu^{R_k})$ is relatively compact in both $C([0,T];H')$ and $L^2(0,T;L^2_{\rho(y)^{-2}\,dy}(\Omega))$. Extracting a subsequence if needed, we may assume that

$$v_\nu^{R_k} \to v_\nu \quad \text{in } C\big([0,T];H'\big) \cap L^2\big(0,T;L^2_{\rho(y)^{-2}\,dy}(\Omega)\big)$$

as $k \to +\infty$. Therefore, we conclude that

$$v_\nu(0) = a, \tag{4.11}$$

$$\omega_\nu(0) = \operatorname{curl} a. \tag{4.12}$$

Finally, we show that v_ν satisfies a variational equation associated with the system (3.1)–(3.8) (with $R = +\infty$). To this end we introduce two families of Hilbert spaces. For all $R > R_0$, let

$$\mathcal{H}_R := \big\{\varphi \in \mathcal{H} \mid \varphi = 0 \text{ for } |y| \geqslant R\big\},$$
$$\mathcal{V}_R := \big\{\varphi \in \mathcal{V} \mid \varphi = 0 \text{ for } |y| \geqslant R\big\}.$$

$\mathcal{H}_R$ and $\mathcal{V}_R$ are closed subspaces of $\mathcal{H}$ and $\mathcal{V}$, respectively. Noticing that $\mathcal{V}_R$ is dense in $\mathcal{H}_R$ and identifying $\mathcal{H}_R$ with $\mathcal{H}_R'$, we obtain the diagram

$$\mathcal{V}_R \subset \mathcal{H}_R \equiv \mathcal{H}_R' \subset \mathcal{V}_R'$$

where $\mathcal{V}_R'$ denotes the dual space of $\mathcal{V}_R$ with respect to the pivot space $\mathcal{H}_R$. Therefore, we may write for any $\varphi \in \mathcal{V}_R$ and any $\psi \in \mathcal{H}_R$

$$(\psi,\varphi)_\gamma = \langle\psi,\varphi\rangle_R,$$

where the symbol $\langle\cdot,\cdot\rangle_R$ denotes the duality pairing between $\mathcal{V}_R'$ and $\mathcal{V}_R$. The following result reveals that (v_ν, l_ν, r_ν) is a *weak* solution of the Navier–Stokes problem (3.1)–(3.8) (with y_R replaced by y).

Proposition 4.3. *For all $R > R_0$, $v_\nu' = (v_\nu)_t$ is bounded in $L^{3/2}(0,T;(\mathcal{V}_R)')$, and for any*

$$\varphi \in L^3(0,T;\mathcal{V}_R) \cap L^\infty\big(0,T;L^\infty(\Omega)\big)$$

we have that

$$\int_0^T \Big\{\big\langle v_\nu',\varphi\big\rangle_R + \nu\int_\Omega \nabla v_\nu : \nabla\varphi\,dy\Big\}\,dt$$
$$- \nu\int_0^T\int_{\partial S} \big\{\kappa\big(v_\nu - l_\nu - r_\nu y^\perp\big)\cdot\big(\varphi - l_\varphi - r_\varphi y^\perp\big) + (r_\varphi v_\nu + r_\nu\varphi)\cdot\tau + r_\nu r_\varphi\, y\cdot n\big\}\,d\Gamma\,dt$$
$$= \int_0^T\int_\Omega \big\{\big(l_\nu + r_\nu y^\perp - v_\nu\big)\cdot\nabla v_\nu - r_\nu v_\nu^\perp\big\}\cdot\varphi\,dy\,dt. \tag{4.13}$$

Proof. By using (3.1)–(3.8) and the fact that $v_\nu^{R_k}$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;\mathcal{V}) \cap L^3(0,T;L^4(\Omega))$, we easily check that $(v_\nu^{R_k})_t$ is bounded in $L^{3/2}(0,T;(\mathcal{V}_R)')$ and that

$$\int_0^T \left\{ \left(\left(v_\nu^{R_k}\right)', \varphi \right)_\gamma + \nu \int_\Omega \nabla v_\nu^{R_k} : \nabla \varphi \, \mathrm{d}y \right\} \mathrm{d}t$$

$$- \nu \int_0^T \int_{\partial S} \left\{ \kappa \left(v_\nu^{R_k} - l_\nu^{R_k} - r_\nu^{R_k} y^\perp \right) \cdot \left(\varphi - l_\varphi - r_\varphi y^\perp \right) + \left(r_\varphi v_\nu^{R_k} + r_\nu^{R_k} \varphi \right) \cdot \tau + r_\nu^{R_k} r_\varphi \, y \cdot n \right\} \mathrm{d}\Gamma \, \mathrm{d}t$$

$$= \int_0^T \int_\Omega \left\{ \left(l_\nu^{R_k} + r_\nu^{R_k} y_{R_k}^\perp - v_\nu^{R_k} \right) \cdot \nabla v_\nu^{R_k} - r_\nu^{R_k} \left(v_\nu^{R_k} \right)^\perp \right\} \cdot \varphi \, \mathrm{d}y \, \mathrm{d}t \tag{4.14}$$

for any $\varphi \in L^3(0, T; \mathcal{V}_R)$. If in addition $\varphi \in L^\infty(Q_T)$, then (4.14) yields (4.13) in the limit $k \to \infty$. □

5. Some estimates for the Navier–Stokes problem

In this section, we prove some estimates for the velocities v_ν, l_ν, r_ν and for the vorticity

$$\omega_\nu = \operatorname{curl} v_\nu. \tag{5.1}$$

Recall that ω_ν fulfills the following system

$$\frac{\partial \omega_\nu}{\partial t} + \left(v_\nu - l_\nu - r_\nu y^\perp \right) \cdot \nabla \omega_\nu - \nu \Delta \omega_\nu = 0 \quad \text{in } \Omega \times [0, T], \tag{5.2}$$

$$\omega_\nu = 0 \quad \text{on } \partial S \times [0, T], \tag{5.3}$$

$$\omega_\nu(y, 0) = \omega_0(y) \quad \forall y \in \Omega. \tag{5.4}$$

These estimates will be used in the next section to pass to the limit in (5.2) and in (4.13) as $\nu \to 0$.

5.1. Energy estimate

The following (energy) estimate for the functions v_ν, l_ν, r_ν is an obvious consequence of Proposition 4.1.

Proposition 5.1. *Let $a \in H^1(\Omega)$ be a function satisfying (4.1). Then there exists a positive constant $C = C(S, m, J, \|\kappa\|_{L^\infty(\partial S)})$ such that for any $\nu > 0$ and for a.e. $t \in [0, T]$*

$$\int_\Omega \left| v_\nu(y, t) \right|^2 \mathrm{d}y + m \left| l_\nu(t) \right|^2 + J \left| r_\nu(t) \right|^2 \leqslant \mathrm{e}^{C\nu t} \left[\int_\Omega \left| a(y) \right|^2 \mathrm{d}y + m|b|^2 + J|c|^2 \right]. \tag{5.5}$$

5.2. Vorticity estimates

We have the following estimate.

Proposition 5.2. *Let $\omega_0 \in L^1(\Omega) \cap L^\infty(\Omega)$. Then for all $p \in [1, +\infty]$ and for all $t \in [0, T]$ we have that*

$$\left\| \omega_\nu(t) \right\|_{L^p(\Omega)} \leqslant \|\omega_0\|_{L^p(\Omega)}. \tag{5.6}$$

Proof. Multiplying (4.5) by $\varphi \in L^2(0, T; H_0^1(\Omega))$ and integrating with respect to time, we have that

$$\int_0^t \left\langle \frac{\partial \omega_\nu^R}{\partial t}, \varphi \right\rangle_{H^{-1} \times H_0^1} \mathrm{d}s - \nu \int_0^t \left\langle \Delta \omega_\nu^R, \varphi \right\rangle_{H^{-1} \times H_0^1} \mathrm{d}s + \int_0^t \left\langle \left(v_\nu^R - l_\nu^R - r_\nu^R y_R^\perp \right) \cdot \nabla \omega_\nu^R, \varphi \right\rangle_{H^{-1} \times H_0^1} \mathrm{d}s$$

$$- \int_0^t r_\nu^R \left\langle D_R : \nabla v_\nu^R, \varphi \right\rangle_{H^{-1} \times H_0^1} \mathrm{d}s = 0.$$

The above equation easily yields that

$$\int_0^t \left\langle \frac{\partial \omega_\nu^R}{\partial t}, \varphi \right\rangle_{H^{-1}\times H_0^1} \mathrm{d}s + \nu \int_0^t \int_\Omega \nabla \omega_\nu^R \cdot \nabla \varphi \,\mathrm{d}y\,\mathrm{d}s$$

$$- \int_0^t \int_\Omega \omega_\nu^R \big(v_\nu^R - l_\nu^R - r_\nu^R y_R^\perp\big) \cdot \nabla \varphi \,\mathrm{d}y\,\mathrm{d}s - \int_0^t r_\nu^R \int_\Omega \big(D_R : \nabla v_\nu^R\big)\varphi \,\mathrm{d}y\,\mathrm{d}s = 0. \tag{5.7}$$

Now, we analyze each integral and consider two cases:

Case 1 ($p \geqslant 2$). As the function $|\omega_\nu|^{p-2}\omega_\nu$ cannot be a priori taken as a test function, we are led to truncate it. For any $M > 0$ let $T_M \in C(\mathbb{R})$ denote the function

$$T_M(a) = \begin{cases} M & \text{if } a > M, \\ a & \text{if } -M \leqslant a \leqslant M, \\ -M & \text{if } a < -M. \end{cases}$$

For every $\delta > 0$, let $\psi_\delta \in C^\infty(\Omega)$ be the function defined by

$$\psi_\delta(y) = \exp\big(-\delta\rho(y)\big), \tag{5.8}$$

where $\rho(y)$ is defined in (2.28). We have that

$$\nabla \psi_\delta(y) = -\delta \frac{y}{\rho(y)} \psi_\delta(y), \tag{5.9}$$

and that

$$\Delta \psi_\delta(y) = \left(\delta^2 \frac{|y|^2}{\rho^2} + \frac{\delta}{\rho}\left(\frac{|y|^2}{\rho^2} - 2\right)\right)\psi_\delta(y). \tag{5.10}$$

We consider for each $M > 0$ and for each $\delta > 0$, the following test function

$$\varphi = \big|T_M\big(\omega_\nu^R\big)\big|^{p-2} T_M\big(\omega_\nu^R\big)\psi_\delta.$$

We easily check that $\varphi \in L^2(0, T; H_0^1(\Omega))$ and that

$$\nabla \varphi(y) = \left[(p-1)\big(\nabla \omega_\nu^R\big) 1_{|\omega_\nu^R| \leqslant M} - \delta \frac{y}{\rho(y)} T_M\big(\omega_\nu^R\big)\right] \big|T_M\big(\omega_\nu^R\big)\big|^{p-2}\psi_\delta.$$

By using this test function in (5.7), we obtain that for every $t \in [0, T]$

$$\int_\Omega F_M\big(\omega_\nu^R(t)\big)\psi_\delta \,\mathrm{d}y - \int_\Omega F_M(\omega_0)\psi_\delta \,\mathrm{d}y + \nu(p-1)\int_0^t \int_\Omega \big|\nabla \omega_\nu^R\big|^2 \big|\omega_\nu^R\big|^{p-2} 1_{|\omega_\nu^R| \leqslant M} \psi_\delta \,\mathrm{d}y\,\mathrm{d}s$$

$$- \nu\delta \int_0^t \int_\Omega \frac{y}{\rho(y)} \cdot \big(\nabla \omega_\nu^R\big) \big|T_M\big(\omega_\nu^R\big)\big|^{p-2} T_M\big(\omega_\nu^R\big)\psi_\delta \,\mathrm{d}y\,\mathrm{d}s$$

$$+ \int_0^t \int_\Omega \delta\big(v_\nu^R - l_\nu^R\big) \cdot \frac{y}{\rho(y)} \left(\omega_\nu^R T_M\big(\omega_\nu^R\big) - \frac{p-1}{p}\big|T_M\big(\omega_\nu^R\big)\big|^2\right) \big|T_M\big(\omega_\nu^R\big)\big|^{p-2}\psi_\delta(y)\,\mathrm{d}y\,\mathrm{d}s$$

$$- \int_0^t r_\nu^R \int_\Omega \big(D_R : \nabla v_\nu^R\big) \big|T_M\big(\omega_\nu^R\big)\big|^{p-2} T_M\big(\omega_\nu^R\big)\psi_\delta \,\mathrm{d}y\,\mathrm{d}s = 0, \tag{5.11}$$

where we have defined the function

$$F_M(a) := \int_0^a \big|T_M(\sigma)\big|^{p-2} T_M(\sigma)\,\mathrm{d}\sigma = \frac{1}{p}\big|T_M(a)\big|^p + M^{p-1}\big(|a| - M\big)^+ \quad \forall a \in \mathbb{R}.$$

Using Cauchy–Schwarz inequality in (5.11), we obtain that

$$\int_\Omega F_M\big(\omega_\nu^R(t)\big)\psi_\delta\,\mathrm{d}y \leqslant \int_\Omega F_M(\omega_0)\psi_\delta\,\mathrm{d}y + C\Bigg\{\nu\delta\int_0^t \big\|\nabla\omega_\nu^R\big\|_{L^2(\Omega)}\big\|\omega_\nu^R\big\|_{L^2(\Omega)}\,\mathrm{d}s + \delta\int_0^t \big\|v_\nu^R\big\|_{L^2(\Omega)}\big\|\omega_\nu^R\big\|_{L^2(\Omega)}\,\mathrm{d}s + \delta\int_0^t \big|l_\nu^R\big|\big\|\omega_\nu^R\big\|^2_{L^2(\Omega)}\,\mathrm{d}s + \big\|r_\nu^R\big\|_{L^\infty(0,T)}\int_0^t \big\|\nabla v_\nu^R\big\|_{L^2(\Omega)}\,\mathrm{d}s\bigg(\int_\Omega 1_{|y|\geqslant R}\psi_\delta(y)^2\,\mathrm{d}y\bigg)^{1/2}\Bigg\} \tag{5.12}$$

where $C = C(\nu, m)$ is a constant. Taking the limit $R \to \infty$ in above equation, using (4.2), (4.7) and the convexity of F_M, we obtain that there exists a constant $C' = C'(\nu)$ such that

$$\int_\Omega F_M\big(\omega_\nu(t)\big)\psi_\delta\,\mathrm{d}y \leqslant \int_\Omega F_M(\omega_0)\psi_\delta\,\mathrm{d}y + C'\delta.$$

Taking $M > \|\omega_0\|_{L^\infty(\Omega)}$ and letting $\delta \to 0$, we get

$$\int_\Omega F_M\big(\omega_\nu(t)\big)\,\mathrm{d}y \leqslant \frac{1}{p}\int_\Omega \big|\omega_0(y)\big|^p\,\mathrm{d}y. \tag{5.13}$$

Finally, letting $M \nearrow \infty$, we obtain by the monotone convergence theorem

$$\big\|\omega_\nu(t)\big\|_{L^p(\Omega)} \leqslant \|\omega_0\|_{L^p(\Omega)}.$$

Thus, taking the limit as $p \to \infty$ we conclude that

$$\big\|\omega_\nu(t)\big\|_{L^\infty(\Omega)} \leqslant \|\omega_0\|_{L^\infty(\Omega)}, \quad \text{for any } t \in [0; T].$$

Case 2 ($1 \leqslant p < 2$). Let us now consider the test function

$$\varphi = \big(\big|\omega_\nu^R\big| + \varepsilon\big)^{p-2}\omega_\nu^R\psi_\delta$$

whose gradient is

$$\nabla\varphi = \big(\big|\omega_\nu^R\big| + \varepsilon\big)^{p-3}\big((p-1)\big|\omega_\nu^R\big| + \varepsilon\big)\nabla\omega_\nu^R\psi_\delta - \frac{\delta y}{\rho(y)}\big(\big|\omega_\nu^R\big| + \varepsilon\big)^{p-2}\omega_\nu^R\psi_\delta.$$

Replacing φ and $\nabla\varphi$ by their new expression in (5.7), we obtain after some calculation that

$$\begin{aligned}&\int_\Omega H_\varepsilon\big(\omega_\nu^R(y,t)\big)\psi_\delta\,\mathrm{d}y + \nu\int_0^t\int_\Omega \big(\big|\omega_\nu^R\big| + \varepsilon\big)^{p-3}\big((p-1)\big|\omega_\nu^R\big| + \varepsilon\big)\big|\nabla\omega_\nu^R\big|^2\psi_\delta\,\mathrm{d}y\,\mathrm{d}s \\ &\quad= \int_\Omega H_\varepsilon\big(\omega_0(y)\big)\psi_\delta\,\mathrm{d}y + \nu\int_0^t\int_\Omega H_\varepsilon\big(\omega_\nu^R\big)\Delta\psi_\delta\,\mathrm{d}y\,\mathrm{d}s - \delta\int_0^t\int_\Omega \frac{y}{\rho(y)}\cdot\big(v_\nu^R - l_\nu^R\big)\big|\omega_\nu^R\big|^2\big(\big|\omega_\nu^R\big| + \varepsilon\big)^{p-2}\psi_\delta\,\mathrm{d}y\,\mathrm{d}s \\ &\qquad+ \int_0^t r_\nu^R\int_\Omega \big(D_R : \nabla v_\nu^R\big)\big(\big|\omega_\nu^R\big| + \varepsilon\big)^{p-2}\omega_\nu^R\psi_\delta\,\mathrm{d}y\,\mathrm{d}s,\end{aligned} \tag{5.14}$$

where

$$H_\varepsilon(a) = \int_0^a \big(|\sigma| + \varepsilon\big)^{p-2}\sigma\,\mathrm{d}\sigma = \frac{|a|(|a| + \varepsilon)^{p-1}}{(p-1)} - \frac{(|a| + \varepsilon)^p - \varepsilon^p}{p(p-1)} \quad \forall a \in \mathbb{R}. \tag{5.15}$$

Using Cauchy–Schwarz inequality in (5.14), we obtain for some constant $C > 0$ (independent of ε, δ, ν, R)

$$\int_\Omega H_\varepsilon\big(\omega_\nu^R(y,t)\big)\psi_\delta\,\mathrm{d}y \leqslant \int_\Omega H_\varepsilon\big(\omega_0(y)\big)\psi_\delta\,\mathrm{d}y + C\nu\delta\int_0^t\int_\Omega H_\varepsilon\big(\omega_\nu^R\big)\psi_\delta\,\mathrm{d}y\,\mathrm{d}s$$
$$+\frac{\delta}{\varepsilon^{2-p}}\int_0^t\big(\|v_\nu^R\|_{L^2(\Omega)}\|\omega_\nu^R\|_{L^4(\Omega)}^2+|l_\nu^R|\|\omega_\nu^R\|_{L^2(\Omega)}^2\big)\,\mathrm{d}s$$
$$+\frac{C}{\varepsilon^{2-p}}\|r_\nu^R\|_{L^\infty(0,T)}\,\mathrm{e}^{-\delta R}\int_0^t\|\nabla v_\nu^R\|_{L^2(\Omega)}^2\,\mathrm{d}s. \tag{5.16}$$

By using (4.2), (4.7) we deduce from the above equation that there exists a constant $C'=C'(\nu,\varepsilon,p)$ such that

$$\int_\Omega H_\varepsilon\big(\omega_\nu^R(y,t)\big)\psi_\delta\,\mathrm{d}y \leqslant \int_\Omega H_\varepsilon\big(\omega_0(y)\big)\psi_\delta\,\mathrm{d}y + C'\delta + C'\,\mathrm{e}^{-\delta R}.$$

From the convexity of H_ε we get, by letting $R\to\infty$, that

$$\int_\Omega H_\varepsilon\big(\omega_\nu(y,t)\big)\psi_\delta\,\mathrm{d}y \leqslant \int_\Omega H_\varepsilon\big(\omega_0(y)\big)\psi_\delta\,\mathrm{d}y + C'\delta.$$

Therefore, letting $\delta\to 0$, we obtain

$$\int_\Omega H_\varepsilon\big(\omega_\nu(y,t)\big)\,\mathrm{d}y \leqslant \int_\Omega H_\varepsilon\big(\omega_0(y)\big)\,\mathrm{d}y.$$

Finally, taking the limit as $\varepsilon\to 0$, we conclude that

$$\int_\Omega \big|\omega_\nu(y,t)\big|^p\,\mathrm{d}y \leqslant \int_\Omega \big|\omega_0(y)\big|^p\,\mathrm{d}y, \quad \forall t\in[0,T]$$

and by letting $p\to 1$, we have that for all $p\in[1,2)$ and all $t\in[0,T]$

$$\big\|\omega_\nu(t)\big\|_{L^p(\Omega)} \leqslant \|w_0\|_{L^p(\Omega)}. \qquad \square$$

Proposition 5.3. *Let $\omega_0\in L^1(\Omega)\cap L^\infty(\Omega)$ be a function such that*

$$\int_\Omega \big|\omega_0(y)\big|\,\rho(y)^\theta\,\mathrm{d}y<\infty,$$

for a positive constant $\theta>1$. Then there exists a positive constant $C>0$ such that for all $\nu>0$ we have that

$$\int_\Omega \big|\omega_\nu(y,t)\big|\,\rho(y)^\theta\,\mathrm{d}y \leqslant \mathrm{e}^{Ct}\int_\Omega \big|\omega_0(y)\big|\rho(y)^\theta\,\mathrm{d}y \quad \forall t\in[0,T]. \tag{5.17}$$

Proof. To prove this result we proceed as above by choosing a convenient test function ϕ. Pick two numbers $p>1$ and $\varepsilon>0$, and take

$$\phi(y,t)=\varphi(y,t)\rho(y)^\theta=\big(\big|\omega_\nu^R(y,t)\big|+\varepsilon\big)^{p-2}\omega_\nu^R(y,t)\rho(y)^\theta\psi_\delta(y).$$

We can easily check that $\phi\in L^2(0,T;H_0^1(\Omega))$ with

$$\nabla\phi=\big(\varepsilon+(p-1)\big|\omega_\nu^R\big|\big)\big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-3}\nabla\omega_\nu^R\rho(y)^\theta\psi_\delta(y)+\big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-2}\omega_\nu^R\nabla\big[\rho^\theta\psi_\delta\big](y).$$

Therefore, replacing φ by ϕ in (5.7), we obtain after some calculations that

$$\int_\Omega H_\varepsilon\big(\omega_\nu^R(y,t)\big)\rho(y)^\theta\psi_\delta(y)\,\mathrm{d}y - \int_\Omega H_\varepsilon\big(\omega_0(y)\big)\rho(y)^\theta\psi_\delta(y)\,\mathrm{d}y$$

$$+\nu\int_0^t\int_\Omega \big(\varepsilon+(p-1)\big|\omega_\nu^R\big|\big)\big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-3}\big|\nabla\omega_\nu^R\big|^2\rho(y)^\theta\psi_\delta(y)\,\mathrm{d}y\,\mathrm{d}s$$

$$+\nu\int_0^t\int_\Omega \big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-2}\omega_\nu^R\nabla\omega_\nu^R\cdot\nabla\big[\rho^\theta\psi_\delta\big](y)\,\mathrm{d}y\,\mathrm{d}s$$

$$+\int_0^t\int_\Omega \big(v_\nu^R-l_\nu^R-r_\nu^R y_R^\perp\big)\cdot\nabla\big[\rho^\theta\psi_\delta\big](y)G_\varepsilon\big(\omega_\nu^R\big)\,\mathrm{d}y\,\mathrm{d}s$$

$$-\int_0^t\int_\Omega \big(\big(v_\nu^R-l_\nu^R-r_\nu^R y_R^\perp\big)\cdot\nabla\big[\rho^\theta\psi_\delta\big](y)\big)\big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-2}\big(\omega_\nu^R\big)^2\,\mathrm{d}y\,\mathrm{d}s$$

$$-\int_0^t r_\nu^R\int_\Omega \big(D_R:\nabla v_\nu^R\big)\big(\big|\omega_\nu^R(y,t)\big|+\varepsilon\big)^{p-2}\omega_\nu^R(y,t)\rho(y)^\theta\psi_\delta(y)\,\mathrm{d}y\,\mathrm{d}s=0 \tag{5.18}$$

where H_ε is defined by (5.15) and G_ε is defined by

$$G_\varepsilon(a)=\int_0^a \sigma\big(\varepsilon+(p-1)|\sigma|\big)\big(|\sigma|+\varepsilon\big)^{p-3}\,\mathrm{d}\sigma=|a|^2\big(|a|+\varepsilon\big)^{p-2}-H_\varepsilon(a).$$

We shall use the following relations:

$$\nabla\big[\rho^\theta\psi_\delta\big](y)=\left\{\theta\frac{y}{\rho(y)^2}-\delta\frac{y}{\rho(y)}\right\}\rho(y)^\theta\psi_\delta(y), \tag{5.19}$$

$$\Delta\big[\rho^\theta\psi_\delta\big](y)=\left\{\frac{2\theta+\theta^2|y|^2}{\rho(y)^4}-\frac{\delta(2\theta-1)|y|^2}{\rho(y)^3}-\frac{2\delta}{\rho}+\frac{\delta^2|y|^2}{\rho(y)^2}\right\}\rho(y)^\theta\psi_\delta(y). \tag{5.20}$$

In the expression (5.18), some terms can be treated by using the following equalities:

$$\int_\Omega \big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-2}\omega_\nu^R\nabla\omega_\nu^R\cdot\nabla\big[\rho^\theta\psi_\delta\big](y)\,\mathrm{d}y=-\frac{1}{p}\int_\Omega\big(\big|\omega_\nu^R\big|+\varepsilon\big)^p\Delta\big[\rho^\theta\psi_\delta\big](y)\,\mathrm{d}y$$
$$+\frac{\varepsilon}{p-1}\int_\Omega\big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-1}\Delta\big[\rho^\theta\psi_\delta\big](y)\,\mathrm{d}y+\frac{\varepsilon^p}{p(p-1)}\int_{\partial S}\big(\delta\rho(y)-\theta\big)\rho(y)^{\theta-2}\psi_\delta(y)y\cdot n\,\mathrm{d}\Gamma, \tag{5.21}$$

$$\int_0^t\int_\Omega \big(v_\nu^R-l_\nu^R-r_\nu^R y_R^\perp\big)\cdot\nabla\big[\rho^\theta\psi_\delta\big]G_\varepsilon\big(\omega_\nu^R\big)\,\mathrm{d}y\,\mathrm{d}s=\int_0^t\int_\Omega\big(v_\nu^R-l_\nu^R\big)\cdot\nabla\big[\rho^\theta\psi_\delta\big]G_\varepsilon\big(\omega_\nu^R\big)\,\mathrm{d}y\,\mathrm{d}s \tag{5.22}$$

and

$$\int_\Omega \big(\big(v_\nu^R-l_\nu^R-r_\nu^R y_R^\perp\big)\cdot\nabla\big[\rho^\theta\psi_\delta\big]\big)\big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-2}\big(\omega_\nu^R\big)^2\,\mathrm{d}y$$
$$=\int_\Omega\big(\big(v_\nu^R-l_\nu^R\big)\cdot\nabla\big[|y|^\theta\psi_\delta\big]\big)\big(\big|\omega_\nu^R\big|+\varepsilon\big)^{p-2}\big(\omega_\nu^R\big)^2\,\mathrm{d}y. \tag{5.23}$$

Consequently, by passing to the limit $R\to\infty$, we obtain that

$$\int_\Omega H_\varepsilon\big(\omega_\nu(y,t)\big)\rho(y)^\theta\psi_\delta\,\mathrm{d}y \leqslant \int_\Omega H_\varepsilon\big(\omega_0(y)\big)\rho(y)^\theta\psi_\delta\,\mathrm{d}y + \left\{C_1+\nu\frac{C_2}{p}\right\}\int_0^t\int_\Omega\big(|\omega_\nu|+\varepsilon\big)^p\rho(y)^\theta\psi_\delta\,\mathrm{d}y\,\mathrm{d}s$$
$$+C_1\int_0^t\int_\Omega G_\varepsilon(\omega_\nu)\rho(y)^\theta\psi_\delta\,\mathrm{d}y\,\mathrm{d}s+\frac{\varepsilon\nu\,C_2}{p-1}\int_0^t\int_\Omega\big(|\omega_\nu|+\varepsilon\big)^{p-1}\rho(y)^\theta\psi_\delta\,\mathrm{d}y\,\mathrm{d}s$$
$$+\frac{\varepsilon^p}{p(p-1)}\int_{\partial S}\big|\big(\delta\rho(y)-\theta\big)\rho(y)^{\theta-2}\psi_\delta(y)y\cdot n\big|\,\mathrm{d}\Gamma, \tag{5.24}$$

where

$$C_1=(\theta+\delta)\big\|v_\nu(t)-l_\nu(t)\big\|_{L^\infty(Q_T)}\quad\text{and}\quad C_2=2\big(\theta^2+\theta+\delta^2\big).$$

The fact that $\|v_\nu-l_\nu\|_{L^\infty(Q_T)}$ remains bounded as $\nu\searrow 0$ readily follows from Propositions 2.2, 5.1 and 5.2. (See below Section 6.1.) Now, letting $\varepsilon\to 0^+$, we obtain thus for some constant $C_3>0$

$$\frac{1}{p}\int_\Omega\big|\omega_\nu(y,t)\big|^p\rho(y)^\theta\psi_\delta\,\mathrm{d}y\leqslant\frac{1}{p}\int_\Omega\big|\omega_0\big|^p\rho(y)^\theta\psi_\delta\,\mathrm{d}y+\left\{C_1+\nu\frac{C_2}{p}+C_3\right\}\int_0^t\int_\Omega|\omega_\nu|^p\rho(y)^\theta\psi_\delta\,\mathrm{d}y\,\mathrm{d}s. \tag{5.25}$$

Applying Gronwall's Lemma and then using the monotone convergence theorem in the limit $\delta\to 0$, we obtain that there exists a positive constant $C>0$ such that for all $p\in(1,2)$, $\nu\in(0,1)$ and $t\in[0,T]$

$$\int_\Omega\big|\omega_\nu(y,t)\big|^p\rho(y)^\theta\,\mathrm{d}y\leqslant \mathrm{e}^{Ct}\int_\Omega\big|\omega_0(y)\big|^p\rho(y)^\theta\,\mathrm{d}y.$$

Now, we have in the limit $p\to 1^+$

$$\int_\Omega\big|\omega_\nu(y,t)\big|\rho(y)^\theta\,\mathrm{d}y\leqslant \mathrm{e}^{Ct}\int_\Omega\big|\omega_0(y)\big|\rho(y)^\theta\,\mathrm{d}y$$

and the proof is complete. □

6. Proof of Theorem 1.1

6.1. Passage to the limit $\nu\to 0$

It follows from (5.5) that l_ν and r_ν are bounded in $L^\infty(0,T)$. On the other hand, the quantity $\int_{\partial S}v_\nu\cdot\tau\,\mathrm{d}\Gamma$ is also bounded in $L^\infty(0,T)$. Indeed, using Lemma 2.5 ($v_\nu\in L^2(\Omega)$ by (5.5)), we obtain

$$\int_{\partial S}v_\nu\cdot\tau\,\mathrm{d}\Gamma=-\int_\Omega\operatorname{curl}v_\nu\,\mathrm{d}y=-\int_\Omega\omega_\nu\,\mathrm{d}y$$

hence, by (5.6),

$$\left|\int_{\partial S}v_\nu\cdot\tau\,\mathrm{d}\Gamma\right|\leqslant\|\omega_\nu\|_{L^1(\Omega)}\leqslant\|\omega_0\|_{L^1(\Omega)}. \tag{6.1}$$

As ω_ν is bounded in $L^\infty(0,T;L^1(\Omega)\cap L^\infty(\Omega))$ by (5.6), it follows from Propositions 2.2 and 5.1 that v_ν is bounded in $L^\infty(Q_T)$ and in $L^\infty(0,T;W^{1,p}(\Omega))$ $\forall p\in[2,+\infty)$. Using in addition Proposition 5.3 with $\theta>2$, we see that $(y^\perp\cdot\nabla)v_\nu$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Therefore, we infer that for some sequence $\nu_k\searrow 0$ and some functions

$$v\in L^\infty\big(0,T;W^{1,p}(\Omega)\big)\quad\forall p\in[2,+\infty)\quad\text{with}$$
$$(y^\perp\cdot\nabla)v\in L^\infty\big(0,T;L^2(\Omega)\big),$$
$$\omega\in L^\infty\big(0,T;L^1_\theta(\Omega)\cap L^\infty(\Omega)\big),$$
$$l\in L^\infty(0,T)\quad\text{and}\quad r\in L^\infty(0,T),$$

we have that

$$\begin{aligned}
&v_{\nu_k} \rightharpoonup v \quad \text{in } L^\infty\big(0,T;W^{1,p}(\Omega)\big)\text{-weak}*,\ \forall p\in[2,+\infty),\\
&\big(y^\perp\cdot\nabla\big)v_{\nu_k} \rightharpoonup (y^\perp\cdot\nabla)v \quad \text{in } L^\infty\big(0,T;L^2(\Omega)\big)\text{-weak}*,\\
&\omega_{\nu_k} \rightharpoonup \omega \quad \text{in } L^\infty(Q_T)\text{-weak}*,\\
&l_{\nu_k} \rightharpoonup l \quad \text{in } L^\infty(0,T)\text{-weak}*,\\
&r_{\nu_k} \rightharpoonup r \quad \text{in } L^\infty(0,T)\text{-weak}*
\end{aligned}$$

as $k\to+\infty$. (To see that $\omega\in L^\infty(0,T;L^1_\theta(\Omega))$ it is sufficient to notice that $\|\omega_{\nu_k}\|_{L^\infty(0,T;L^q_\theta(\Omega))}\leqslant C$ for some constant $C>0$ independent of k and q, and to do $k\to\infty$ and next $q\to 1$. Here, $\|\omega\|^q_{L^q_\theta(\Omega)}:=\int_\Omega|\omega(y)|^q|y|^\theta\,\mathrm{d}y$.)

We now turn to the pointwise convergence of (v_{ν_k}). According to Proposition 4.3 (v'_ν) is bounded in $L^{3/2}(0,T;\mathcal{V}'_R)$. Pick any $p>2$. Observing that the first embedding in

$$W^{1,p}\big(B_R(0)\big)\cap\mathcal{H}_R\subset C\big(\overline{B_R(0)}\big)\cap\mathcal{H}_R\subset\mathcal{H}_R\subset\mathcal{V}'_R$$

is compact, we deduce from [31, Corollary 4] that $(v_\nu)_{\nu>0}$ is relatively compact in $C(\overline{\Omega_R}\times[0,T])$ for any $R>R_0$. Therefore, we obtain that

$$v\in\mathcal{B}(\overline{Q_T}) \tag{6.2}$$

and that v_{ν_k} converges to v uniformly on each compact subset of $\overline{\Omega}\times[0,T]$ as $k\to+\infty$.

We now aim to establish the uniform convergence of the sequences (l_{ν_k}) and (r_{ν_k}). The key point is that the correspondence which to $(l,r)\in\mathbb{R}^2\times\mathbb{R}$ associates the continuous map $y\mapsto(l+ry^\perp)\cdot n$ on ∂S is a one-to-one linear map (hence an *isomorphism* onto its image) when S is *not* a ball, as is showed by the next result.

Lemma 6.1. *Let S be a bounded simply connected domain which is of class C^1 and piecewise C^2, and which is different from a ball. Then the only couple $(l,r)\in\mathbb{R}^2\times\mathbb{R}$ for which $(l+ry^\perp)\cdot n=0$ for all $y\in\partial S$ is $(l,r)=(0,0)$.*

Proof of Lemma 6.1. Let $y=y(\sigma)$, $\sigma\in[0,\sigma_p]$, be an anticlockwise parametrization by arc length of ∂S, y being of class C^2 on each interval $[\sigma_i,\sigma_{i+1}]$ where $0=\sigma_0<\sigma_1<\cdots<\sigma_p$ denotes an appropriate subdivision of $[0,\sigma_p]$. Thus $\tau=\mathrm{d}y/\mathrm{d}\sigma$ on $[0,\sigma_p]$ and $\mathrm{d}\tau/\mathrm{d}\sigma=\kappa n$ on each interval (σ_i,σ_{i+1}), where $n=\tau^\perp$ denotes the unit outer normal vector to $\partial(\mathbb{R}^2\setminus S)$ and κ denotes the curvature of ∂S, which may assume *nonpositive* values. Derivating with respect to $\sigma\in(\sigma_i,\sigma_{i+1})$ in

$$\big(l+ry^\perp\big)\cdot n=0 \tag{6.3}$$

yields

$$r-\big(l+ry^\perp\big)\cdot\kappa\tau=0. \tag{6.4}$$

If $(l+ry^\perp)\cdot\tau=0$ for some $\sigma\in[\sigma_i,\sigma_{i+1}]$ $(0\leqslant i\leqslant p-1)$, then $r=0$ and $l\cdot n=0$ on $[0,\sigma_p]$, which gives $l=0$. If $(l+ry^\perp)\cdot\tau\neq 0$ for each $\sigma\in[0,\sigma_p]$. then $\kappa=r/((l+ry^\perp)\cdot\tau)$ is continuous and piecewise C^1, hence $y=y(\sigma)$ is of class C^2 on $[0,\sigma_p]$. A second derivation gives then

$$\big(l+ry^\perp\big)\cdot\left(\frac{\mathrm{d}\kappa}{\mathrm{d}\sigma}\tau+\kappa^2 n\right)+rn\cdot\kappa\tau=0. \tag{6.5}$$

Combining (6.3), (6.4) and (6.5) we obtain that

$$r\frac{\mathrm{d}\kappa}{\mathrm{d}\sigma}=0 \quad \text{on } (\sigma_i,\sigma_{i+1}).$$

If $r\neq 0$, then κ has to be constant on each interval (σ_i,σ_{i+1}), and also on $[0,\sigma_p]$ by continuity. This implies that S is a ball, contradicting the hypotheses. Therefore $r=0$, and we conclude as before that $l=0$. □

Modifying (if necessary) l_{ν_k} and r_{ν_k} on a zero measure set, we infer from (4.9), Lemma 6.1 and from the continuity of v_{ν_k} on $\partial S\times[0,T]$ that l_{ν_k} and r_{ν_k} are both continuous on $[0,T]$. Furthermore, using the uniform convergence of

v_{ν_k} to v on $\partial S \times [0,T]$, we infer that l_{ν_k} (resp., r_{ν_k}) converges uniformly to l (resp., r) on $[0,T]$, hence $l \in C([0,T])$ (resp., $r \in C([0,T])$).

It is then easy to see that the sequence (v'_{ν_k}) is bounded in $L^2(0,T;\mathcal{V}'_R)$ and that $v'_{\nu_k} \rightharpoonup v'$ in $L^2(0,T;\mathcal{V}'_R)$ for any $R > R_0$.

Therefore, taking the limit $k \to \infty$ in (4.13) yields

$$\int_0^T \left\{ \langle v', \varphi\rangle_R + \int_\Omega \left(\left(v - l - ry^\perp\right) \cdot \nabla v + r v^\perp \right) \cdot \varphi \, \mathrm{d}y \right\} \mathrm{d}t = 0 \tag{6.6}$$

for all $\varphi \in L^2(0,T;\mathcal{V}_R)$. As

$$\left| \int_0^T \langle v', \varphi\rangle_R \, \mathrm{d}t \right| \leqslant C \|\varphi\|_{L^2(0,T;\mathcal{V})}$$

for some constant $C > 0$ independent of φ and R, we see that $v' \in L^2(0,T;\mathcal{V}')$ ($\mathcal{V}'$ denoting the dual space of $\mathcal{V}$ with respect to the pivot space $\mathcal{H}$), and that

$$\int_0^T \left\{ \langle v', \varphi\rangle + \int_\Omega \left(\left(v - l - ry^\perp\right) \cdot \nabla v + r v^\perp \right) \cdot \varphi \, \mathrm{d}y \right\} \mathrm{d}t = 0 \quad \forall \varphi \in L^2(0,T;\mathcal{V}), \tag{6.7}$$

where $\langle \cdot,\cdot \rangle$ denotes the duality pairing between $\mathcal{V}$ and $\mathcal{V}'$. Obviously, (1.11), (1.12), (1.15) and (1.16) hold true. On the other hand $[\mathcal{V},\mathcal{V}']_{1/2} = \mathcal{H}$, hence by a classical result in [24]

$$v \in C\big([0,T],\mathcal{H}\big), \tag{6.8}$$

and we infer from Hölder inequality and (6.2) that

$$v \in C\big([0,T], L^p(\Omega)\big) \quad \forall p \in [2,+\infty).$$

In particular, it follows from [36, Lemma 1.4] that

$$v \in C_w\big([0,T], W^{1,p}(\Omega)\big) \quad \forall p \in [2,+\infty)$$

and that

$$\lim_{|y| \to +\infty} v(y,t) = 0 \quad \forall t \in [0,T].$$

We now turn to the equation satisfied by ω. Using (4.8), (5.2) may be rewritten as

$$\omega'_{\nu_k} + \operatorname{div}\big(\big(v_{\nu_k} - l_{\nu_k} - r_{\nu_k} y^\perp\big)\omega_{\nu_k}\big) - \nu_k \Delta \omega_{\nu_k} = 0. \tag{6.9}$$

Clearly, $(v_{\nu_k} - l_{\nu_k} - r_{\nu_k} y^\perp)\omega_{\nu_k} \rightharpoonup (v - l - ry^\perp)\omega$ in $L^\infty(\Omega_R \times (0,T))$-weak$*$ for any R, hence, letting $k \to +\infty$ in (6.9) we obtain

$$\omega' + \operatorname{div}\big(\big(v - l - ry^\perp\big)\omega\big) = 0 \quad \text{in } \mathcal{D}'(Q_T). \tag{6.10}$$

Finally, passing to the limit in (5.1), (6.1) we get

$$\omega = \operatorname{curl} v, \tag{6.11}$$

$$\left| \int_{\partial S} v \cdot \tau \, \mathrm{d}\Gamma \right| \leqslant \|\omega_0\|_{L^1(\Omega)}. \tag{6.12}$$

6.2. Existence of a classical solution of (1.10)–(1.16)

In this section we prove that all the equations in (1.10)–(1.16) are satisfied in the classical sense. More precisely, we prove that v, ∇v, and v_t belong to $\mathcal{B}(\overline{Q_T})$ and that $\nabla q \in C(\overline{Q_T})$. We begin with the following result.

Lemma 6.2. *There exists a constant $H > 0$ such that for all $y, z \in \overline{\Omega}$ and all $t \in [0, T]$*

$$\big|v(y,t) - v(z,t)\big| \leqslant H|y - z|\chi\big(|y - z|\big), \tag{6.13}$$

where

$$\chi(r) := \begin{cases} 1 & \text{for } r \geqslant 1, \\ 1 + \ln(1/r) & \text{for } 0 < r < 1. \end{cases}$$

Proof. Applying Lemma 2.3 with $l = l(t)$, $r = r(t)$, we may write

$$v(y,t) = d_1(y,t) + v_1(y,t),$$

where $d_1 \in C([0,T], W^{1,\infty}(\Omega))$ satisfies $d_1(y,t) = l(t) + r(t)y^\perp$ for $|y| \leqslant R_0$, $d_1(y,t) = 0$ for $|y| \geqslant R_0 + 1$, and v_1 fulfils $\operatorname{curl} v_1 = \omega_1 := \omega - \operatorname{curl} d_1$ and $\operatorname{div} v_1 = 0$ in Ω, $v_1 \cdot n = 0$ on ∂S, $\lim_{|y|\to+\infty} v_1(y,t) = 0$, and

$$\int_{\partial S} v_1 \cdot \tau \, \mathrm{d}\Gamma = \int_{\partial S} v \cdot \tau \, \mathrm{d}\Gamma - r(t) \int_{\partial S} y^\perp \cdot \tau \, \mathrm{d}\Gamma := C_1(t).$$

(Note that the function $C_1(t)$ is continuous.) Then, by virtue of [23, Lemma 2.14],

$$v_1(y,t) = \operatorname{curl} G(\omega_1)(y,t) + \lambda_1(t)u_1(y),$$

where $\operatorname{curl} G(\omega_1)(y,t) = (2\pi)^{-1} \int_\Omega \operatorname{curl} G(y,z)\omega_1(z,t)\,\mathrm{d}z$, $G(y,z)$ denoting the Green function for the exterior zero-Dirichlet problem, $\lambda_1(t) = C_1(t) - \int_{\partial S} \operatorname{curl} G(\omega_1) \cdot \tau \, \mathrm{d}\Gamma$, and $u_1 \in W^{1,\infty}(\Omega)$ is some irrotational and solenoidal flow satisfying $u_1 \cdot n = 0$ on ∂S, $\int_{\partial S} u_1 \cdot \tau \, \mathrm{d}\Gamma = 1$ and $u_1(y) \to 0$ as $|y| \to +\infty$. (See [23, Lemma 1.5] for the existence of the vector field u_1.) Then, by virtue of [23, Lemma 2.4], v_1 and v satisfy (6.13). □

Remark 6.3. In [23], λ_1 takes the following form

$$\lambda_1(t) = \int_{\partial B} v_1(y,0) \cdot \tau \, \mathrm{d}\Gamma - \int_{\partial B} \operatorname{curl} G(\omega_1) \cdot \tau \, \mathrm{d}\Gamma.$$

The result in [23, Lemma 2.4] remains nevertheless valid with this new definition of $\lambda_1(t)$.

The vector field v being quasi-Lipschitz (see (6.13)), it follows from Osgood's criterion (see e.g. [17, Corollary 6.2]) that the Cauchy problem

$$\begin{cases} \dfrac{\mathrm{d}y}{\mathrm{d}t} = v(y,t) - l(t) - r(t)y^\perp, \\ y(t_0) = y_0 \end{cases}$$

has a *unique* solution $y(t)$. We may therefore define the flow associated with $v - l - ry^\perp$ as the solution of the following system

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}s} U_{s,t}(y) = v\big(U_{s,t}(y), s\big) - l(s) - r(s)U_{s,t}(y)^\perp, \\ U_{t,t}(y) = y. \end{cases}$$

As $v \in \mathcal{B}(\overline{Q_T})$, $l \in C([0,T])$, $r \in C([0,T])$, and $(v - l - ry^\perp) \cdot n = 0$ on $\partial S \times [0,T]$, we see that $U_{s,t}(y)$ is defined for all $(s,t,y) \in [0,T]^2 \times \overline{\Omega}$. The following result comes from [22].

Lemma 6.4. *There exist two constants $\delta > 0$ and $L > 0$ such that*

$$\big|U_{s,t}(y) - U_{\bar{s},\bar{t}}(\bar{y})\big| \leqslant L\big(|s - \bar{s}|^\delta + |t - \bar{t}|^\delta + |y - \bar{y}|^\delta\big) \quad \forall s, \bar{s}, t, \bar{t} \in [0,T],\ \forall y, \bar{y} \in \overline{\Omega}.$$

The following uniqueness result is similar to a result given in [25, Proof of Theorem 2.5]. Its proof is left to the reader.

Lemma 6.5. *Let $v \in L^\infty(Q_T)$ be such that* $\operatorname{div} v = 0$ *on $\Omega \times (0,T)$ and $v \cdot n = 0$ on $\partial S \times (0,T)$, and let $\omega \in L^\infty(0,T;L^1(\Omega) \cap L^\infty(\Omega))$ be a solution of*

$$\frac{\partial \omega}{\partial t} + \operatorname{div}(v\omega) = 0 \quad \text{in } \mathcal{D}'\big(\Omega \times (0,T)\big) \tag{6.14}$$

such that $\omega|_{t=0} = 0$. Then $\omega \equiv 0$.

Let $\overline{\omega}(y,t) := \omega_0(U_{0,t}(y))$, where $\omega_0 := \operatorname{curl} a$. As $\operatorname{div}(v - l - ry^\perp) = 0$, we infer as in [23, pp. 70–71] that for all $t \in [0,T]$

$$\int_\Omega \big|\overline{\omega}(y,t)\big| \,\mathrm{d}y = \int_\Omega \big|\omega_0(y)\big| \,\mathrm{d}y,$$

hence $\overline{\omega} \in L^\infty(0,T;L^1(\Omega) \cap L^\infty(\Omega))$. It follows from Lemma 6.5 (applied to $\omega - \overline{\omega}$ and $v - l - ry^\perp$) that

$$\omega(y,t) = \overline{\omega}(y,t) = \omega_0\big(U_{0,t}(y)\big).$$

Using once again the fact that the Lebesgue measure is preserved by $U_{s,t}(y)$, one may show that

$$\int_\Omega \omega(y,t) \,\mathrm{d}y = \int_\Omega \omega_0(y) \,\mathrm{d}y,$$

hence

$$C(t) = \int_{\partial S} v \cdot \tau \,\mathrm{d}\Gamma = -\int_\Omega \omega_0(y) \,\mathrm{d}y = Const.$$

On the other hand, we infer from Lemma 6.4 that

$$\omega \in C^{\delta\lambda,0}\big(\overline{Q_T}\big). \tag{6.15}$$

Then we derive the following result.

Lemma 6.6. $\frac{\partial v}{\partial y_j} \in \mathcal{B}(\overline{Q_T})$ *for $j = 1,2$.*

The proof is virtually the same as the one for [23, Lemma 2.10].

The following result contains the fact that $\nabla v \in C([0,T], L^2(\Omega))$, which will be used later when proving that $v' = \mathrm{d}v/\mathrm{d}t \in \mathcal{B}(\overline{Q_T})$.

Lemma 6.7. *$v \in C([0,T];L^p(\Omega))$ for any $p \in [2,+\infty]$, $\nabla v \in C([0,T];L^p(\Omega))$ for any $p \in (1,+\infty)$, and $(y^\perp \cdot \nabla)v \in C([0,T];L^2(\Omega))$.*

Proof. We need the following

Claim. *$\omega \in C([0,T], L^1_{\theta'}(\Omega) \cap L^\infty(\Omega))$ for any $\theta' \in (2,\theta)$.*

Pick any number $\theta' \in (2,\theta)$. We readily infer from (5.17) and (6.15) that $\omega \in C([0,T];L^1_{\theta'}(\Omega))$. On the other hand, ω_0 is Hölder continuous on $\overline{\Omega}$ by assumption, and we infer from Lemma 6.4 that

$$\big|U_{0,t}(y) - U_{0,t'}(y)\big| \leqslant L|t-t'|^\delta \quad \forall y \in \overline{\Omega},\ \forall t,t' \in [0,T].$$

Thus, the vorticity $\omega(y,t) = \omega_0(U_{0,t}(y))$ belongs to the space $C([0,T], L^\infty(\Omega))$. The claim is proved.

The proof of the lemma is completed by using Proposition 2.2, (6.8), the claim, and the fact that $l \in C([0,T])$, $r \in C([0,T])$, and $\int_{\partial B} v \cdot \tau \, d\Gamma = Const.$ □

Lemma 6.8. $v' \in \mathcal{B}(\overline{Q_T}) \cap C([0,T], L^2(\Omega))$.

Proof. Let f denote the function $((v - l - ry^{\perp}) \cdot \nabla)v + rv^{\perp}$ extended by 0 on S, and let $\mathbb{P}$ denote the orthogonal projector from $L^2(\mathbb{R}^2)$ (endowed with the $(\cdot,\cdot)_\gamma$ scalar product) onto $\mathcal{H}$. We infer from Lemma 6.7 that $f \in C([0,T]; L^2(\mathbb{R}^2))$, hence $\mathbb{P}f \in C([0,T];\mathcal{H})$. It follows that

$$\int_\Omega \left(\left(v - l - ry^{\perp}\right) \cdot \nabla v + rv^{\perp}\right) \cdot \varphi \, \mathrm{d}y = (f,\varphi)_\gamma = (\mathbb{P}f,\varphi)_\gamma = \langle \mathbb{P}f,\varphi\rangle \quad \forall \varphi \in \mathcal{V}.$$

Thus (6.7) may be rewritten

$$\int_0^T \left\langle v' + \mathbb{P}f, \varphi\right\rangle \mathrm{d}t = 0 \quad \forall \varphi \in L^2(0,T;\mathcal{V}),$$

which implies that

$$v' + \mathbb{P}f = 0 \quad \text{in } L^2(0,T;\mathcal{V}').$$

Thus $v' \in C([0,T],\mathcal{H})$, $l' = l_{v'} \in C([0,T])$ and $r' = r_{v'} \in C([0,T])$. We now decompose v as

$$v(y,t) = v_2(y,t) + d_1(y,t) + C_1 d_2(y), \tag{6.16}$$

where v_2 solves (2.20)–(2.24) (with $\omega_2(y,t) = \omega(y,t) - \operatorname{curl} d_1(y,t) - C_1 \operatorname{curl} d_2(y)$), d_1 (resp. d_2) is given by Lemma 2.3 (resp. Lemma 2.4), and $C_1 = \int_{\partial S} a \cdot \tau \, \mathrm{d}\Gamma - r \int_{\partial S} y^{\perp} \cdot \tau \, \mathrm{d}\Gamma$. Derivating with respect to time in (6.16), we obtain

$$v' = v_2' + d_1' - \left(r' \int_{\partial S} y^{\perp} \cdot \tau \, \mathrm{d}\Gamma\right) d_2(y).$$

As $l' \in C([0,T])$ and $r' \in C([0,T])$, $d_1' \in \mathcal{B}(\overline{Q_T})$. The fact that $v_2' \in \mathcal{B}(\overline{Q_T})$ may be found in [23, Proof of Lemma 2.11]. Therefore, $v' \in \mathcal{B}(\overline{Q_T})$. □

Corollary 6.9. $v \in C^1([0,T], L^2(\Omega)) \cap C([0,T], H^1(\Omega))$.

Proof. This is a direct consequence of Lemmas 6.7 and 6.8. □

It remains to prove the existence of a pressure $q(y,t)$ satisfying (1.10) and (1.13)–(1.14) in a classical sense. As $v' \in C([0,T],\mathcal{H})$, we infer from (6.7) that for every $t \in [0,T]$

$$ml' \cdot l_\phi + Jr' r_\phi + \int_\Omega \left(v' + \left(v - l - ry^{\perp}\right) \cdot \nabla v + rv^{\perp}\right) \cdot \phi \, \mathrm{d}y = 0 \quad \forall \phi \in \mathcal{V}. \tag{6.17}$$

In particular,

$$\int_\Omega \left(v' + \left(v - l - ry^{\perp}\right) \cdot \nabla v + rv^{\perp}\right) \cdot \phi \, \mathrm{d}y = 0 \quad \forall \phi \in C_0^\infty(\Omega) \text{ with } \operatorname{div}\phi = 0.$$

By [36, Propositions 1.1 and 1.2], there exists a function $q \in L^2(0,T;\widehat{H}^1(\Omega))$ such that for a.e. $t \in (0,T)$

$$v' + \left(v - l - ry^{\perp}\right) \cdot \nabla v + rv^{\perp} + \nabla q = 0. \tag{6.18}$$

As v', $(v - l - ry^{\perp}) \cdot \nabla v$ and $rv^{\perp}$ belong to $C(\overline{Q_T}) \cap C([0,T]; L^2(\Omega))$, adding a function of time to q if necessary we see that $q \in C([0,T];\widehat{H}^1(\Omega))$ and $\nabla q \in C(\overline{Q_T})$. Picking any $\phi \in \mathcal{V}$, we infer from (6.17)–(6.18) and the divergence formula that

$$ml' \cdot l_\phi + Jr' r_\phi = \int_\Omega \nabla q \cdot \phi \, \mathrm{d}y = \int_\Omega \operatorname{div}(q\phi)\, \mathrm{d}y = \int_{\partial S} qn \cdot \phi \, \mathrm{d}\Gamma.$$

Therefore (1.13) and (1.14) hold true.

6.3. Uniqueness of the solution

Finally, we prove the uniqueness of a classical solution to the problem (1.10)–(1.16).

Assume given two solutions (v^1, q^1, l^1, r^1) and (v^2, q^2, l^2, r^2) of (1.10)–(1.16) with the regularity depicted in Theorem 1.1. We introduce the functions

$$v = v^1 - v^2, \quad q = q^1 - q^2, \quad r = r^1 - r^2, \quad l = l^1 - l^2, \tag{6.19}$$

which fulfil the following system

$$\frac{\partial v}{\partial t} + \big(\big(v^1 - l^1 - r^1 y^{\perp}\big) \cdot \nabla\big)v + \big(\big(v - l - r y^{\perp}\big) \cdot \nabla\big)v^2 + r^1 v^{\perp} + r v^{2\perp} + \nabla q = 0 \quad \text{in } \Omega \times [0, T], \tag{6.20}$$

$$\operatorname{div} v = 0 \quad \text{in } \Omega \times [0, T], \tag{6.21}$$

$$v \cdot n = \big(l + r y^{\perp}\big) \cdot n \quad \text{on } \partial S \times [0, T], \tag{6.22}$$

$$m l' = \int_{\partial S} q n \, \mathrm{d}\Gamma - m\big(r^1 l^{\perp} + r l^{2\perp}\big) \quad \text{in } [0, T], \tag{6.23}$$

$$J r' = \int_{\partial S} q n \cdot y^{\perp} \, \mathrm{d}\Gamma \quad \text{in } [0, T], \tag{6.24}$$

$$v(y, 0) = 0 \quad \forall y \in \Omega, \tag{6.25}$$

$$l(0) = 0 \quad r(0) = 0. \tag{6.26}$$

In order to prove that $(v, l, r) = (0, 0, 0)$, we establish some energy estimate for (6.20)–(6.26).

Multiplying (6.20) by v and integrating over $\Omega \times (0, t)$, we obtain that

$$0 = \int_0^t \int_\Omega v_t \cdot v \, \mathrm{d}y \, \mathrm{d}s + \int_0^t \int_\Omega \big(\big(v^1 - l^1 - r^1 y^{\perp}\big) \cdot \nabla\big)v \cdot v \, \mathrm{d}y \, \mathrm{d}s + \int_0^t \int_\Omega \big(\big(v - l - r y^{\perp}\big) \cdot \nabla\big)v^2 \cdot v \, \mathrm{d}y \, \mathrm{d}s$$
$$+ \int_0^t \int_\Omega r v^{2\perp} \cdot v \, \mathrm{d}y \, \mathrm{d}s + \int_0^t \int_\Omega \nabla q \cdot v \, \mathrm{d}y \, \mathrm{d}s = I_1 + I_2 + I_3 + I_4 + I_5.$$

We now study each integral term. We easily have that

$$I_1 = \frac{1}{2} \int_\Omega \big|v(t)\big|^2 \, \mathrm{d}y.$$

Next, some integrations by part give that

$$I_2 = 0. \tag{6.27}$$

On the other hand, we have that

$$I_3 = \int_0^t \int_\Omega (v \cdot \nabla)v^2 \cdot v \, \mathrm{d}y \, \mathrm{d}s - \int_0^t \int_\Omega (l \cdot \nabla)v^2 \cdot v \, \mathrm{d}y \, \mathrm{d}s - \int_0^t \int_\Omega \big(r y^{\perp} \cdot \nabla\big)v^2 \cdot v \, \mathrm{d}y \, \mathrm{d}s = I_{31} + I_{32} + I_{33}.$$

We can estimate each part:

$$|I_{31}| \leqslant \|\nabla v^2\|_{L^\infty(Q_T)} \int_0^t \int_\Omega |v|^2 \, \mathrm{d}y \, \mathrm{d}s,$$

$$|I_{32}| \leqslant \int_0^t \big|l(s)\big| \bigg(\int_\Omega \big|\nabla v^2\big|^2 \, \mathrm{d}y\bigg)^{1/2} \bigg(\int_\Omega |v|^2 \, \mathrm{d}y\bigg)^{1/2} \mathrm{d}s \leqslant \frac{1}{2} \big\|v^2\big\|_{L^\infty(0,T; H^1(\Omega))} \Bigg[\int_0^t \bigg(\int_\Omega |v|^2 \, \mathrm{d}y + |l|^2\bigg) \mathrm{d}s\Bigg]$$

and

$$|I_{33}| \leqslant \int_0^t |r(s)| \left(\int_\Omega |y|^2 |\nabla v^2|^2 \, \mathrm{d}y \right)^{1/2} \left(\int_\Omega |v|^2 \, \mathrm{d}y \right)^{1/2} \mathrm{d}s$$
$$\leqslant \frac{1}{2} \| |y| \nabla v^2 \|_{L^\infty(0,T;L^2(\Omega))} \left[\int_0^t \left(\int_\Omega |v|^2 \, \mathrm{d}y + |r|^2 \right) \mathrm{d}s \right].$$

On the other hand,

$$|I_4| \leqslant \int_0^t |r(s)| \left(\int_\Omega |v^2|^2 \, \mathrm{d}y \right)^{1/2} \left(\int_\Omega |v|^2 \, \mathrm{d}y \right)^{1/2} \mathrm{d}s \leqslant \frac{1}{2} \|v^2\|_{L^\infty(0,T;L^2(\Omega))} \left[\int_0^t \left(\int_\Omega |v|^2 \, \mathrm{d}y + |r|^2 \right) \mathrm{d}s \right].$$

Finally we have that

$$I_5 = \int_0^t \int_{\partial S} q(l + r y^\perp) \cdot n \, \mathrm{d}\Gamma \, \mathrm{d}s = \frac{m}{2} |l(t)|^2 + \frac{J}{2} |r(t)|^2 + m \int_0^t l \cdot (r l^{2\perp}) \, \mathrm{d}s = I_{51} + I_{52} + I_{53}$$

with

$$|I_{53}| \leqslant \frac{m}{2} \|l^2\|_{L^\infty(0,T)} \int_0^t (|l|^2 + |r|^2) \, \mathrm{d}s.$$

Thus, we have that

$$\int_\Omega |v(t)|^2 \, \mathrm{d}y + m |l(t)|^2 + J |r(t)|^2 \leqslant C \left[\int_0^t \left(\int_\Omega |v|^2 \, \mathrm{d}y + m|l|^2 + J|r|^2 \right) \mathrm{d}s \right]$$

and by Gronwall's Lemma, we obtain that

$$v = 0 \quad \text{in } \Omega \times (0,T) \quad \text{and} \quad (l,r) = (0,0) \quad \text{in } (0,T).$$

Using (6.20) we conclude that $\nabla q = 0$ in $\Omega \times (0,T)$, which proves that the solution of our problem is unique (up to an arbitrary function of t for q). The proof of Theorem 1.1 is achieved. □

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