Thermoelastic Systems

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Thermoelasic systems

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u_{tt}(t,x) - u_{xx}(t,x) + m\theta_x(t,x) = 0, & 0 \le x \le 1, \\
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$$A := -\Delta_D. \qquad C := -\triangle_D,$$

$$B := m\nabla, \qquad D(B) = H_0^1([0,1]).$$

Abstract thermoelasic systems

$$(1) \quad \begin{cases} u_{tt} + Au + B\theta = 0 & (\text{in } H_1) \\ \theta_t + C\theta - B^* u_t = 0 & (\text{in } H_2) \end{cases}$$

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- $B : D(B) \subset H_2 \longrightarrow H_1$: closed operator.
- $D(C^{\frac{1}{2}}) \subset D(B)$ and $D(A^{\frac{1}{2}}) \subset D(B^*)$.

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The decoupled abstract system :

$$(2) \quad \begin{cases} u_{tt} + Au + BC^{-1}B^* u_t = 0 & (\text{ in } H_1) \\ \theta_t + C\theta - B^* u_t = 0 & (\text{ in } H_2) \end{cases}$$

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$$(2) \longleftrightarrow \mathcal{A}_2 = \begin{pmatrix} 0 & I & 0 \\ -A & -BC^{-1}B^* & 0 \\ 0 & B^* & -C \end{pmatrix}, \quad D(\mathcal{A}_2) = D(\mathcal{A}_1)$$

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Theorem 1 :

$\mathcal{A}_1$ and $\mathcal{A}_2$ generate contraction semigroups

$(T_1(t))_{t\geq 0}$ and $(T_2(t))_{t\geq 0}$.

D.B. Henry, Lopes, Perissinitto

Theorem 2 :

If $BC^{-\gamma}$ **is compact** for some $0 < \gamma < 1$,

then

$T_1(t) - T_2(t)$ is compact on $\mathbb{H}$ for $t > 0$.

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Theorem 2 :

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(ii) $R(\lambda, \mathcal{A}_1) - R(\lambda, \mathcal{A}_2)$ is compact $\forall \lambda \in \rho(\mathcal{A}_1) \cap \rho(\mathcal{A}_2)$,

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$\Delta(t)$ is compact for $t \geq 0$.

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The map $t \longmapsto T_1(t) - T_2(t)$ is norm continuous on $(0, \infty)$.

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Proof : $x_0 = (u_0, v_0, \theta_0) \in D(L) : \ \|x_0\| \leq 1$.

$$T_1(t)(u_0, v_0, \theta_0) - T_2(t)(u_0, v_0, \theta_0) = \begin{pmatrix} u(t) - \overline{u}(t) \\ v(t) - \overline{v}(t) \\ \theta(t) - \overline{\theta}(t) \end{pmatrix}.$$

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- $f(s) = B\overline{\theta}(s) - B^{-1} C B^* \overline{v}(s)$

Lemma 5 :

Assume $A^{-\frac{1}{2}}BC^{-1}$ is compact from H_2 to H_1.

Then,

$R(\lambda, \mathcal{A}_1) - R(\lambda, \mathcal{A}_2)$ is also compact on $\mathbb{H}$ for $\lambda \in \rho(\mathcal{A}_1) \cap \rho(\mathcal{A}_2)$.

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Proof :

$$\mathcal{A}_1^{-1} - \mathcal{A}_2^{-1} = \begin{pmatrix} 0 & 0 & A^{-1}BC^{-1} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

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Assume that $A^{-\frac{1}{2}}BC^{-1}$ is compact. Then

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Remark :

$BC^{-\gamma}$ is compact $\quad\Longrightarrow\quad$ $A^{-\frac{1}{2}}BC^{-1}$ is compact

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- Δ_D, $H_1 = L_2(\Omega, \mathbb{R}^2)$, has compact resolvent.

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\begin{cases}
u_{tt}(t,x) - \Delta u(t,x) - \beta \nabla(\text{ div } u(t,x)) + m \nabla \theta(t,x) = 0, & x \in \Omega, \\
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β, m, k are positive constants.

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$$C = k - \triangle_N, \qquad B = m\nabla, \qquad D(B) = H^1(\Omega).$$

$$B^* = -m \text{ div }, \qquad D(B^*) = \{u \in H^1(\Omega, \mathbb{R}^2) : u \cdot \overrightarrow{n} = 0 \text{ in } \partial\Omega\}$$

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$$\begin{aligned} C^{-1}B^*BC^{-1} &= m^2C^{-1}(-\Delta_N)C^{-1} \\ &= m^2\, C^{-1}(C-kI)C^{-1} \\ &= m^2\,(C^{-1}-kC^{-2}). \end{aligned}$$

We have obtained the same results for the thermo-viscoelastic systems (cf. W. J. Liu)

$$\begin{cases} u_{tt}(t,x) - \Delta u(t,x) + \int_{-\infty}^{t} g(t-s)\triangle u(s,x)\,ds + \nabla\theta(t,x) = 0, x \in \Omega, \\ \\ \theta_t(t,x) - \Delta\theta(t,x) + k\theta(t,x) + \text{ div } u_t(t,x) = 0, x \in \Omega, \\ \\ u(t,x) = 0, \quad \frac{\partial\theta}{\partial n}(t,x) = 0, t \geq 0,\ x \in \partial\Omega, \end{cases}$$