Global controllability for Burgers equation

Jean-Pierre Puel *

Work in collaboration with O.Yu. Imanuvilov

Introduction

Case of Navier-Stokes Equations

$(\bar{\mathbf{y}}, \bar{p})$: "ideal" solution of Navier-Stokes equations (for example a stationnary solution).

$$\left\{\begin{array}{l} \frac{\partial \bar{\mathbf{y}}}{\partial t} - \nu \Delta \bar{\mathbf{y}} + \bar{\mathbf{y}}.\nabla \bar{\mathbf{y}} + \nabla \bar{p} = \mathbf{f} \text{ in } \Omega \times (0,T), \\ \operatorname{div} \bar{\mathbf{y}} = 0 \text{ in } \Omega \times (0,T), \\ \bar{\mathbf{y}} = 0 \text{ on } \Gamma \times (0,T) \\ \bar{\mathbf{y}}(0) = \bar{\mathbf{y}}_0 \text{ in } \Omega. \end{array}\right. \tag{1}$$

Consider a solution of the controlled system, starting from a different initial value

$$\left\{\begin{array}{l} \frac{\partial \mathbf{y}}{\partial t} - \nu \Delta \mathbf{y} + \mathbf{y}.\nabla \mathbf{y} + \nabla p = \mathbf{f} + \mathbf{v}.1\!\!\mathrm{I}_\omega \text{ in } \Omega \times (0,T), \\ \operatorname{div} \mathbf{y} = 0 \text{ in } \Omega \times (0,T), \\ \mathbf{y} = 0 \quad \text{on} \quad \Gamma \times (0,T) \\ \mathbf{y}(0) = \mathbf{y}_0 \text{ in } \Omega, \end{array}\right. \tag{2}$$

$1\!\!\mathrm{I}_\omega$: characteristic function of a (little) subset ω of Ω.

Exact Controllability to Trajectories :

Can we find a control $\mathbf{v}$ such that

$$\mathbf{y}(T) = \bar{\mathbf{y}}(T) \ ?$$

i.e can we reach exactly in finite time the "ideal" trajectory $\bar{\mathbf{y}}$?

Local version : same result provided $||\mathbf{y}_0 - \bar{\mathbf{y}}_0||$ is small enough.

Last result (Fernandez-Cara, Guerrero, Imanuvilov, Puel, Journal de Math. Pures et Appl., 2004) (dimension 3) : Local exact controllability to trajectories.

$$H = \{\mathbf{y} \in L^2(\Omega)^3,\ \mathrm{div}\,\mathbf{y} = 0,\ \mathbf{y}.\nu = 0 \text{ on } \Gamma\}.$$

Theorem 1 *Let us assume that*

$$\bar{\mathbf{y}}_\mathbf{0} \in H \cap L^4(\Omega)^3, \quad \bar{\mathbf{y}} \in L^\infty(\Omega \times (0,T))^3$$

and

$$\frac{\partial \bar{\mathbf{y}}}{\partial t} \in L^2(0,T;L^\sigma(\Omega))^3,\ \sigma > \frac{6}{5}$$

then there exists $\eta > 0$ *such that for every* $\mathbf{y}_0 \in H \cap L^4(\Omega)^3$ *such that* $||\mathbf{y}_0 - \bar{\mathbf{y}}_0||_{L^4(\Omega)^3} \leq \eta$, *there exists a control* $\mathbf{v} \in L^2(0,T;L^2(\omega))^3$ *and a solution* $(\mathbf{y},p)$ *of (2) such that*

$$\mathbf{y}(T) = \bar{\mathbf{y}}(T).$$

Among open problems :

Can the result be global (at least to achieve 0)?

Open problem except for control on the whole boundary : combining results of Coron for approximate controllability and a local exact controllability result (Fursikov-Imanuvilov or result mentionned above).

Can we use a more “nonlinear” method ?

Case of Burgers Equations

For 1-d Burgers equation : counter-example due to Guerrero-Imanuvilov.
Therefore no global exact controllability.

Global exact boundary controllability for the 2-d Burgers equation

$$\frac{\partial u}{\partial t} - \Delta u + \frac{\partial u^2}{\partial x_1} + \frac{\partial u^2}{\partial x_2} = f \quad \text{in } Q = (0,T) \times \Omega, \tag{3}$$

$$u|_{\Gamma_0} = 0, \quad u|_{\Gamma_1} = h, \tag{4}$$

$$u(0,\cdot) = u_0, \tag{5}$$

$$u(T,\cdot) = 0. \tag{6}$$

Without loss of generality we may assume that Ω is included in the rectangle $0 \leq x_2 - x_1 \leq A, \quad -B \leq x_1 + x_2 \leq B$ with A and B two positive constants.

Theorem 2 *Let us assume that*

$$\Gamma_0 \subset \{x \in \Gamma \mid x_1 - x_2 = 0\} \tag{7}$$

(or Γ_0 *is empty which is allowed). Suppose that* $f \in L^2(0,T;L^2(\Omega))$ *and that there exists* $T_0 \in (0,T)$ *such that* $f(t,x) = 0, \forall t \geq T_0$*. Then for every* $u_0 \in L^2(\Omega)$ *there exists a solution* $u \in L^2(0,T;H^1_{\Gamma_0}(\Omega)) \cap C([0,T];L^2(\Omega))$ *such that* $t^2.u \in H^{1,2}(Q) = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega))$ *to problem (3)-(5) satisfying (6) (and a corresponding control* h*).*

Proof : related to the return method by Coron but different. Use of a special solution of Burgers equation that we can drive to zero whenever we want.

First of all some existence and regularity results for Burgers equations (good exercises !!)

Proposition 3 *For every* $f \in L^2(0,T;H^{-1}(\Omega))$ *and* $u_0 \in L^2(\Omega)$ *there exists a unique solution* u *to 2-D Burgers equation with* $u \in L^2(0,T;H_0^1(\Omega)) \cap C([0,T];L^2(\Omega))$ *and we have*

$$||u||_{L^2(0,T;H_0^1(\Omega))} + ||u||_{C([0,T];L^2(\Omega))} \le C(|u_0|_{L^2(\Omega)} + ||f||_{L^2(0,T;H^{-1}(\Omega))}).$$

If $f \in L^2(0,T;L^2(\Omega))$ *and* $u_0 \in H_0^1(\Omega)$ *then* $u \in H^{1,2}(Q) = H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^2(\Omega) \cap H_0^1(\Omega))$ *and we have*

$$||u||_{H^{1,2}(Q)} \le C(||u_0||_{H_0^1(\Omega)} + |f|_{L^2(0,T;L^2(\Omega))} + ||u_0||^5_{H_0^1(\Omega)} + |f|^5_{L^2(0,T;L^2(\Omega))}).$$

Proposition 4 *Let us assume that* $f \in L^2(0,T;L^2(\Omega))$ *and that* $u_0 \in L^2(\Omega)$. *Then* $t^2.u \in H^{1,2}(Q)$ *which implies that for every* $\eta > 0$, $u \in C([\eta,T];H_0^1(\Omega)) \cap L^2(\eta,T;H^2(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^2(\eta,T;L^2(\Omega))$. *Moreover we have the following estimate*

$$||t^2.u||_{H^{1,2}(Q)} \leq C(|u_0|_{L^2(\Omega)} + |f|_{L^2(0,T;L^2(\Omega))} + \\ +|u_0|^{13}_{L^2(\Omega)} + |f|^{13}_{L^2(0,T;L^2(\Omega))}). \tag{8}$$

On the time interval $(0, T_0)$ set $h(t,x) = 0$ and leave the system evolve without control. For every $\eta > 0$, we have

$$u \in C([\eta, T_0]; H_0^1(\Omega)) \cap L^2(\eta, T_0; H^2(\Omega)), \; \frac{\partial u}{\partial t} \in L^2(\eta, T_0; L^2(\Omega))$$

and we write

$$u(T_0, \cdot) = u_1 \in H_0^1(\Omega) \subset L^p(\Omega), \; \forall p, \; 1 \leq p < +\infty.$$

Now we set

$$\delta_0 = \frac{T - T_0}{4} > 0.$$

We will construct a solution u in the interval $(T_0, T_0 + 3\delta_0)$ (and a corresponding control) such that $u(T_0 + 3\delta_0, \cdot)$ is as small as desired in the norm $H_0^1(\Omega)$.

First of all we construct a very specific solution U of the 2-d Burgers equation.

Let $w(t,z)$ be a solution to the heat equation

$$\frac{\partial w}{\partial t} - 2\frac{\partial^2 w}{\partial z^2} = 0 \quad z \in (0,A),\ t > T_0, \tag{9}$$

$$w(t,0) = 0, \quad w(t,A) = v(t), \tag{10}$$

$$w(T_0,\cdot) = 0, \tag{11}$$

where $v(\cdot)$ is a boundary control which will be determined later on. This control will be chosen regular so that w will also be regular.

We now set

$$U(t,x) = w(t, x_2 - x_1). \tag{12}$$

We have

$$\frac{\partial U}{\partial x_1} + \frac{\partial U}{\partial x_2} = 0, \ \frac{\partial U^2}{\partial x_1} + \frac{\partial U^2}{\partial x_2} = 0$$

so that for every $N > 0$, $N.U$ is a regular solution of the 2-d Burgers equation

$$\begin{aligned} &\frac{\partial (N.U)}{\partial t} - \Delta(N.U) + \frac{\partial (N.U)^2}{\partial x_1} + \frac{\partial (N.U)^2}{\partial x_2} = 0 \quad \text{in } (T_0, T) \times \Omega, \\ &N.U|_{\Gamma_0} = 0, \\ &N.U(T_0, \cdot) = 0. \end{aligned}$$

Notice that the value of $N.U$ on $(T_0, T) \times \Gamma_1$, which will be a boundary control h and which depends on v, does not appear explicitely. If δ is any number such that $0 < \delta \leq \delta_0$, from the controllability results for the heat equation, we can choose this control h (and in fact v) on $(T_0 + \delta, T_0 + 2\delta_0)$ such that

$$N.U(T_0 + 2\delta_0, \cdot) = 0.$$

On the interval $(T_0, T_0 + 2\delta_0)$ we look for u in the form

$$u = y + N.U, \tag{13}$$

where N is a large parameter to be determined later on and y is chosen to vanish on the whole boundary Γ.

Therefore, y must satisfy the following equation

$$\frac{\partial y}{\partial t} - \Delta y + 2N.U(\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2}) + \frac{\partial y^2}{\partial x_1} + \frac{\partial y^2}{\partial x_2} = 0 \quad \text{in } (T_0, T_0 + 2\delta_0) \times \Omega, \tag{14}$$

$$y|_\Gamma = 0, \tag{15}$$

$$y(T_0, \cdot) = u_1. \tag{16}$$

Lemma 5 *There exists a unique solution y to (14), (15), (16) with $y \in C([T_0, T_0+2\delta_0]; H_0^1(\Omega)) \cap L^2(T_0, T_0+2\delta_0; H^2(\Omega))$, $\frac{\partial y}{\partial t} \in L^2(T_0, T_0+2\delta_0; L^2(\Omega))$ and for every t_0, t_1 with $T_0 \leq t_0 \leq t_1 \leq T_0 + 2\delta_0$ and every $p \geq 1$ we have*

$$\|y(t_1, \cdot)\|_{L^p(\Omega)} \leq \|y(t_0, \cdot)\|_{L^p(\Omega)}. \tag{17}$$

Proof.

Existence, uniqueness and regularity of y is classical as (14) is essentially a Burgers equation. To show that the L^p-norm of y is decreasing, multiply equation (14) by $|y|^{p-2}y$ with $p \geq 1$. We obtain

$$\frac{1}{p}\frac{d}{dt}\int_\Omega |y|^p dx + (p-1)\int_\Omega |y|^{p-2}|\nabla y|^2 dx = 0$$

since

$$\int_\Omega U.(\frac{\partial y}{\partial x_1} + \frac{\partial y}{\partial x_2})|y|^{p-2}y dx = \frac{1}{p}\int_\Omega U.(\frac{\partial |y|^p}{\partial x_1} + \frac{\partial |y|^p}{\partial x_2})dx = 0$$

and

$$\int_\Omega (\frac{\partial y^2}{\partial x_1} + \frac{\partial y^2}{\partial x_2})|y|^{p-2}y dx = \frac{2}{p+1}\int_\Omega (\frac{\partial |y|^p y}{\partial x_1} + \frac{\partial |y|^p y}{\partial x_2})dx = 0.$$

Let us now define a function β by

$$\beta(x) = C_0 - x_1 - x_2,$$

where C_0 is chosen such that

$$\exists \beta_0 > 0,\ \forall x \in \Omega,\ \beta(x) \geq \beta_0.$$

We also write

$$\beta_1 = \max_{x \in \overline{\Omega}} \beta(x).$$

Lemma 6 *The solution y of (14), (15), (16) satisfies the following differential inequality*

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \beta|y|^2 dx + \int_\Omega \beta|\nabla y|^2 dx + \frac{2}{\beta_1}\int_\Omega (N.U)\beta|y|^2 dx \le \frac{4}{3}\int_\Omega |u_1|^3 dx. \quad (18)$$

Proof.

Multiply equation (14) by βy. We obtain, as $\Delta\beta = 0$ and $\frac{\partial\beta}{\partial x_1} + \frac{\partial\beta}{\partial x_2} = -2$,

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \beta|y|^2 dx + \int_\Omega \beta|\nabla y|^2 dx + 2\int_\Omega (N.U)|y|^2 dx + \frac{4}{3}\int_\Omega |y|^2 y dx = 0.$$

Using Lemma 5 with $p = 3$ we obtain the desired result.

Notice that up to this point the control v has not been chosen.

In the case when Γ_0 is empty which means that we can apply a control on the whole boundary, we don't have to take the boundary condition $w(t,0) = 0$ and we can take w such that $\min_{x\in\Omega} U(t,x) \geq \min_{z\in(0,A)} w(t,z) \geq \alpha(t) > 0$ if $t > T_0$, which ensures that U has a strictly positive minimum when $t > T_0$.

When Γ_0 is not empty, due to the boundary condition $w(t,0) = 0$ we cannot have a strictly positive minimum for U over Ω.

Let us now make a choice for w and v. On the interval $(T_0, T_0 + \delta)$, where $0 < \delta \leq \delta_0$, we set

$$w(t,z) = \frac{1}{\sqrt{(t - T_0)}}(e^{-\frac{(z-5A)^2}{8(t-T_0)}} - e^{-\frac{(z+5A)^2}{8(t-T_0)}}). \tag{19}$$

We can see that w satisfies (9), (10) with a suitable control v and (11).

For $0 < a \leq z \leq A$ we have

$$w(t,z) \geq w(t,a) = \frac{2}{\sqrt{(t-T_0)}} e^{-\frac{(a^2+25A^2)}{8(t-T_0)}} \sinh(\frac{5Aa}{4(t-T_0)})$$

$$\geq \frac{5Aa}{2(t-T_0)^{\frac{3}{2}}} e^{-\frac{(a^2+25A^2)}{8(t-T_0)}}.$$

At the same time we also have

$\exists C_0 > 0,\ \forall a \in (0, A),\ \forall t \in (T_0, T_0{+}\delta),\ \forall z,\ 0 \le z \le a,\ w(t,z) \le w(t,a) \le C_0 a.$

We will write

$$\Omega_a = \{x \in \Omega,\ 0 \le x_2 - x_1 \le a\}$$

and we have

$$|\Omega_a| \le Ca,$$

and

$$\min_{x \in \Omega \setminus \Omega_a} U(t,x) \ge w(t,a) \ge \frac{5Aa}{2(t-T_0)^{\frac{3}{2}}} e^{-\frac{(a^2+25A^2)}{8(t-T_0)}}.$$

Therefore, from (18), we obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega \beta|y|^2dx + \int_\Omega \beta|\nabla y|^2dx + \frac{5NAa}{\beta_1(t-T_0)^{\frac{3}{2}}}e^{-\frac{(a^2+25A^2)}{8(t-T_0)}}\int_\Omega \beta|y|^2dx$$
$$\leq \frac{4}{3}\int_\Omega |u_1|^3dx + 2N\int_{\Omega_a} w(t,a)|y|^2dx$$
$$\leq \frac{4}{3}\int_\Omega |u_1|^3dx + 2Nw(t,a)|\Omega_a|^{\frac{1}{3}}(\int_\Omega |y|^3dx)^{\frac{2}{3}}$$
$$\leq \frac{4}{3}\int_\Omega |u_1|^3dx + CNa^{\frac{4}{3}}(\int_\Omega |u_1|^3dx)^{\frac{2}{3}}.$$

We now take

$$a = \frac{1}{N^{\frac{3}{4}}}$$

which implies the following differential inequality

$$\frac{d}{dt}\int_\Omega \beta|y|^2 dx \le -\frac{10N^{\frac{1}{4}}A}{\beta_1(t-T_0)^{\frac{3}{2}}} e^{-\frac{26A^2}{8(t-T_0)}} \int_\Omega \beta|y|^2 dx + C(\|u_1\|_{L^3(\Omega)}).$$

Using Gronwall Lemma, integrating this inequality on $(T_0, T_0+\delta)$, we obtain

$$\int_\Omega \beta|y(T_0+\delta, x)|^2 dx \le (\int_\Omega \beta|u_1|^2 dx) e^{-N^{\frac{1}{4}}g(\delta)} + \delta C(\|u_1\|_{L^3(\Omega)})$$

where for δ small enough

$$g(\delta) = \int_{T_0}^{T_0+\delta} \frac{10A}{\beta_1(t-T_0)^{\frac{3}{2}}} e^{-\frac{26A^2}{8(t-T_0)}} dt \ge Ce^{-\frac{A^2}{\delta}} > 0$$

This implies

$$\int_\Omega |y(T_0+\delta,x)|^2 dx \le \frac{\beta_1}{\beta_0}\|u_1\|^2_{L^2(\Omega)} e^{-N^{\frac{1}{4}}g(\delta)} + \frac{\delta}{\beta_0} C(\|u_1\|_{L^3(\Omega)})$$

and, choosing first δ sufficiently small then N sufficiently large we have proved the following

Proposition 7 *Given u_1 in $H_0^1(\Omega)$ (in fact $u_1 \in L^3(\Omega)$ would be enough), for every $\delta_0 > 0$ and for every $\epsilon_0 > 0$, there exists δ with $0 < \delta \le \delta_0$ and there exists N sufficiently large such that*

$$\|y(T_0+\delta,\cdot)\|_{L^2(\Omega)} \le \epsilon_0.$$

Now we choose the control v on the time interval $(T_0+\delta, T_0+2\delta_0)$ in (10) such that w satisfies

$$w(T_0+2\delta_0,\cdot) = 0.$$

This is possible using classical results on null controllability for the heat equation. Then we also have

$$U(T_0 + 2\delta_0, \cdot) = 0.$$

Therefore,

$$\|u(T_0 + 2\delta_0, \cdot)\|_{L^2(\Omega)} = \|y(T_0 + 2\delta_0, \cdot)\|_{L^2(\Omega)} \leq \|y(T_0 + \delta, \cdot)\|_{L^2(\Omega)} \leq \epsilon_0.$$

Notice that ϵ_0 can be chosen as small as we wish. At this point we only know that the $L^2(\Omega)$-norm of $u(T_0 + 2\delta_0, \cdot)$ is as small as we wish.

On the interval $(T_0+2\delta_0, T_0+3\delta_0)$ we let the system evolve freely and we take the boundary control equal zero. Then using the regularizing effect of Burgers equation we see that at time $T_0+3\delta_0$ we have

$$||u(T_0+3\delta_0,\cdot)||_{H_0^1(\Omega)} \leq \epsilon_1,$$

where ϵ_1 can be taken as small as we wish provided ϵ_0 is small enough.

Therefore, on the time interval $(T_0+3\delta_0, T)$ we can use a result of local exact controllability to trajectories for 2-d Burgers equations (not completely trivial !) to find a boundary control h such that

$$u(T,\cdot)=0.$$

A situation without global controllability

Theorem 2 was proved under the restrictive assumption (7) on the boundary Γ_0. The next result shows that without this assumption the global controllability property may fail.

Let us suppose that the geometrical situation is such that there exists a function $\rho(x) \in C^2(\bar{\Omega})$ such that

$$\rho|_{\Gamma_1} = 0, \quad \rho(x) > 0 \text{ in } \Omega, \quad \frac{\partial \rho}{\partial x_1} + \frac{\partial \rho}{\partial x_2} < 0 \quad \forall x \in \bar{\Omega}. \tag{20}$$

Of course this cannot occur in the situation considered in the previous section, but there are many cases where such a function ρ exists, for example when $\Omega = \{(x_1, x_2),\ 0 < x_2 - x_1 < 1,\ -1 < x_1 + x_2 < 1\}$ and $\Gamma_1 = \{(x_1, x_2),\ 0 < x_2 - x_1 < 1,\ x_1 + x_2 = 1\}$.

For a function v defined on Ω or $(0, T) \times \Omega$ we set

$$v^+ = \max(v, 0),\ v^- = (-v)^+.$$

Theorem 8 *Suppose that condition (20) holds true. Let* $f \in L^2(0,T;L^2(\Omega))$ *and* $u_0 \in H_0^1(\Omega)$ *such that* $u_0^- \neq 0$. *Then there exists a time* $T_0(u_0^-, f) > 0$ *such that for each* $T \leq T_0(u_0^-, f)$ *there is no solution to problem (3)-(5) in the space* $u \in H^{1,2}(Q)$ *satisfying (6).*

Proof. We argue by contradiction. Let $u_0 \in H_0^1(\Omega)$ and $f \in L^2(0,T;L^2(\Omega))$ be given functions. Suppose that there exists a solution u to (3)-(6). Then we consider the function $y(t,x) = u(t,x) - u_0(x)$ which satisfies the following system of equations

$$\frac{\partial y}{\partial t} - \Delta y + \frac{\partial y^2}{\partial x_1} + \frac{\partial y^2}{\partial x_2} + 2\frac{\partial(yu_0)}{\partial x_1} + 2\frac{\partial(yu_0)}{\partial x_2} = q \quad \text{in } (0,T) \times \Omega,$$
$$y|_{\Gamma_0} = 0, \quad y|_{\Gamma_1} = h \quad y(0,\cdot) = 0,$$
$$y(T,\cdot) = -u_0,$$

where

$$q = \Delta u_0 - \frac{\partial u_0^2}{\partial x_1} - \frac{\partial u_0^2}{\partial x_2} + f.$$

We set

$$\rho_1(x) = \rho(x)^4.$$

Multiplying the equation by $\rho_1 y^+$ and integrating by parts we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho_1|y^+|^2dx + \int_\Omega (\rho_1|\nabla y^+|^2 - \frac{\Delta\rho_1}{2}|y^+|^2 - \frac{2}{3}(\frac{\partial\rho_1}{\partial x_1} + \frac{\partial\rho_1}{\partial x_2})(y^+)^3)dx \\
&+ \int_{\Gamma_0}\frac{1}{2}\frac{\partial\rho_1}{\partial n}|y^+|^2d\sigma - 2\int_\Omega ((\frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2})\rho_1 u_0 y^+ - u_0(\frac{\partial\rho_1}{\partial x_1} + \frac{\partial\rho_1}{\partial x_2})|y^+|^2)dx \\
&= \int_\Omega f\rho_1 y^+ dx - \int_\Omega \nabla u_0.\nabla y^+ \rho_1 dx - \int_\Omega \nabla u_0.\nabla\rho_1 y^+ dx \\
&+ \int_\Omega u_0^2 y^+(\frac{\partial\rho_1}{\partial x_1} + \frac{\partial\rho_1}{\partial x_2})dx + \int_\Omega u_0^2\rho_1(\frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2})dx \\
&\leq \int_\Omega f\rho_1 y^+ dx - \int_\Omega \nabla u_0.\nabla y^+ \rho_1 dx - \int_\Omega \nabla u_0.\nabla\rho_1 y^+ dx \\
&+ \int_\Omega u_0^2\rho_1(\frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2})dx.
\end{aligned}
$$

By (20) we have $\int_{\Gamma_0} \frac{1}{2}\frac{\partial \rho_1}{\partial \vec{n}} |y^+|^2 d\sigma = 0$. Again using (20) we may assume that for some positive constant M we have $-\frac{2}{3}(\frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2}) > M\rho_1^{\frac{3}{4}}$ for all $x \in \bar{\Omega}$. Then denoting by C_i various constants independent of y and u_0 we have

$$\int_\Omega (-\frac{\Delta \rho_1}{2}|y^+|^2 - \frac{2}{3}(\frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2})(y^+)^3)dx \geq \int_\Omega (-C_0 \rho_1^{\frac{1}{2}}|y^+|^2 + M\rho_1^{\frac{3}{4}}(y^+)^3)dx$$

$$\geq -C_1(\int_\Omega \rho_1^{\frac{3}{4}}(y^+)^3 dx)^{\frac{2}{3}} + M\int_\Omega \rho_1^{\frac{3}{4}}(y^+)^3 dx \geq \frac{3M}{4}\int_\Omega \rho_1^{\frac{3}{4}}(y^+)^3 dx - C_2.$$

Then we have

$$2\int_\Omega (\frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2})\rho_1 u_0 y^+ dx \le \frac{1}{4}\int_\Omega \rho_1 |\nabla y^+|^2 dx + C_3 \int_\Omega u_0^2 \rho_1 |y^+|^2 dx$$

$$\le \frac{1}{4}\int_\Omega \rho_1 |\nabla y^+|^2 dx + C_4 ||u_0||^2_{H_0^1(\Omega)} . (\int_\Omega \rho_1^{\frac{3}{4}} (y^+)^3 dx)^{\frac{2}{3}}$$

$$\le \frac{1}{4}\int_\Omega \rho_1 |\nabla y^+|^2 dx + \frac{M}{4}\int_\Omega \rho_1^{\frac{3}{4}} (y^+)^3 dx + C_5 ||u_0||^6_{H_0^1(\Omega)} .$$

Also

$$2\int_\Omega u_0 (\frac{\partial \rho_1}{\partial x_1} + \frac{\partial \rho_1}{\partial x_2}) |y^+|^2 dx \le C_6 ||u_0||_{H_0^1(\Omega)} . (\int_\Omega \rho_1^{\frac{3}{4}} (y^+)^3 dx)^{\frac{2}{3}}$$

$$\le \frac{M}{4}\int_\Omega \rho_1^{\frac{3}{4}} (y^+)^3 dx + C_7 ||u_0||^3_{H_0^1(\Omega)} .$$

We also obtain

$$
\begin{aligned}
&\int_\Omega f\rho_1 y^+ dx - \int_\Omega \nabla u_0.\nabla y^+ \rho_1 dx \\
&- \int_\Omega \nabla u_0.\nabla \rho_1 y^+ dx + \int_\Omega u_0^2 \rho_1 (\frac{\partial y^+}{\partial x_1} + \frac{\partial y^+}{\partial x_2}) dx \\
&\leq C_8(\|f\|^2_{L^2(Q)} + ||u_0||^2_{H_0^1(\Omega)} + ||u_0||^4_{H_0^1(\Omega)}) \\
&+ \frac{1}{2}\int_\Omega \rho_1 |y^+|^2 dx + \frac{1}{4}\int_\Omega \rho_1 |\nabla y^+|^2 dx.
\end{aligned}
$$

Using all these inequalities we obtain

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega \rho_1 |y^+|^2 dx + \int_\Omega \rho_1 |\nabla y^+|^2 dx + \int_\Omega \frac{M}{2}\rho_1^{\frac{3}{4}} (y^+)^3 dx \\
&\leq C_9(1 + \|f\|^2_{L^2(Q)} + ||u_0||^2_{H_0^1(\Omega)} + ||u_0||^6_{H_0^1(\Omega)}) + \int_\Omega \rho_1 |y^+|^2 dx.
\end{aligned}
$$

Applying Gronwall's inequality we obtain, as $y^+(0,\cdot) = 0$,

$$\sup_{t\in(0,T)} \int_\Omega \rho_1 |y^+|^2 dx \leq C_{10}(1 + \|f\|^2_{L^2(Q)} + ||u_0||^2_{H_0^1(\Omega)} + ||u_0||^6_{H_0^1(\Omega)})Te^T.$$

Since the right hand side goes to zero as T goes to zero and $y^+(T) = u_0^-$, we immediately arrive to a contradiction and the proof of Theorem 8 is complete.