Discontinuous Galerkin Approximation for Advection-Diffusion-Reaction problems

Blanca Ayuso

Universidad Autónoma de Madrid

Joint work with L. Donatella Marini
Università degli Studi di Pavia & IMATI-CNR

Benasque, 29 August 2007

A bit of History

Introduced for purely hyperbolic problems (Reed-Hill 70's, Lesaint-Raviart)

Used for second order elliptic (Douglas-Dupont school, mid 70's) and for fourth order (Baker).

Abandoned because of the big size of the final system.

Great revival some 10 years ago (mainly by Cockburn-Shu) also for applications to problems where the elliptic part is present but it is not dominant.

(example: strongly advection-dominated equations, very thin Reissner-Mindlin plates)

Advection-Diffusion-Reaction Problems

$\Omega \subset \mathbb{R}^d$ $(d = 2, 3)$ bounded convex (polygonal or polyhedral)
Let $f \in L^2(\Omega)$, $g \in H^{3/2}(\Gamma)$. Consider the problem

$$\begin{aligned} \mathrm{div}(-\varepsilon \nabla u + \beta u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

Goal: Numerical Schemes for $\varepsilon \downarrow 0$

- *High order Accurate*
- *Stable*
- *No Problem Dependent*

Plain Conforming Approximation Fails!

Introduction & Motivation

DG for strongly advection-dominated equations

- *Reed-Hill, Lesaint-Raviart (70's)*
- *Johnson & Pitkäranta, Nävert (80's): DG+SUPG*
- *Johnson & Pitkäranta (87): DG*
- *Cockburn & Shu (90's)*
- . . . .
- *Houston Schwab & Süli (SINUM 2002)*
- *Kanschat & Gopalakrishnan (Numer. Matem. 2005)*
- *Hughes, Scovazzi, Bochev & Buffa (CMAME 2006)*

Aim:

- *Identify the stabilization procedures?*
- *Characterization & validation of methods?*
- *Analysis for variable coefficients case*

Introduction & Motivation

DG for strongly advection-dominated equations

- *Reed-Hill, Lesaint-Raviart (70's)*
- *Johnson & Pitkäranta, Nävert (80's): DG+SUPG*
- *Johnson & Pitkäranta (87): DG*
- *Cockburn & Shu (90's)*
- . . . .
- *Houston Schwab & Süli (SINUM 2002)*
- *Kanschat & Gopalakrishnan (Numer. Matem. 2005)*
- *Hughes, Scovazzi, Bochev & Buffa (CMAME 2006)*

Aim:

- *Identify the stabilization procedures?*
- *Characterization & validation of methods?*
- *Analysis for variable coefficients case*

Introduction & Motivation

DG for strongly advection-dominated equations

- *Reed-Hill, Lesaint-Raviart (70's)*
- *Johnson & Pitkäranta, Nävert (80's): DG+SUPG*
- *Johnson & Pitkäranta (87): DG*
- *Cockburn & Shu (90's)*
- ....
- *Houston Schwab & Süli (SINUM 2002)*
- *Kanschat & Gopalakrishnan (Numer. Matem. 2005)*
- *Hughes, Scovazzi, Bochev & Buffa (CMAME 2006)*

Aim:

- *Identify the stabilization procedures?*
- *Characterization & validation of methods?*
- *Analysis for variable coefficients case*

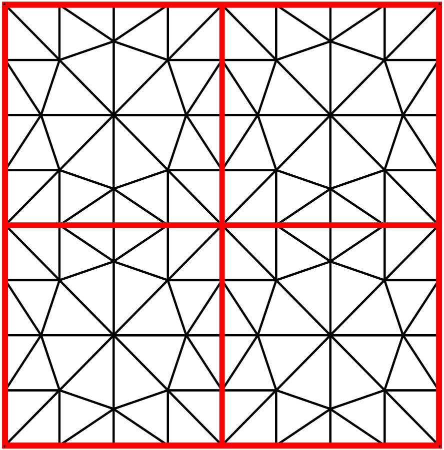
Outline

1. Introduction & Motivation of DG Methods
2. DG Methods as Weighted Residuals
3. Advection-Diffusion-Reaction Problems
4. Analysis
5. Numerical Results

What are Discontinuos Galerkin Methods?

- *Finite Element (FE) Method for Approximating PDE*

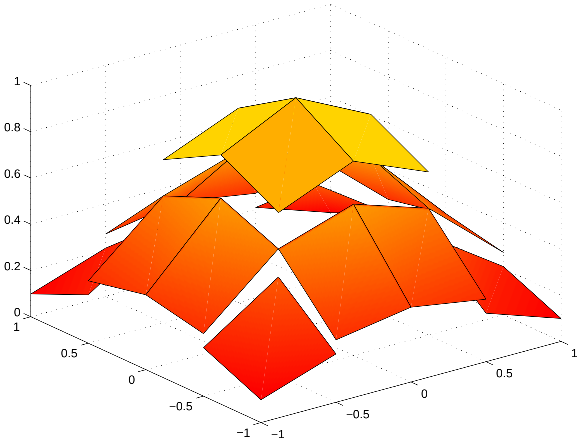
- $\mathscr{T}_h$ FE partition of Ω into elements K (triangles or tetrahedron)



What are Discontinuos Galerkin Methods?

- *Finite Element (FE) Method for Approximating PDE*

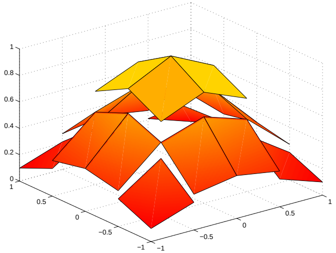
IDEA: piecewise polynomial DISCONTINUOS functions $(V_h \not\subset V)$



Why DG Approximation?

IDEA: piecewise polynomial DISCONTINUOS functions $(V_h \not\subset V)$

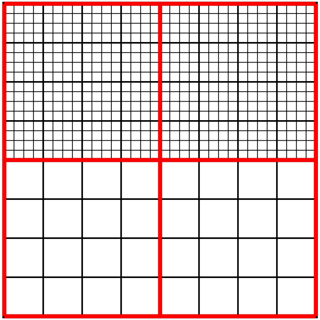
- √ Wide Range of PDE's
- √ Weak approximation of b.c.
- √ Non-matching-grids
- √ Adaptivity: *hp*-strategies
- √ Flexible choice of approximation spaces
- √ Conservativity



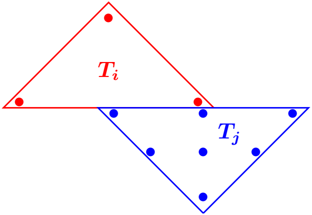
- X more of degrees of freedom (d.o.f.)!
- X Systems of equations difficult to solve
- X Techniques under development
- X Stabilization mechanisms?

Why DG Approximation?

- √ Wide Range of PDE's
- √ Weak approximation of b.c.
- √ Non-matching-grids
- √ Adaptivity: *hp*-strategies
- √ Flexible choice of approximation spaces
- √ Conservativity

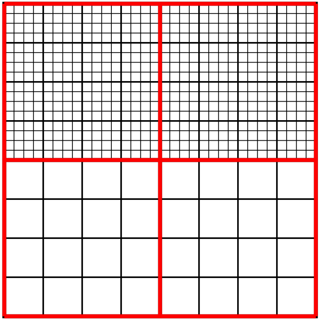


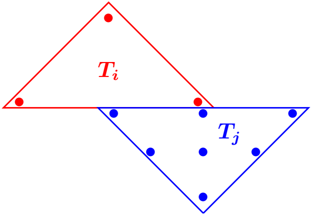
- X more of degrees of freedom (d.o.f.)!
- X Systems of equations difficult to solve
- X Techniques under development
- X Stabilization mechanisms?



Why DG Approximation?

- √ Wide Range of PDE's
- √ Weak approximation of b.c.
- √ Non-matching-grids
- √ Adaptivity: *hp*-strategies
- √ Flexible choice of approximation spaces
- √ Conservativity

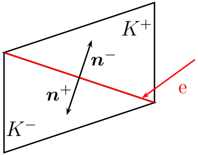




- X more of degrees of freedom (d.o.f.)!
- X Systems of equations difficult to solve
- X Techniques under development
- X Stabilization mechanisms?

Averages & Jumps

$\mathscr{T}_h$: partition of Ω into triangles/tetrahedra K



$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

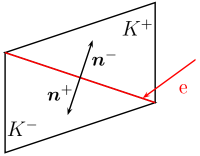
On boundary edges $[\![v]\!] = v\mathbf{n}$; $\quad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

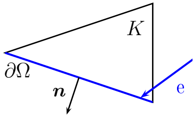
Crucial Formula

$$\sum_{K \in \mathscr{T}_h} \int_{\partial K} v \boldsymbol{\tau} \cdot \mathbf{n}_T = \sum_{e \in \mathcal{E}_h} \int_e [\![v]\!] \cdot \{\boldsymbol{\tau}\} + \sum_{e \in \mathcal{E}_h^\circ} \int_e \{v\} [\![\boldsymbol{\tau}]\!]$$

Averages & Jumps

$\mathscr{T}_h$: partition of Ω into triangles/tetrahedra K





$$\{v\} = \frac{v^+ + v^-}{2}; \qquad [\![v]\!] = v^+\mathbf{n}^+ + v^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

$$\{\boldsymbol{\tau}\} = \frac{\boldsymbol{\tau}^+ + \boldsymbol{\tau}^-}{2}; \qquad [\![\boldsymbol{\tau}]\!] = \boldsymbol{\tau}^+\mathbf{n}^+ + \boldsymbol{\tau}^-\mathbf{n}^- \quad \forall e \in \mathcal{E}_h^\circ \equiv \text{internal edges}$$

On boundary edges $[\![v]\!] = v\mathbf{n}$; $\qquad \{\boldsymbol{\tau}\} = \boldsymbol{\tau}$

Crucial Formula

$$\sum_{K\in\mathscr{T}_h} \int_{\partial K} v\boldsymbol{\tau}\cdot\mathbf{n}_T = \sum_{e\in\mathcal{E}_h} \int_e [\![v]\!]\cdot\{\boldsymbol{\tau}\} + \sum_{e\in\mathcal{E}_h^\circ} \int_e \{v\}\,[\![\boldsymbol{\tau}]\!]$$

DG Methods as Weighted Residuals (Brezzi-Cockburn-Marini-Süli, CMAME 2006)

A is a second order elliptic operator

$$Au \equiv -\mathrm{div}(\kappa \nabla u) = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

- $\sigma = -\kappa \nabla u \;$ in Ω,
- $\nabla \cdot \sigma = f \;$ in Ω,
- $u = 0 \;$ on $\partial\Omega$.

Define $\quad H^2(\mathscr{T}_h) := \{v \in L^2(\Omega) : \; v_{|K} \in H^2(K) \; \forall K \in \mathscr{T}_h\}$.

If you allow, a priori, your solution to be discontinuous $(u \in H^2(\mathscr{T}_h))$, then the equations to be required are:

- $A\,u - f \; = \; 0$ in each element
- $[\![u]\!] = 0$ on each edge
- $[\![\sigma]\!] = 0$ on each internal edge

DG Methods as Weighted Residuals (Brezzi-Cockburn-Marini-Süli, CMAME 2006)

A is a second order elliptic operator

$$Au \equiv -\mathrm{div}(\kappa \nabla u) = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

- $\sigma = -\kappa \nabla u$ in Ω,
- $\nabla \cdot \sigma = f$ in Ω,
- $u = 0$ on $\partial\Omega$.

Define $H^2(\mathscr{T}_h) := \{v \in L^2(\Omega) : \ v_{|K} \in H^2(K) \ \forall K \in \mathscr{T}_h\}$.

If you allow, a priori, your solution to be discontinuous $(u \in H^2(\mathscr{T}_h))$, then the equations to be required are:

- $A\, u - f \ = \ 0$ in each element
- $[\![u]\!] = 0$ on each edge
- $[\![\sigma]\!] = 0$ on each internal edge

DG Methods as Weighted Residuals (Brezzi-Cockburn-Marini-Süli, CMAME 2006)

A is a second order elliptic operator

$$Au \equiv -\mathrm{div}(\kappa \nabla u) = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

- $\sigma = -\kappa \nabla u$ in Ω,
- $\nabla \cdot \sigma = f$ in Ω,
- $u = 0$ on $\partial\Omega$.

Define $H^2(\mathscr{T}_h) := \{v \in L^2(\Omega) : \; v_{|K} \in H^2(K) \; \forall K \in \mathscr{T}_h\}$.

If you allow, a priori, your solution to be discontinuous $(u \in H^2(\mathscr{T}_h))$, then the equations to be required are:

- $A\,u - f \; = \; 0$ in each element
- $[\![u]\!] = 0$ on each edge
- $[\![\sigma]\!] = 0$ on each internal edge

DG Methods as Weighted Residuals (Brezzi-Cockburn-Marini-Süli, CMAME 2006)

A is a second order elliptic operator

$$Au \equiv -\mathrm{div}(\kappa\nabla u) = f \quad \text{in } \Omega \qquad u = 0 \quad \text{on } \partial\Omega$$

- $\sigma = -\kappa\nabla u$ in Ω,
- $\nabla\cdot\sigma = f$ in Ω,
- $u = 0$ on $\partial\Omega$.

Define $H^2(\mathscr{T}_h) := \{v \in L^2(\Omega) : \; v_{|K} \in H^2(K) \; \forall K \in \mathscr{T}_h\}$.

If you allow, a priori, your solution to be discontinuous $(u \in H^2(\mathscr{T}_h))$, then the equations to be required are:

- $A\,u - f \;=\; 0$ in each element
- $[\![u]\!] = 0$ on each edge
- $[\![\boldsymbol{\sigma}]\!] = 0$ on each internal edge

Starting Point

We take three operators B_0, B_1, B_2 from $H^2(\mathscr{T}_h)$ to $L^2(\mathscr{T}_h)$, $[L^2(\mathcal{E}_h)]^d$ and $L^2(\mathcal{E}_h^\circ)$ respectively.

Then we consider the following *variational* formulation

$$\text{find } u \in H^2(\mathscr{T}_h) \text{ such that, } \forall v \in H^2(\mathscr{T}_h) :$$
$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!]\,,\, B_1(v) >_h + < [\![\boldsymbol{\sigma}]\!]\,,\, B_2(v) >_h^0 = 0,$$

where

$$(u,\, v)_h = \sum_{K \in \mathscr{T}_h} \int_K u\, v\; \mathrm{d}x \qquad < u,\, v >_h = \sum_{e \in \mathcal{E}_h} \int_e u\, v\; \mathrm{d}s$$

and $< u,\, v >_h^0$ runs only on internal edges

Conditions on $B_j' s$

Advection-Diffusion-Reaction Problems

Let $f \in L^2(\Omega)$, $g \in H^{3/2}(\Gamma)$. Consider the problem

$$\begin{aligned} \operatorname{div}(-\varepsilon \nabla u + \beta u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

Introducing the flux $\sigma(u) = -\varepsilon \nabla u + \beta u$ we can write

$$\begin{aligned} \operatorname{div} \sigma(u) + \gamma u &= f \quad \text{in } \Omega, \\ u &= g \quad \text{on } \Gamma = \partial\Omega, \end{aligned}$$

(For simplicity of exposition) ε, β, γ constants: $\varepsilon > 0$, $\gamma \geq 0$.
Unique solution $u \in H^2(\Omega)$.

$\Gamma = \Gamma^+ \cup \Gamma^-$, $\Gamma^- =$ inflow $(\beta \cdot \mathbf{n} < 0)$, $\Gamma^+ =$ outflow $(\beta \cdot \mathbf{n} \geq 0)$

The Residuals

$$H^2(\mathscr{T}_h) := \{v \in L^2(\Omega) : \; v_{|K} \in H^2(K) \; \forall K \in \mathscr{T}_h\}$$

- $R_0(u) := \operatorname{div} \boldsymbol{\sigma}(u) + \gamma u - f = 0$ in each $K \in \mathscr{T}_h$,
- $R_1(u) := [\![u]\!] = 0$ on each $e \in \mathcal{E}_h^\circ$,
- $R_2(u) := [\![\boldsymbol{\sigma}(u)]\!] = 0$ on each $e \in \mathcal{E}_h^\circ$,
- $R_1^D(u) := u - g = 0$ on each $e \in \Gamma$

We need four operators $B_0, B_1, \; B_2, \; B_1^D$ such that:

$$(R_0(u), B_0(v))_h \quad + < R_1(u), B_1(v) >_h^0 + < R_2(u), B_2(v) >_h^0$$
$$+ < R_1^D(u), B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

The Residuals

$$H^2(\mathscr{T}_h) := \{v \in L^2(\Omega) : \; v_{|K} \in H^2(K) \; \forall K \in \mathscr{T}_h\}$$

- $R_0(u) := \mathrm{div}\, \boldsymbol{\sigma}(u) + \gamma u - f = 0$ in each $K \in \mathscr{T}_h$,
- $R_1(u) := [\![u]\!] = 0$ on each $e \in \mathcal{E}_h^\circ$,
- $R_2(u) := [\![\boldsymbol{\sigma}(u)]\!] = 0$ on each $e \in \mathcal{E}_h^\circ$,
- $R_1^D(u) := u - g = 0$ on each $e \in \Gamma$

We need four operators B_0, B_1, B_2, B_1^D such that:

$$(R_0(u), B_0(v))_h + < R_1(u), B_1(v) >_h^0 + < R_2(u), B_2(v) >_h^0 + < R_1^D(u), B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

Choices of the operators

Taking $B_0 v = v$ we have:

$$(\mathrm{div}\,\boldsymbol{\sigma}(u) + \gamma u - f, v)_h \quad + < [\![u]\!], B_1(v) >_h^0 + < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) >_h^0$$
$$+ < u - g, B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

Integrating by parts and using the crucial formula:

$$\int_\Omega \mathrm{div}_h \boldsymbol{\sigma}(u) v = -\int_\Omega \boldsymbol{\sigma}(u) \cdot \nabla_h v + \sum_{K \in \mathscr{T}_h} \int_{\partial K} \boldsymbol{\sigma}(u) \cdot \mathbf{n} v$$
$$= -(\boldsymbol{\sigma}(u), \nabla v)_h + < [\![\boldsymbol{\sigma}(u)]\!], \{v\} >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$

Substituting in the equation we obtain:

$$(\gamma u, v) - (\boldsymbol{\sigma}(u), \nabla v)_h + < [\![u]\!], B_1(v) >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$
$$+ < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) + \{v\} >_h^0 + < u, B_1^D(v) >_\Gamma$$
$$= (f, v) + < g, B_1^D(v) >_\Gamma$$

Choices of the operators

Taking $B_0 v = v$ we have:

$$(\mathrm{div}\,\boldsymbol{\sigma}(u) + \gamma u - f, v)_h \quad + < [\![u]\!]\,, B_1(v) >_h^0 + < [\![\boldsymbol{\sigma}(u)]\!]\,, B_2(v) >_h^0$$
$$+ < u - g, B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

Integrating by parts and using the crucial formula:

$$\int_\Omega \mathrm{div}_h \boldsymbol{\sigma}(u) v = -\int_\Omega \boldsymbol{\sigma}(u) \cdot \boldsymbol{\nabla}_h v + \sum_{K \in \mathscr{T}_h} \int_{\partial K} \boldsymbol{\sigma}(u) \cdot \mathbf{n} v$$
$$= -(\boldsymbol{\sigma}(u), \boldsymbol{\nabla} v)_h + < [\![\boldsymbol{\sigma}(u)]\!]\,, \{v\} >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$

Substituting in the equation we obtain:

$$(\gamma u, v) - (\boldsymbol{\sigma}(u), \boldsymbol{\nabla} v)_h + < [\![u]\!]\,, B_1(v) >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$
$$+ < [\![\boldsymbol{\sigma}(u)]\!]\,, B_2(v) + \{v\} >_h^0 + < u, B_1^D(v) >_\Gamma$$
$$= (f, v) + < g, B_1^D(v) >_\Gamma$$

Choices of the operators

Taking $B_0 v = v$ we have:

$$(\mathrm{div}\,\boldsymbol{\sigma}(u) + \gamma u - f, v)_h \quad + < [\![u]\!]\,, B_1(v) >_h^0 + < [\![\boldsymbol{\sigma}(u)]\!]\,, B_2(v) >_h^0$$
$$+ < u - g, B_1^D(v) >_\Gamma = 0 \qquad \forall v \in H^2(\mathscr{T}_h)$$

Integrating by parts and using the crucial formula:

$$\int_\Omega \mathrm{div}_h \boldsymbol{\sigma}(u) v = -\int_\Omega \boldsymbol{\sigma}(u)\cdot \boldsymbol{\nabla}_h v + \sum_{K\in\mathscr{T}_h} \int_{\partial K} \boldsymbol{\sigma}(u)\cdot \mathbf{n} v$$
$$= -(\boldsymbol{\sigma}(u), \boldsymbol{\nabla} v)_h + < [\![\boldsymbol{\sigma}(u)]\!]\,, \{v\} >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$

Substituting in the equation we obtain:

$$(\gamma u, v) - (\boldsymbol{\sigma}(u), \boldsymbol{\nabla} v)_h + < [\![u]\!]\,, B_1(v) >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$
$$+ < [\![\boldsymbol{\sigma}(u)]\!]\,, B_2(v) + \{v\} >_h^0 + < u, B_1^D(v) >_\Gamma$$
$$= (f, v) + < g, B_1^D(v) >_\Gamma$$

Guidelines for choosing the operators

$$(k \geq 1 \longrightarrow V_h^k = \{v \in L^2(\Omega) : \; v_{|K} \in P_k(K) \quad \forall K \in \mathscr{T}_h\})$$

$$a_h(u,v) := (\gamma u, v) - (\boldsymbol{\sigma}(u), \nabla v)_h + < [\![u]\!], B_1(v) >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$
$$+ < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) + \{v\} >_h^0 + < u, B_1^D(v) >_\Gamma$$

- Key: $\boxed{\text{Stability:} \quad a_h(v,v) \geq \alpha |||v|||^2 \qquad \forall v \in V_h^k}$

Guidelines for choosing the operators

$$(k \geq 1 \longrightarrow V_h^k = \{v \in L^2(\Omega) : \; v_{|K} \in P_k(K) \quad \forall K \in \mathscr{T}_h\})$$

$$a_h(u,v) := (\gamma u, v) - (\boldsymbol{\sigma}(u), \nabla v)_h + < [\![u]\!], B_1(v) >_h^0 + < \{\boldsymbol{\sigma}(u)\}, [\![v]\!] >_h$$
$$+ < [\![\boldsymbol{\sigma}(u)]\!], B_2(v) + \{v\} >_h^0 + < u, B_1^D(v) >_\Gamma$$

- Key: **Stability:** $a_h(v,v) \geq \alpha |||v|||^2 \qquad \forall v \in V_h^k$

$$|||v|||^2 = \gamma \|v\|_{0,\Omega}^2 + \varepsilon |v|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} \frac{\varepsilon}{|e|} \| [\![v]\!] \|_{0,e}^2 + \sum_{e \in \mathcal{E}_h} |\boldsymbol{\beta} \cdot \mathbf{n}| \| [\![v]\!] \|_{0,e}^2$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma \|v\|_{0,\Omega}^2 + \varepsilon |v|_{1,h}^2 - \boxed{\int_\Omega (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h v) v}$$

$$\boxed{-\int_\Omega (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h v) v} = -\frac{1}{2} \int_\Omega \boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h (v^2) = -\sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{\boldsymbol{\beta} \cdot \mathbf{n}}{2} v^2$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma \|v\|^2_{0,\Omega} + \varepsilon |v|^2_{1,h} - \boxed{\int_\Omega (\boldsymbol{\beta} \cdot \nabla_h v) v}$$

$$\boxed{-\int_\Omega (\boldsymbol{\beta} \cdot \nabla_h v) v} = -\frac{1}{2} \int_\Omega \boldsymbol{\beta} \cdot \nabla_h (v^2) = -\sum_{K \in \mathscr{T}_h} \int_{\partial K} \frac{\boldsymbol{\beta} \cdot \mathbf{n}}{2} v^2$$

$$= -\frac{1}{2} < \{\boldsymbol{\beta}\}, [[v^2]] >_h - \frac{1}{2} < [\![\boldsymbol{\beta}]\!], \{v^2\} >_h^0$$

$$\text{but } [\![\boldsymbol{\beta}]\!] = 0$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \boxed{\int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v)v}$$

$$\boxed{-\int_\Omega (\boldsymbol{\beta}\cdot\boldsymbol{\nabla}_h v)v} = -\frac{1}{2} < \{\boldsymbol{\beta}\}, [[v^2]] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma \|v\|^2_{0,\Omega} + \varepsilon |v|^2_{1,h} - \boxed{\int_\Omega (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h v) v} + \boxed{< \{\boldsymbol{\sigma}(v)\}, [\![v]\!] >_h} + \cdots$$

$$\boxed{-\int_\Omega (\boldsymbol{\beta} \cdot \boldsymbol{\nabla}_h v) v} = -\frac{1}{2} < \{\boldsymbol{\beta}\}, [[v^2]] >^0_h - \frac{1}{2} < \boldsymbol{\beta} \cdot \mathbf{n}, v^2 >_\Gamma$$

$$\boxed{< \{\boldsymbol{\sigma}(v)\}, [\![v]\!] >_h} = - < \{\varepsilon \boldsymbol{\nabla}_h v\}, [\![v]\!] >_h + < \{\boldsymbol{\beta} v\}, [\![v]\!] >_h$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|_{0,\Omega}^2 + \varepsilon|v|_{1,h}^2 - \boxed{\int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v} + \boxed{<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h} + \cdots$$

$$\boxed{-\int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v} = -\frac{1}{2} < \{\boldsymbol{\beta}\}, [[v^2]] >_h^0 - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$\boxed{<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h} = -<\{\varepsilon\nabla_h v\}, [\![v]\!]>_h + <\{\boldsymbol{\beta} v\}, [\![v]\!]>_h^0 + <\boldsymbol{\beta}\cdot\mathbf{n}, v^2>_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \boxed{\int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v} + \boxed{<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h} + \cdots$$

$$\boxed{-\int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v} = -\frac{1}{2} < \{\boldsymbol{\beta}\}, [[v^2]] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$\boxed{<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h} = - <\{\varepsilon\nabla_h v\}, [\![v]\!]>_h + <\{\boldsymbol{\beta} v\}, [\![v]\!]>^0_h + <\boldsymbol{\beta}\cdot\mathbf{n}, v^2>_\Gamma$$

$$< \{\boldsymbol{\beta} v\}, [\![v]\!] >^0_h = \frac{1}{2} <\{\boldsymbol{\beta}\}, [[v^2]] >^0_h$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} - \boxed{\int_\Omega(\boldsymbol{\beta}\cdot\nabla_h v)v} + \boxed{<\{\boldsymbol{\sigma}(v)\},[\![v]\!]>_h} + \cdots$$

$$\boxed{-\int_\Omega(\boldsymbol{\beta}\cdot\nabla_h v)v} = -\frac{1}{2} < \{\boldsymbol{\beta}\}, [[v^2]] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$\boxed{<\{\boldsymbol{\sigma}(v)\},[\![v]\!]>_h} = - < \{\varepsilon\nabla_h v\}, [\![v]\!] >_h + \frac{1}{2} < \{\boldsymbol{\beta}\}, [[v^2]] >^0_h + < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} + \boxed{\frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma} - \boxed{<\{\varepsilon\nabla_h v\}, [\![v]\!]>_h} + \cdots$$

$$\boxed{-\int_\Omega (\boldsymbol{\beta}\cdot\nabla_h v)v} = -\frac{1}{2} < \{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h - \frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma$$

$$\boxed{<\{\boldsymbol{\sigma}(v)\}, [\![v]\!]>_h} = -<\{\varepsilon\nabla_h v\}, [\![v]\!]>_h + \frac{1}{2}<\{\boldsymbol{\beta}\}, [\![v^2]\!] >^0_h + <\boldsymbol{\beta}\cdot\mathbf{n}, v^2>_\Gamma$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} + \boxed{\frac{1}{2} < \boldsymbol{\beta}\cdot\mathbf{n}, v^2 >_\Gamma} - <\{\varepsilon\nabla_h v\}, [\![v]\!]>_h + \cdots$$

$$+ \boxed{< [\![\boldsymbol{\sigma}(v)]\!]\,, B_2(v) + \{v\} >^0_h} + \cdots$$

$$\boxed{< [\![\boldsymbol{\sigma}(v)]\!]\,, B_2(v) + \{v\} >^0_h} = - < [\![\varepsilon\nabla_h v]\!]\,, B_2(v) + \{v\} >^0_h$$

$$+ < [\![\boldsymbol{\beta} v]\!]\,, B_2(v) + \{v\} >^0_h$$

Guidelines for choosing the operators. (Towards stability)

$$a_h(v,v) = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} + \boxed{\frac{1}{2} < \beta\cdot\mathbf{n}, v^2 >_\Gamma} - <\{\varepsilon\nabla_h v\}, [\![v]\!]>_h + \cdots$$

$$+ \boxed{< [\![\sigma(v)]\!], B_2(v) + \{v\} >^0_h} + \boxed{< [\![v]\!], B_1(v) >^0_h} + \boxed{< v, B^D_1(v) >_\Gamma}$$

$$\boxed{< [\![\sigma(v)]\!], B_2(v) + \{v\} >^0_h} = - < [\![\varepsilon\nabla_h v]\!], B_2(v) + \{v\} >^0_h$$

$$+ < [\![\beta v]\!], B_2(v) + \{v\} >^0_h$$

$$\boxed{< [\![v]\!], B_1(v) >^0_h}, \qquad \boxed{< v, B^D_1(v) >_\Gamma}$$

Guidelines for Choosing the Operators. (Towards Stability)

$$|||v|||^2 = \gamma \|v\|^2_{0,\Omega} + \boxed{\varepsilon |v|^2_{1,h} + \sum_{e \in \mathcal{E}_h} \frac{\varepsilon}{|e|} \| [\![v]\!] \|^2_{0,e}} + \sum_{e \in \mathcal{E}_h} |\beta \cdot \mathbf{n}| \| [\![v]\!] \|^2_{0,e}$$

$B_2(v) =, \qquad B_1(v) =$

$$B_1^D(v) = \frac{c\varepsilon}{|e|} v + Q_1^D(v) \text{ on } \Gamma^-, \quad B_1^D(v) = \frac{c\varepsilon}{|e|} v + Q_1^D(v) \text{ on } \Gamma^+$$

with $Q_1(v)$, $Q_2(v)$, $Q_1^D(v)$ to be chosen such that

$$< [\![v]\!], Q_1(v) >_h^0 + < [\![\beta v]\!], Q_2(v) >_h^0 \geq \sum_{e \in \mathcal{E}_h^\circ} |\beta \cdot \mathbf{n}| \| [\![v]\!] \|^2_{0,e}$$

$Q_1^D(v)$ not depending on β.

In other words, we can upwind either with B_1 or with B_2.

Guidelines for Choosing the Operators. (Towards Stability)

$$|||v|||^2 = \gamma \|v\|^2_{0,\Omega} + \boxed{\varepsilon |v|^2_{1,h} + \sum_{e \in \mathcal{E}_h} \frac{\varepsilon}{|e|} \| [\![v]\!] \|^2_{0,e}} + \sum_{e \in \mathcal{E}_h} |\boldsymbol{\beta} \cdot \mathbf{n}| \| [\![v]\!] \|^2_{0,e}$$

$$B_2(v) = \boxed{-\{v\}}, \qquad B_1(v) = \boxed{\frac{c\varepsilon}{|e|} [\![v]\!]}$$

$$B_1^D(v) = \frac{c\varepsilon}{|e|} v + Q_1^D(v) \text{ on } \Gamma^-, \quad B_1^D(v) = \frac{c\varepsilon}{|e|} v + Q_1^D(v) \text{ on } \Gamma^+$$

with $Q_1(v)$, $Q_2(v)$, $Q_1^D(v)$ to be chosen such that

$$< [\![v]\!], Q_1(v) >_h^0 + < [\![\beta v]\!], Q_2(v) >_h^0 \geq \sum_{e \in \mathcal{E}_h^\circ} |\boldsymbol{\beta} \cdot \mathbf{n}| \| [\![v]\!] \|^2_{0,e}$$

$Q_1^D(v)$ not depending on β.

In other words, we can upwind either with B_1 or with B_2.

Guidelines for Choosing the Operators. (Towards Stability)

$$|||v|||^2 = \gamma\|v\|_{0,\Omega}^2 + \varepsilon|v|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} \frac{\varepsilon}{|e|}\| [\![v]\!] \|_{0,e}^2 + \boxed{\sum_{e\in\mathcal{E}_h} |\beta\cdot\mathbf{n}|\| [\![v]\!] \|_{0,e}^2}$$

$$B_2(v) = -\{v\} + Q_2(v), \qquad B_1(v) = \frac{c\varepsilon}{|e|}[\![v]\!] + Q_1(v)$$

$$B_1^D(v) = \frac{c\varepsilon}{|e|}v - \boxed{\beta\cdot\mathbf{n}v} + Q_1^D(v) \text{ on } \Gamma^-, \quad B_1^D(v) = \frac{c\varepsilon}{|e|}v + Q_1^D(v) \text{ on } \Gamma^+$$

with $Q_1(v)$, $Q_2(v)$, $Q_1^D(v)$ to be chosen such that

$$< [\![v]\!], Q_1(v) >_h^0 + < [\![\beta v]\!], Q_2(v) >_h^0 \geq \sum_{e\in\mathcal{E}_h^\circ} |\beta\cdot\mathbf{n}|\| [\![v]\!] \|_{0,e}^2$$

$Q_1^D(v)$ not depending on β.

In other words, we can upwind either with B_1 or with B_2.

Guidelines for Choosing the Operators. (Towards Stability)

$$|||v|||^2 = \gamma\|v\|^2_{0,\Omega} + \varepsilon|v|^2_{1,h} + \sum_{e\in\mathcal{E}_h} \frac{\varepsilon}{|e|}\|[\![v]\!]\|^2_{0,e} + \boxed{\sum_{e\in\mathcal{E}_h} |\boldsymbol{\beta}\cdot\mathbf{n}|\|[\![v]\!]\|^2_{0,e}}$$

$$B_2(v) = -\{v\} + Q_2(v), \qquad B_1(v) = \frac{c\varepsilon}{|e|}[\![v]\!] + Q_1(v)$$

$$B_1^D(v) = \frac{c\varepsilon}{|e|}v - \boxed{\boldsymbol{\beta}\cdot\mathbf{n}v} + Q_1^D(v) \text{ on } \Gamma^-, \quad B_1^D(v) = \frac{c\varepsilon}{|e|}v + Q_1^D(v) \text{ on } \Gamma^+$$

with $Q_1(v)$, $Q_2(v)$, $Q_1^D(v)$ to be chosen such that

$$< [\![v]\!], Q_1(v) >_h^0 + < [\![\boldsymbol{\beta}v]\!], Q_2(v) >_h^0 \geq \sum_{e\in\mathcal{E}_h^\circ} |\boldsymbol{\beta}\cdot\mathbf{n}|\|[\![v]\!]\|^2_{0,e}$$

$Q_1^D(v)$ not depending on $\boldsymbol{\beta}$.

In other words, we can upwind either with B_1 or with B_2.

Guidelines for Choosing the Operators. (Towards Stability)

$$B_2(v) = \boxed{-\{v\}} + Q_2(v), \qquad B_1(v) = \boxed{S_e [\![v]\!]} + Q_1(v) \qquad \left(S_e = \frac{c\varepsilon}{|e|}\right)$$

$$B_1^D(v) = S_e v - \boxed{\boldsymbol{\beta}\cdot\mathbf{n} v} + Q_1^D(v) \text{ on } \Gamma^-, \quad B_1^D(v) = S_e v + Q_1^D(v) \text{ on } \Gamma^+$$

$$< [\![v]\!], Q_1(v) >_h^0 + < [\![\beta v]\!], Q_2(v) >_h^0 \geq \sum_{e\in\mathcal{E}_h^\circ} |\boldsymbol{\beta}\cdot\mathbf{n}| \| [\![v]\!] \|_{0,e}^2$$

$Q_1^D(v)$ not depending on $\boldsymbol{\beta}$.

In other words, we can upwind either with B_1 or with B_2.

Guidelines for Choosing the Operators. (Towards Stability)

$$B_2(v) = \boxed{-\{v\}} + Q_2(v), \qquad B_1(v) = \boxed{S_e [\![v]\!]} + Q_1(v) \qquad \left(S_e = \frac{c\varepsilon}{|e|}\right)$$

$$B_1^D(v) = S_e v - \boxed{\boldsymbol{\beta} \cdot \mathbf{n} v} + Q_1^D(v) \text{ on } \Gamma^-, \qquad B_1^D(v) = S_e v + Q_1^D(v) \text{ on } \Gamma^+$$

$$< [\![v]\!], Q_1(v) >_h^0 + < [\![\boldsymbol{\beta} v]\!], Q_2(v) >_h^0 \geq \sum_{e \in \mathcal{E}_h^\circ} |\boldsymbol{\beta} \cdot \mathbf{n}| \| [\![v]\!] \|_{0,e}^2$$

$Q_1^D(v)$ not depending on $\boldsymbol{\beta}$.

In other words, we can upwind either with B_1 or with B_2.

Let $e = \partial K^+ \cap \partial K^-$, with K^+ an upwind element $(\boldsymbol{\beta} \cdot \mathbf{n}^+ > 0)$

$$\text{on} \qquad e \qquad (\varphi)_{upw} = \varphi^+ = \{\varphi\} + \frac{\mathbf{n}^+}{2} [\![\varphi]\!]$$

First choice - minimal choice

$$Q_2(v) = 0, \quad Q_1(v) = \frac{\mathbf{n}^+}{2} [\![\boldsymbol{\beta} v]\!] \quad Q_1^D(v) = 0$$

$$B_2(v) = -\{v\} \qquad B_1(v) = S_e [\![v]\!] + \frac{n^+}{2} [\![\boldsymbol{\beta} v]\!]$$

$$B_1^D(v) = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v \text{ on } \Gamma^-, \quad B_1^D(v) = S_e v \text{ on } \Gamma^+$$

First choice - minimal choice

$$B_2(v) = -\{v\} \qquad B_1(v) = S_e [\![v]\!] + \frac{n^+}{2} [\![\boldsymbol{\beta} v]\!]$$

$$B_1^D(v) = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v \text{ on } \Gamma^-, \quad B_1^D(v) = S_e v \text{ on } \Gamma^+$$

$$\begin{cases} \text{Find } u \in H^2(\mathscr{T}_h) \text{ such that } \forall v \in H^2(\mathscr{T}_h) \\ \int_\Omega (\gamma uv - \boldsymbol{\sigma}(u) \cdot \boldsymbol{\nabla}_h v) + \sum_{e \in \mathcal{E}_h} S_e \int_e [\![u]\!] \cdot [\![v]\!] + \sum_{e \in \mathcal{E}_h^\circ} \int_e (\boldsymbol{\beta} u)_{upw} [\![v]\!] \\ - \sum_{e \in \mathcal{E}_h} \int_e \{\varepsilon \boldsymbol{\nabla}_h u\} \cdot [\![v]\!] + \int_{\Gamma^+} \boldsymbol{\beta} \cdot \mathbf{n} uv \\ = \int_\Omega fv + \sum_{e \in \Gamma} S_e \int_e g\, v - \sum_{e \in \Gamma^-} \int_e \boldsymbol{\beta} \cdot \mathbf{n} g v. \end{cases}$$

For the diffusive part this choice corresponds to the IIP scheme
Dawson, Sun & Wheeler (CMAME04), Sun & Wheeler (SINUM05)

Other Choices

Süli et. al.

Hughes et. al..

Weighted Average

Another Choice

Other Choices

Süli et. al.

Hughes et. al..

Weighted Average

Another Choice

Other Choices

Süli et. al.

Hughes et. al..

Weighted Average

Another Choice

Other Choices

Süli et. al.

Hughes et. al..

Weighted Average

Another Choice

Approximation

- Consistency $\surd$
- Stability $\surd$

$$|||v|||^2 = \gamma \|v\|_{0,\Omega}^2 + \varepsilon |v|_{1,h}^2 + \sum_{e \in \mathcal{E}_h} \frac{\varepsilon}{|e|} \| [\![v]\!] \|_{0,e}^2 + \sum_{e \in \mathcal{E}_h} |\boldsymbol{\beta} \cdot \mathbf{n}| \| [\![v]\!] \|_{0,e}^2$$

Approximation

- Consistency $\surd$
- Stability $\surd$

$$|||v|||^2 = \gamma \|v\|_{0,\Omega}^2 + \varepsilon |v|_{1,h}^2 + \sum_{e\in\mathcal{E}_h} \frac{\varepsilon}{|e|} \| [\![v]\!] \|_{0,e}^2 + \sum_{e\in\mathcal{E}_h} |\boldsymbol{\beta}\cdot\mathbf{n}| \| [\![v]\!] \|_{0,e}^2$$

- Stability in stronger norms:

$$|||v|||_{DG}^2 := |||v|||^2 + \boxed{\frac{\|\boldsymbol{\beta}\|_{0,\infty}}{L} \|v\|_{0,\Omega}^2}$$ [JP(87)]

$$|||v|||_{\mathrm{SUPG}}^2 := |||v|||^2 + \boxed{\sum_{K\in\mathscr{T}_h} C_{0,K} \frac{h_K}{|\boldsymbol{\beta}|} \|\boldsymbol{\beta}\cdot\nabla v\|_{0,K}^2}$$ [JP(87); KG(05);BHS(07)]

- Error Estimates

$$|||u-u_h|||\,,\ |||u-u_h|||_{\mathrm{SUPG}} \leq C \begin{cases} h^{k+1/2} & \text{if advection dominates,} \\ h^k & \text{if diffusion dominates.} \end{cases}$$

Analysis for Variable Coefficients case

$\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^d\ ,\quad \gamma \in L^{\infty}(\Omega)$ (more complicated...)

- Stability in the norm: $|||v|||_{DG}^2 := |||v|||^2 + \boxed{\frac{\|\boldsymbol{\beta}\|_{0,\infty}}{L}\|v\|_{0,\Omega}^2}$

- Stability in "$\mathcal{SUPG}^*$"-norm:

$$|||v|||_{\mathcal{SUPG}^*}^2 = |||v|||_{DG}^2 + \boxed{\sum_{K\in\mathcal{T}_h} C_{0,K}\frac{h_K}{|\boldsymbol{\beta}|}\|P_h^k(\boldsymbol{\beta}\cdot\nabla v)\|_{0,K}^2}$$

- if $\boldsymbol{\beta}\cdot\nabla v \in V_h^k \quad \Longrightarrow \quad |||v|||_{\mathcal{SUPG}^*}^2 \equiv |||v|||_{\mathrm{SUPG}}^2$

- $|||u-u_h|||_{\mathcal{SUPG}}^* \leq C\begin{cases} h^{k+1/2} & \text{if advection dominates,}\\ h^k & \text{if diffusion dominates.}\end{cases}$

Hypotheses

Analysis for Variable Coefficients case

$\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^d \,, \quad \gamma \in L^\infty(\Omega)$ (more complicated...)

- Stability in the norm: $|||v|||^2_{DG} := |||v|||^2 + \boxed{\frac{\|\boldsymbol{\beta}\|_{0,\infty}}{L} \|v\|^2_{0,\Omega}}$
- Stability in "$\mathcal{SUPG}^*$"-norm:

$$|||v|||^2_{\mathcal{SUPG}^*} = |||v|||^2_{DG} + \boxed{\sum_{K \in \mathcal{T}_h} C_{0,K} \frac{h_K}{|\boldsymbol{\beta}|} \|P^k_h(\boldsymbol{\beta} \cdot \nabla v)\|^2_{0,K}}$$

- if $\boldsymbol{\beta} \cdot \nabla v \in V^k_h \quad \Longrightarrow \quad |||v|||^2_{\mathcal{SUPG}^*} \equiv |||v|||^2_{\mathrm{SUPG}}$
- $|||u - u_h|||^*_{\mathcal{SUPG}} \le C \begin{cases} h^{k+1/2} & \text{if advection dominates,} \\ h^k & \text{if diffusion dominates.} \end{cases}$

Hypotheses

Analysis for Variable Coefficients case

$\beta \in W^{1,\infty}(\Omega)^d$, $\gamma \in L^{\infty}(\Omega)$ (more complicated...)

- Stability in the norm: $|||v|||^2_{DG} := |||v|||^2 + \boxed{\frac{\|\beta\|_{0,\infty}}{L}\|v\|^2_{0,\Omega}}$

- Stability in "$\mathcal{SUPG}^*$"-norm:

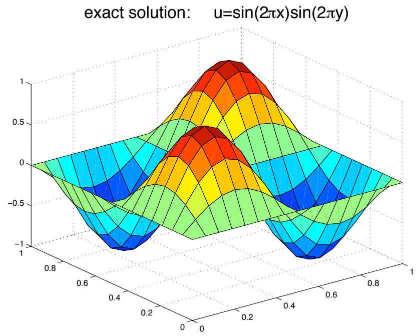
$$|||v|||^2_{\mathcal{SUPG}^*} = |||v|||^2_{DG} + \boxed{\sum_{K\in\mathscr{T}_h} C_{0,K}\frac{h_K}{|\beta|}\|P_h^k(\beta\cdot\nabla v)\|^2_{0,K}}$$

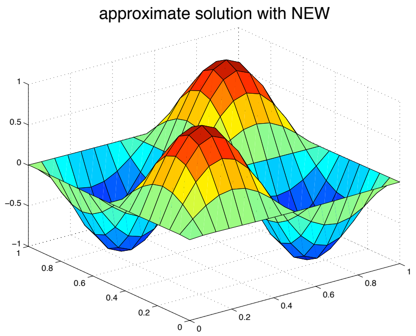
- if $\beta\cdot\nabla v \in V_h^k \implies |||v|||^2_{\mathcal{SUPG}^*} \equiv |||v|||^2_{\mathrm{SUPG}}$

- $|||u-u_h|||^*_{\mathcal{SUPG}} \leq C \begin{cases} h^{k+1/2} & \text{if advection dominates,} \\ h^k & \text{if diffusion dominates.} \end{cases}$

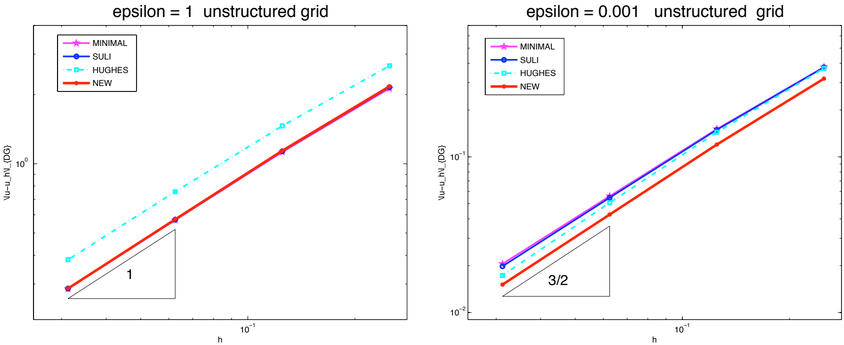
Hypotheses

Approximation to a Smooth Solution

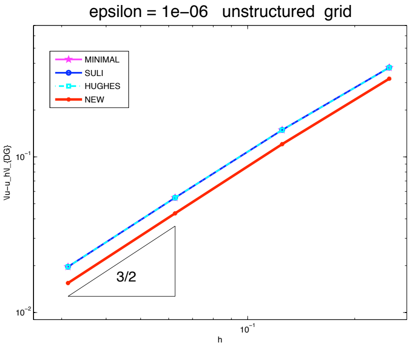


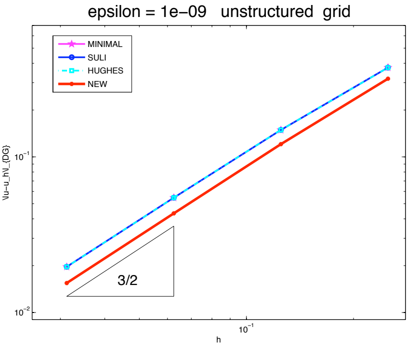


Convergence Diagrams:Unstructured Meshes



Convergence Diagrams: Smooth Solution



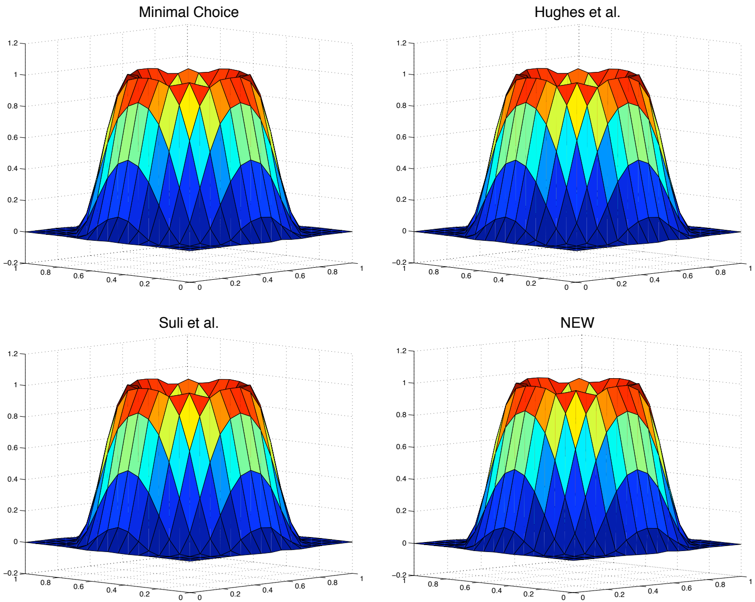


Pure Hyperbolic Problem (*Rotating Flow*)

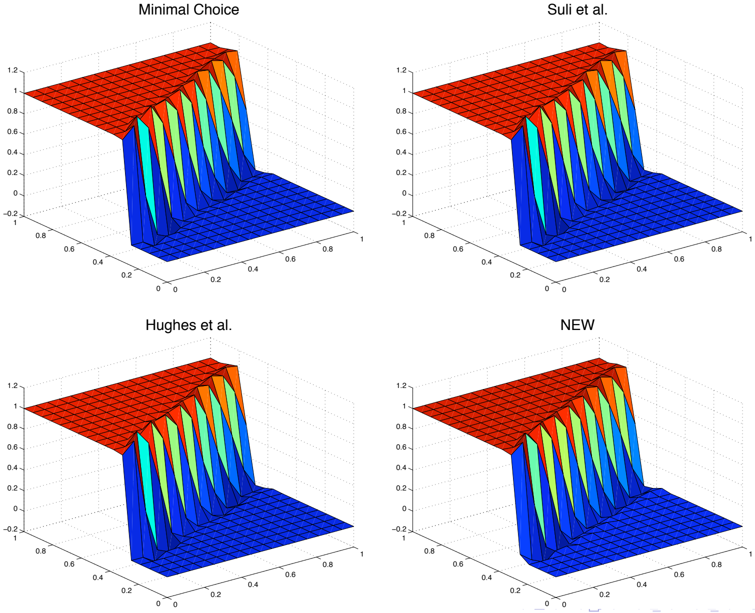
$$\boldsymbol{\beta} = \left[y - \frac{1}{2}, \frac{1}{2} - x \right]^T, \qquad \gamma = 0$$

$$\begin{cases} \operatorname{div}(\boldsymbol{\beta} u) = 0 & \text{in} \quad [0,1]^2 \\ u(1/2, y) = \sin^2(2\pi y) & y \in \left[0, \frac{1}{2}\right]. \end{cases}$$

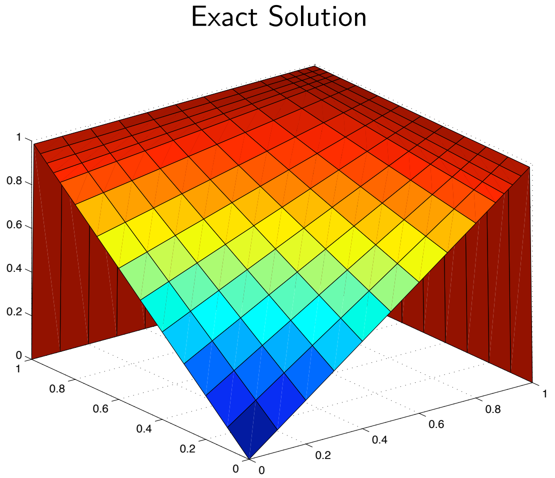
Pure Hyperbolic Problem (*Rotating Flow*)



Internal Layer $\varepsilon = 1e - 09$

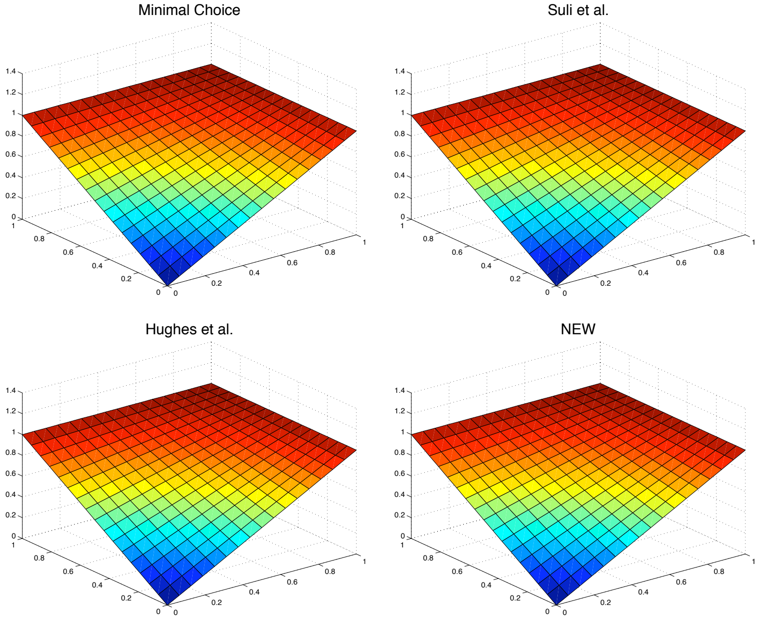


Forcing term with Boundary Layer $\epsilon = 1e - 09$

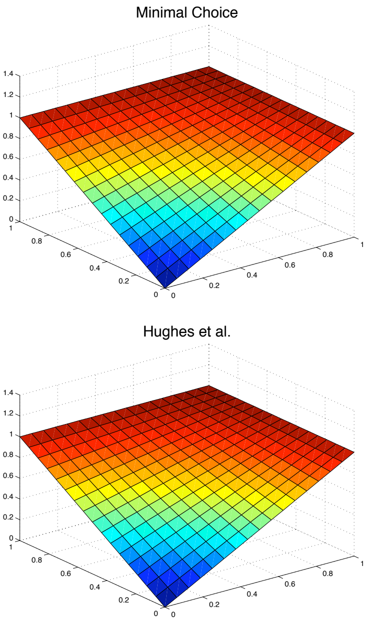


$$u(x,y) = x + y(1-x) + \frac{e^{-1/\varepsilon} - e^{-(1-x)(1-y)/\varepsilon}}{1 - e^{-1/\varepsilon}}\,, \quad (x,y) \in \Omega\,.$$

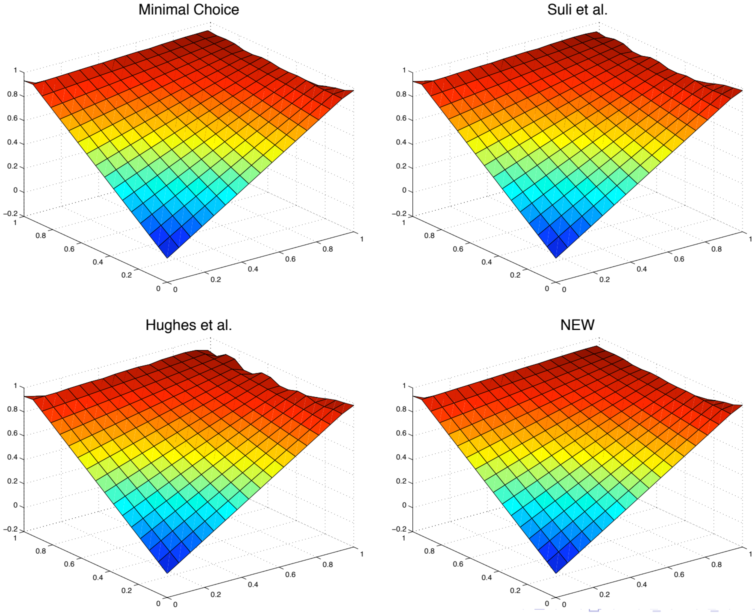
Forcing term with Boundary Layer $\epsilon = 1e - 09$



Forcing term with Boundary Layer $\epsilon = 1e - 06$



Forcing term with Boundary Layer $\epsilon = 1e - 03$



Conclusions & Further Research

- New approach allows to identify the stabilizing mechanisms
- Analysis for the variable coefficient case. Robust Error Analysis
- Surprisingly..... ALL methods perform equally well in the strongly advection-dominated regime
- extension for P^k on quadrilaterals?
- A posteriori error estimates
  - Elliptic problems
  - Advection-Diffusion-Reaction problems

Conclusions & Further Research

- New approach allows to identify the stabilizing mechanisms
- Analysis for the variable coefficient case. Robust Error Analysis
- Surprisingly..... ALL methods perform equally well in the strongly advection-dominated regime

- extension for P^k on quadrilaterals?
- A posteriori error estimates
  - Elliptic problems
  - Advection-Diffusion-Reaction problems

Conclusions & Further Research

- New approach allows to identify the stabilizing mechanisms
- Analysis for the variable coefficient case. Robust Error Analysis
- Surprisingly..... ALL methods perform equally well in the strongly advection-dominated regime

- extension for P^k on quadrilaterals?
- A posteriori error estimates
  - Elliptic problems
  - Advection-Diffusion-Reaction problems

Conditions on the Operators B_i

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!]\,,\, B_1(v) >_h + < [\![\sigma]\!]\,,\, B_2(v) >_h^0 = 0 \quad \forall v$$

The above equation gives back the original three equations on u (that is $A\,u = f$, $[\![u]\!] = 0$, and $[\![\sigma]\!] = 0$) **if** (and, essentially, only if)

- $\forall K \in \mathscr{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathscr{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h$ and $\forall \psi \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_1 v = \psi \text{ on } e, \quad B_1 v = 0 \text{ on } \mathcal{E}_h \setminus e \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h^\circ$ and $\forall \chi \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_2 v = \chi \text{ on } e, \quad B_2 v = 0 \text{ on } \mathcal{E}_h^\circ \setminus e$$

back

Conditions on the Operators B_i

$$(A\,u - f,\, B_0(v))_h + < [\![u]\!]\,,\, B_1(v) >_h + < [\![\sigma]\!]\,,\, B_2(v) >_h^0 = 0 \quad \forall v$$

The above equation gives back the original three equations on u (that is $A\,u = f$, $[\![u]\!] = 0$, and $[\![\sigma]\!] = 0$) **if** (and, essentially, only if)

- $\forall K \in \mathcal{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathcal{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathcal{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h$ and $\forall \psi \in C_0^\infty(e)$ there is a $v \in H^2(\mathcal{T}_h)$ such that

$$B_1 v = \psi \text{ on } e, \quad B_1 v = 0 \text{ on } \mathcal{E}_h \setminus e \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h^\circ$ and $\forall \chi \in C_0^\infty(e)$ there is a $v \in H^2(\mathcal{T}_h)$ such that

$$B_2 v = \chi \text{ on } e, \quad B_2 v = 0 \text{ on } \mathcal{E}_h^\circ \setminus e$$

back

Conditions on the Operators B_i

$$(A\,u - f,\, B_0\,(v))_h + < [\![u]\!]\,,\, B_1(v) >_h + < [\![\sigma]\!]\,,\, B_2(v) >_h^0 = 0 \quad \forall v$$

The above equation gives back the original three equations on u (that is $A\,u = f$, $[\![u]\!] = 0$, and $[\![\sigma]\!] = 0$) **if** (and, essentially, only if)

- $\forall K \in \mathscr{T}_h$ and $\forall \varphi \in C_0^\infty(K)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_0 v = \varphi \text{ in } K, \quad B_0 v = 0 \text{ in } \mathscr{T}_h \setminus K, \quad B_1 v \equiv 0, \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h$ and $\forall \psi \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_1 v = \psi \text{ on } e, \quad B_1 v = 0 \text{ on } \mathcal{E}_h \setminus e \quad B_2 v \equiv 0$$

- $\forall e \in \mathcal{E}_h^\circ$ and $\forall \chi \in C_0^\infty(e)$ there is a $v \in H^2(\mathscr{T}_h)$ such that

$$B_2 v = \chi \text{ on } e, \quad B_2 v = 0 \text{ on } \mathcal{E}_h^\circ \setminus e$$

back

Analysis for the Variable Coefficient Case

- $\varepsilon > 0$ constant
- $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^d$, $\gamma \in L^\infty(\Omega)$
- $\gamma + \frac{1}{2}\,\mathrm{div}(\boldsymbol{\beta}) \geq c_0 \geq 0$
- $\beta(x) \neq 0 \qquad \forall\, x \in \Omega$
- β has no closed curves
-

$$\begin{cases} \exists c_\beta > 0, \text{ and } \psi \in [W^{1,\infty}(\Omega)]^2 \text{ with } \|\psi\|_{1,\infty} = 1, \quad \|\psi\|_{0,\infty} \geq c_\beta \\ \text{such that} \quad \beta(x) = \widetilde{\beta}\psi(x), \qquad \forall x,\ \widetilde{\beta} \in \mathbf{R} \end{cases}$$

back

Analysis for the Variable Coefficient Case

- $\varepsilon > 0$ constant
- $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^d \;,\quad \gamma \in L^\infty(\Omega)$
- $\gamma + \frac{1}{2}\operatorname{div}(\boldsymbol{\beta}) \geq c_0 \geq 0$
- $\boldsymbol{\beta}(x) \neq 0 \qquad \forall\, x \in \Omega$
- $\boldsymbol{\beta}$ has no closed curves
- $$\begin{cases} \exists c_\beta > 0, \text{ and } \boldsymbol{\psi} \in [W^{1,\infty}(\Omega)]^2 \text{ with } \|\boldsymbol{\psi}\|_{1,\infty} = 1, \quad \|\boldsymbol{\psi}\|_{0,\infty} \geq c_\beta \\ \text{such that} \quad \boldsymbol{\beta}(x) = \widetilde{\beta}\boldsymbol{\psi}(x), \qquad \forall x,\; \widetilde{\beta} \in \mathbf{R} \end{cases}$$

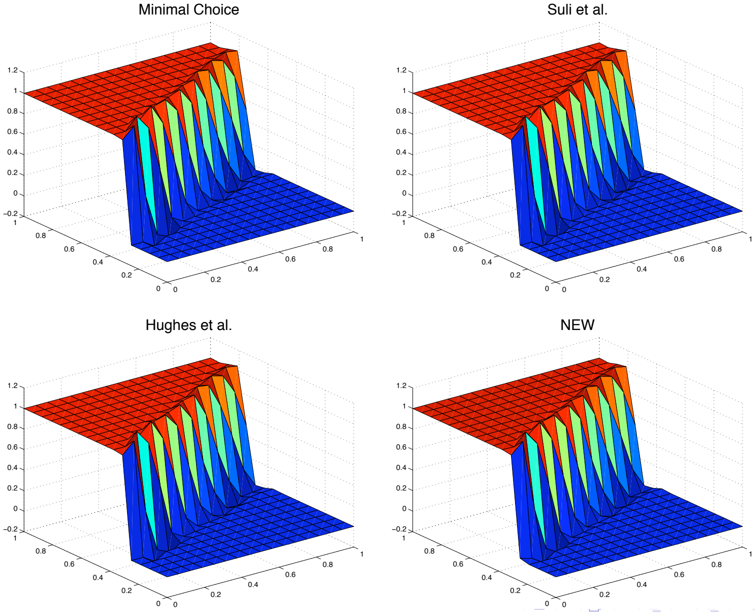
back

Analysis for the Variable Coefficient Case

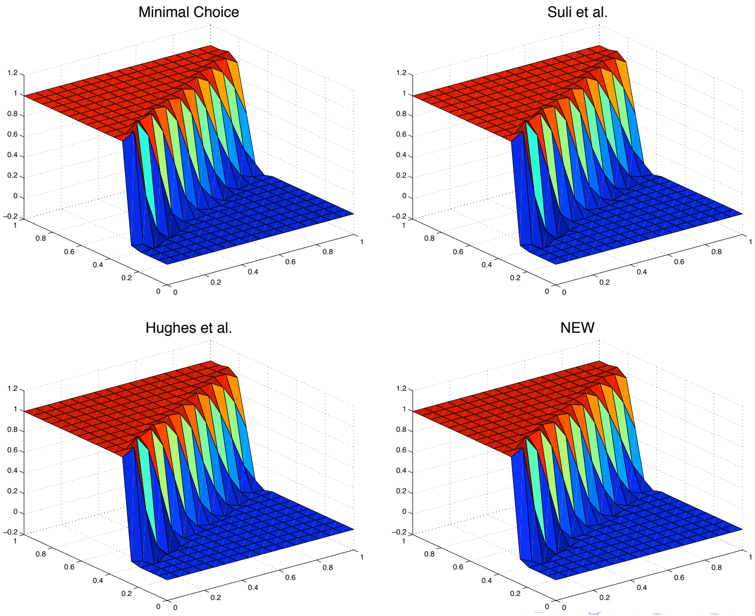
- $\varepsilon > 0$ constant
- $\boldsymbol{\beta} \in W^{1,\infty}(\Omega)^d$, $\gamma \in L^{\infty}(\Omega)$
- $\gamma + \frac{1}{2}\operatorname{div}(\boldsymbol{\beta}) \geq c_0 \geq 0$
- $\boldsymbol{\beta}(x) \neq 0 \qquad \forall\, x \in \Omega$
- $\boldsymbol{\beta}$ has no closed curves
- $$\begin{cases} \exists c_\beta > 0, \text{ and } \boldsymbol{\psi} \in [W^{1,\infty}(\Omega)]^2 \text{ with } \|\boldsymbol{\psi}\|_{1,\infty} = 1, \quad \|\boldsymbol{\psi}\|_{0,\infty} \geq c_\beta \\ \text{such that} \quad \boldsymbol{\beta}(x) = \widetilde{\beta}\boldsymbol{\psi}(x), \qquad \forall x,\ \widetilde{\beta} \in \mathbf{R} \end{cases}$$

back

Internal Layer $\varepsilon = 1e - 06$



Internal Layer $\varepsilon = 0.001$



Choice by Houston-Schwab-Süli (SINUM 2002)

$$\Longrightarrow B_2(v) = -\{v\} \qquad B_1(v) = S_e [\![v]\!] + \{\varepsilon \nabla_h v\} + \frac{n^+}{2} [\![\beta v]\!]$$

$$B_1^D(v) = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v + \varepsilon \nabla_h v \cdot \mathbf{n} \text{ on } \Gamma^-,$$

$$B_1^D(v) = S_e v + \varepsilon \nabla_h v \cdot \mathbf{n} \text{ on } \Gamma^+$$

Choice by Houston-Schwab-Süli (SINUM 2002)

$$
\begin{cases}
\text{Find} \quad u \in H^2(\mathscr{T}_h) \quad \text{such that} \quad \forall\, v \in H^2(\mathscr{T}_h) \\
\displaystyle\int_\Omega (\gamma u v - \boldsymbol{\sigma}(u)\cdot \boldsymbol{\nabla}_h v) + \sum_{e\in\mathcal{E}_h} S_e \int_e [\![u]\!]\cdot[\![v]\!] + \sum_{e\in\mathcal{E}_h^\circ} \int_e (\boldsymbol{\beta} u)_{upw}\cdot [\![v]\!] \\
\displaystyle + \sum_{e\in\mathcal{E}_h} \int_e \left([\![u]\!]\cdot\{\varepsilon\boldsymbol{\nabla}_h v\} - \{\varepsilon \boldsymbol{\nabla}_h u\}\cdot[\![v]\!]\right) + \sum_{e\in\Gamma^+}\int_e \boldsymbol{\beta}\cdot\mathbf{n} u v \\
\displaystyle = \int_\Omega f v + \sum_{e\in\Gamma}\int_e g\left(S_e v + \varepsilon\boldsymbol{\nabla}_h v\cdot\mathbf{n} - \boldsymbol{\beta}\cdot\mathbf{n}\, v\right).
\end{cases}
$$

NIPG for the diffusive part (Wheeler-Rivière-Girault (99'))

Choice by Hughes-Scovazzi-Bochev-Buffa (CMAME 2006)

$$\Longrightarrow B_2(v) = -(v)_{dw} \equiv -\{v\} + \frac{\mathbf{n}^+}{2} \cdot [\![v]\!]$$

$$B_1(v) = S_e\,[\![v]\!] + \theta(\varepsilon \boldsymbol{\nabla}_h v)_{upw}$$

$$B_1^D(v) = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v + \theta \varepsilon \boldsymbol{\nabla}_h v \cdot \mathbf{n} \text{ on } \Gamma^-,$$

$$B_1^D(v) = S_e v + \theta \varepsilon \boldsymbol{\nabla}_h v \cdot \mathbf{n} \text{ on } \Gamma^+$$

Choice by Hughes-Scovazzi-Bochev-Buffa (CMAME 2006)

$$
\begin{cases}
\text{Find } u \in H^2(\mathscr{T}_h) \text{ such that } \forall v \in H^2(\mathscr{T}_h) \\
\int_\Omega (\gamma uv - \boldsymbol{\sigma}(u)\cdot\boldsymbol{\nabla}_h v) + \sum_{e\in\mathcal{E}_h} S_e \int_e [\![u]\!]\cdot[\![v]\!] + \sum_{e\in\mathcal{E}_h^\circ}\int_e (\boldsymbol{\beta} u)_{upw}\cdot[\![v]\!] \\
+ \sum_{e\in\mathcal{E}_h^\circ}\int_e (\theta[\![u]\!]\cdot(\varepsilon\boldsymbol{\nabla}_h v)_{upw} - (\varepsilon\boldsymbol{\nabla}_h u)_{upw}\cdot[\![v]\!]) \\
+ \sum_{e\in\Gamma}\int_e \theta u(\varepsilon\boldsymbol{\nabla}_h v\cdot\mathbf{n}) - (\varepsilon\boldsymbol{\nabla}_h u\cdot\mathbf{n})v + \sum_{e\in\Gamma^+}\int_e \boldsymbol{\beta}\cdot\mathbf{n}uv \\
= \int_\Omega fv + \sum_{e\in\Gamma}\int_e g(S_e v + \theta\varepsilon\boldsymbol{\nabla} v\cdot\mathbf{n}) - \sum_{e\in\Gamma^-}\int_e g\boldsymbol{\beta}\cdot\mathbf{n}v.
\end{cases}
$$

$\theta = -1 \longrightarrow$ symmetric, $\theta = 1 \longrightarrow$ nonsymmetric, $\theta = 0 \longrightarrow$ neutral

Buffa, Hughes & Sangalli (SINUM07)

Another Choice

- Weighted Average: Let $e = \partial K^+ \cap \partial K^-$,

$\alpha = (\alpha^+,\ \alpha^-)$, $\alpha^+,\ \alpha^- \in [0,1]$, with $\alpha^+ + \alpha^- = 1$:

$$\{\varphi\}_\alpha = \alpha^+\varphi^+ + \alpha^-\varphi^-$$

$$\alpha^+ = \alpha^- = 1/2 \longrightarrow \text{ arithmetic average}$$

$$\{\varphi\}_\alpha = \{\varphi\} + \frac{[\![\alpha]\!]}{2}[\![\varphi]\!]$$

Let K^+ be an upwind element $(\boldsymbol{\beta} \cdot \mathbf{n}^+ > 0)$ then $\alpha = (1,0)$

$$\{\varphi\}_\alpha \equiv (\varphi)_{upw} = \varphi^+ \qquad \{\varphi\}_{1-\alpha} \equiv (\varphi)_{dw} = \varphi^-$$

$$\alpha^i = (sign(\boldsymbol{\beta} \cdot \mathbf{n}^i) + 1)/2 \longrightarrow \text{ classical upwind}$$

Another choice

$$B_1 v = S_e [\![v]\!] + \theta(\{\boldsymbol{\sigma}(v)\}_\alpha - \{\boldsymbol{\beta} v\}),$$

$$B_2 v = -\{v\}_{1-\alpha} \equiv -\{v\} + \frac{[\![\alpha]\!]}{2} [\![v]\!],$$

$$B_1^D v = S_e v - \boldsymbol{\beta} \cdot \mathbf{n} v - \theta \varepsilon \boldsymbol{\nabla}_h v \cdot \mathbf{n} \text{ on } \Gamma^-,$$

$$B_1^D v = S_e v - \theta \varepsilon \boldsymbol{\nabla}_h v \cdot \mathbf{n} \text{ on } \Gamma^+ .$$

back

Another choice

$$
\begin{cases}
\text{Find} \quad u \in H^2(\mathscr{T}_h) \quad \text{such that} \quad \forall\, v \in H^2(Th) \\
\int_\Omega (\gamma uv - \boldsymbol{\sigma}(u)\cdot \nabla_h v) + \sum_{e\in\mathcal{E}_h} S_e \int_e [\![u]\!]\cdot[\![v]\!] - \sum_{e\in\mathcal{E}_h^\circ} \int_e \theta\, [\![u]\!]\cdot\{\boldsymbol{\beta} v\} \\
+ \sum_{e\in\mathcal{E}_h^\circ} \int_e (\{\boldsymbol{\sigma}(u)\}_\alpha \cdot [\![v]\!] + \theta\, [\![u]\!]\cdot\{\boldsymbol{\sigma}(v)\}_\alpha) + \sum_{e\in\Gamma^+} \int_e \boldsymbol{\beta}\cdot\mathbf{n} uv \\
- \sum_{e\in\Gamma} \int_e (\varepsilon \nabla_h u\cdot \mathbf{n}\, v + \theta u\, \varepsilon \nabla_h v\cdot\mathbf{n}) \\
= \int_\Omega fv + \sum_{e\in\Gamma}\int_e g(S_e v - \theta\varepsilon\nabla_h v\cdot\mathbf{n}) - \sum_{e\in\Gamma^-}\int_e \boldsymbol{\beta}\cdot\mathbf{n} gv
\end{cases}
$$

$\theta = 1 \longrightarrow$ symmetric, $\theta = 0 \longrightarrow$ neutral

$\theta = -1 \longrightarrow$ nonsymmetric, but stability in a weaker norm $\Longrightarrow$ suboptimal order of convergence.

back