Uniqueness of the Cheeger set of a convex body

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Devoted to the memory of Thomas Lachand-Robert

Abstract

We prove that if $C \subset I\!\!R^N$ is a an open bounded convex set, then there is only one Cheeger set inside C and it is convex. The Cheeger set of C is the set which minimizes for sets inside C the ratio perimeter over volume.

1 Introduction

Given an nonempty open bounded subset Ω of $I\!\!R^N$, we call Cheeger constant of Ω the quantity

$$h_\Omega \; = \; \min_{F \subseteq \Omega} \frac{P(F)}{|F|} \,. \tag{1}$$

Here $|F|$ denotes de N-dimensional volume of F and $P(F)$ denotes the perimeter of F. The minimum in (1) is taken over all nonempty sets of finite perimeter contained in Ω. A Cheeger set of Ω is any set $G \subseteq \Omega$ which minimizes (1). If Ω minimizes (1), we say that it is Cheeger in itself. We observe that the minimum in (1) is attained at a subset G of Ω such that ∂G intersects $\partial\Omega$: otherwise we would diminish the quotient $P(G)/|G|$ by dilating G.

For any set of finite perimeter F in $I\!\!R^N$, let us denote

$$\lambda_F := \frac{P(F)}{|F|}.$$

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Notice that for any Cheeger set G of Ω, $\lambda_G = h_G$. Observe also that G is a Cheeger set of Ω if and only if G minimizes

$$\min_{F \subseteq \Omega} P(F) - \lambda_G |F|. \tag{2}$$

We say that a set $\Omega \subset I\!\!R^N$ is calibrable if Ω minimizes the problem

$$\min_{F \subseteq \Omega} P(F) - \lambda_\Omega |F|. \tag{3}$$

In particular, if G is a Cheeger set of Ω, then G is calibrable. Thus, Ω is a Cheeger set of itself if and only if it is calibrable.

Finding the Cheeger sets of a given Ω is a difficult task. This task is simplified if Ω is a convex set and $N = 2$. In that case, the Cheeger set in Ω is unique and is identified with the set $\Omega^R \oplus B(0, R)$ where $\Omega^R := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > R\}$ is such that $|\Omega^R| = \pi R^2$ and $X \oplus Y := \{x + y : x \in X, y \in Y\}$, $X, Y \subset I\!\!R^2$ [2, 22]. We see in particular that it is convex. Moreover, a convex set $\Omega \subseteq I\!\!R^2$ is Cheeger in itself if and only if $\max_{x \in \partial\Omega} \kappa_\Omega(x) \leq \lambda_\Omega$ where $\kappa_\Omega(x)$ denotes the curvature of $\partial\Omega$ at the the point x. This has been proved in [12, 5, 22, 2, 23], though it was stated in terms of calibrability in [5, 2]. The proof in [12] had also a complement result: if Ω is convex and Cheeger in itself, then Ω is strictly calibrable, that is, for any set $F \subset \Omega$, $F \neq \Omega$, then

$$0 = P(\Omega) - \lambda_\Omega |\Omega| < P(F) - \lambda_\Omega |F|,$$

i.e., there is no other Cheeger set inside Ω, and this implies that the capillary problem in absence of gravity (with vertical contact angle at the boundary)

$$\begin{aligned} -\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) &= \lambda_\Omega \quad \text{in } \Omega \\ -\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu^\Omega &= 1 \qquad \text{in } \partial\Omega \end{aligned} \tag{4}$$

has a solution. Indeed, both problems are equivalent [12, 21].

Assuming that $C \subset I\!\!R^N$ is an open uniformly convex set of class C^2, in [7], the authors proved the uniqueness and convexity of the Cheeger set contained in C. As a consequence, this implies the extension of Giusti's results on existence of solutions of problem (4) when $\Omega \subseteq I\!\!R^2$ is convex and calibrable [12] to the case where Ω is the Cheeger set in an open uniformly convex set $C \subset I\!\!R^N$ of class C^2. Our purpose in this paper is to remove these regularity assumptions proving that the uniqueness result of the Cheeger set holds inside any non-trivial convex body of $I\!R^N$.

Recall that a convex body of $I\!R^N$ is a compact convex subset of $I\!R^N$. We say that a convex body is non-trivial if it has nonempty interior.

Theorem 1. *There is a unique Cheeger set inside any non-trivial convex body of $I\!R^N$. The Cheeger set is convex and of class $C^{1,1}$.*

Moreover, the characterization of a calibrable (i.e. Cheeger in itself) non-trivial convex body $\Omega \subset I\!R^N$ of class $C^{1,1}$ in terms of the mean curvature of its boundary was proved in [1]. The precise result states that such a set Ω is calibrable if and only if

$$(N-1)\mathbf{H}_\Omega(x) \leq \lambda_\Omega \qquad \text{for any } x \in \partial\Omega, \tag{5}$$

where $\mathbf{H}_\Omega(x)$ is the mean curvature of $\partial\Omega$ at x (so that $(N-1)\mathbf{H}_\Omega(x)$ denotes the sum of the principal curvatures of the boundary of Ω at x). We observe that this result can be slightly strengthened to say that a non-trivial convex body of $\Omega \subset I\!R^N$ is calibrable if and only if is of class $C^{1,1}$ and (5) holds.

Collecting these results we obtain the full extension of Giusti's results to $I\!R^N$ $(N \geq 2)$, that is, we obtain that $\Omega \subset I\!R^N$ is the unique Cheeger set of itself, whenever Ω is a non-trivial calibrable convex body and those sets are characterized by the bound on the mean curvature (5). We point out that, by Theorems 1.1 and 4.2 in [12], this uniqueness result is equivalent to the existence of a solution $u \in W^{1,\infty}_{\text{loc}}(\Omega)$ of the capillary problem (4).

Let us explain the plan of the paper. In Section 2 we reduce the proof of Theorem 1 to the case of non-trivial $C^{1,1}$ convex bodies. For that we prove the existence of a $C^{1,1}$ maximal Cheeger set inside any non-trivial convex body of $I\!R^N$. The rest of the paper is devoted to the proof of Theorem 1 for non-trivial convex bodies of class $C^{1,1}$. We start in Section 3 by proving some basic linear algebra inequalities to be used in Section 4 to prove the behavior of the mean curvature of the boundary of the convex combination of two smooth strictly convex sets. In Section 5 we prove an auxiliar property, namely that the free boundary of an isoperimetric region inside a convex body of class C^1 is strictly convex. Finally, in Section 6 we prove the uniqueness of Cheeger sets inside non-trivial convex bodies of class $C^{1,1}$.

2 Nontrivial convex bodies contain a maximal $C^{1,1}$ Cheeger set

The purpose of this Section is to prove the existence of a $C^{1,1}$ maximal Cheeger set inside any non-trivial convex body of $I\!R^N$. This reduces the proof of Theorem 1 to the class of $C^{1,1}$ calibrable sets. Let us recall some results proved in [1].

Lemma 2.1. *([1]) Let C be a bounded convex subset of $I\!R^N$. For any $\mu > 0$, the problem the problem*

$$(P)_\mu : \quad \min_{F\subseteq C} P(F) - \mu|F|. \tag{6}$$

has always a minimizer. The following properties hold:

(i) Let C_λ, C_μ be minimizers of $(P)_\lambda$, and $(P)_\mu$ respectively. If $\lambda < \mu$, then $C_\lambda \subseteq C_\mu$.

(ii) Let $\lambda_n \uparrow \lambda$. Then $C_\lambda^\cup := \bigcup_n C_{\lambda_n}$ is a minimizer of $(P)_\lambda$. Moreover $P(C_{\lambda_n}) \to P(C_\lambda^\cup)$. Similarly, if $\lambda_n \downarrow \lambda$, then $C_\lambda^\cap := \bigcap_n C_{\lambda_n}$ is a minimizer of $(P)_\lambda$, and $P(C_{\lambda_n}) \to P(C_\lambda^\cap)$.

(iii) Assume that C has bounded mean curvature. Let $\Lambda := (N-1)\|\mathbf{H}_C\|_\infty$. Then C is a solution of $(P)_\lambda$ for any $\lambda \geq N\Lambda$.

If $C \subseteq I\!R^N$ is be a non-trivial convex body of class $C^{1,1}$, we denote by $\mathbf{H}_C$ the ($\mathcal{H}^{N-1}$-almost everywhere defined) mean curvature of ∂C, nonnegative for convex sets. If C is of class C^2, then $\mathbf{H}_C$ is defined everywhere on ∂C.

Theorem 2. *([1]) Let $C \subseteq I\!R^N$ be a non-trivial convex body of class $C^{1,1}$. Then there is a convex calibrable set $K \subseteq C$ which is the maximal Cheeger set contained in C. Therefore K minimizes*

$$\min_{F\subseteq C} P(F) - \lambda_K|F| \qquad \text{where } \lambda_K := \tfrac{P(K)}{|K|}. \tag{7}$$

For any $\mu > \lambda_K$, there is a unique minimizer C_μ of $(P)_\mu$, the function $\mu \to C_\mu$ is increasing and continuous and $C_\mu \to K$ as $\mu \to \lambda_K+$. Moreover, we have $C_\mu = C$ if and only if $\mu \geq \max(\lambda_C, (N-1)\|\mathbf{H}_C\|_\infty)$.

As a consequence of Theorem 1 (or Theorem 6) we will be able to say that K is the Cheeger set of C and $\lambda_K = h_C$. Let us refine a result proved in [1].

Proposition 2.2. *Let C be a non-trivial convex body of $I\!R^N$. Let $u \in BV(I\!R^N) \cap L^2(I\!R^N)$ be the (unique) solution of the variational problem*

$$(\mathrm{Q})_{\lambda,C}: \quad \min_{u \in BV(I\!R^N) \cap L^2(I\!R^N)} \left\{ \int_{I\!R^N} |Du| + \frac{\lambda}{2} \int_{I\!R^N} (u - \chi_C)^2 \, dx \right\}. \tag{8}$$

Then $0 \le u \le 1$. Let $E_s := [u \ge s]$, $s \in (0,1]$. Then $E_s \subseteq C$, and, for any $s \in (0,1]$, E_s is a minimum of $(P)_\mu$ for $\mu = \lambda(1-s)$. Moreover, each level set E_s is convex and the function u restricted to $[u > 0]$ is concave.

Proof. The facts that $0 \le u \le 1$ and E_s is a solution of (6) with $\mu = \lambda(1-s)$ coincides with Proposition 4 in [1]. The rest of assertions were proved assuming that C is $C^{1,1}$ and for $\lambda \ge 2N(N-1)\|\mathbf{H}_C\|_\infty$. Let us observe how can they be extended to any convex set and any $\lambda > 0$. First we assume that C is $C^{1,1}$ and $\lambda > 0$. We follow the construction in [1]. Let K be the calibrable set contained in C defined in Theorem 2. For each $\mu \in (0,\infty)$ let C_μ be the solution of $(P)_\mu$. We take $C_\mu = \emptyset$ for any $\mu < \lambda_K$, and, by Theorem 2 we have that $C_\mu = C$ for any $\mu \ge \max(\lambda_C, (N-1)\|\mathbf{H}_C\|_\infty)$. Following the approach in [1] (see also [4, 14]), using the monotonicity of C_μ and $|C \setminus \cup\{C_\mu : \mu > 0\}| = 0$, we may define

$$H_C(x) = \begin{cases} -\inf\{\mu : x \in C_\mu\} & \text{if } x \in C \\ \\ 0 & \text{if } I\!R^N \setminus C. \end{cases} \tag{9}$$

Observe that $H_C(x) = -\lambda_K$ for any $x \in K$. Then as it was proved in [1] $u_\lambda(x) := (1 + \lambda H_C(x))^+ \chi_C$ is the solution of $(\mathrm{Q})_{\lambda,C}$ for any $\lambda > 0$. Moreover for $\lambda \ge 2N(N-1)\|\mathbf{H}_C\|_\infty$ we have that $u_\lambda > 0$ and is concave in C. This amounts to say that $H_C(x)$ is also a concave function. Now, this implies that for any $s \in (0,1]$ the level set $[u_\lambda \ge s]$ is convex and u_λ restricted to $[u_\lambda > 0]$ is concave.

Assume that C is any bounded convex set in $I\!R^N$ and $\lambda > 0$. Let C_n be bounded convex subsets of $I\!R^N$ of class $C^{1,1}$ such that $C \subseteq C_n$ and $C_n \to C$ in the Hausdorff distance (such sets exist, see for instance, [26], pp. 158-160, [3, Proposition 1.9], or Lemma 4.3 below). Let $u_{n,\lambda}$, u_λ be the solutions of $(\mathrm{Q})_{\lambda,C}$ and $(\mathrm{Q})_{\lambda,C_n}$, respectively. We know that $0 \le u_\lambda \le u_{n,\lambda} \le 1$, $u_{n,\lambda} = 0$ outside C_n, $u_\lambda = 0$ outside C, and $u_{n,\lambda} \to u_\lambda$ in $L^2(I\!R^N)$. Since the level sets $[u_{n,\lambda} \ge s]$, $\forall s \in (0,1]$, are convex and $u_{n,\lambda}$ restricted to $[u_{n,\lambda} > 0]$ is concave, we deduce that for almost any $s \in (0,1]$ the level sets $[u_\lambda \ge s]$ are convex and u_λ restricted to $[u_\lambda > 0]$ is concave. Hence u_λ is continuous in $[u_\lambda > 0]$ and the level sets $[u_\lambda \ge s]$ are convex for any $s \in (0,1]$. □

Remark 2.3. Notice that, as proved in [1, Lemma 3], we have that $u_\lambda \neq \chi_C$ for any $\lambda > 0$ and $u_\lambda \to \chi_C$ in $L^2(I\!R^N)$ as $\lambda \to \infty$.

Theorem 3. *Let $C \subseteq I\!R^N$ be a non-trivial convex body. For any $\mu > h_C$, there is a unique solution C_μ of $(P)_\mu$ which is convex. The set $K = \cap_{\mu > h_C} C_\mu$ is a solution of $(P)_{h_C}$ which is convex and a maximal Cheeger set. The function $\mu \in [h_C, \infty) \to C_\mu$ is increasing, continuous and $C_\mu \to C$ as $\mu \to \infty$.*

Proof. Notice that the isoperimetric inequality implies that $h_C > 0$ and any Cheeger set has positive measure. Let K' be a Cheeger set of C. Let $\mu > h_C$. Let $\lambda > 0$ be large enough and $s \in (0, 1]$ be such that $\mu = \lambda(1 - s)$. We observe that, using Remark 2.3, by taking $\lambda > 0$ large enough we may assume that $s < \|u_\lambda\|_\infty$. If u_λ is the solution of $(Q)_{\lambda,C}$, then $[u_\lambda \geq s]$ is a solution of $(P)_\mu$ and, by Lemma 2.1.(i), $[u_\lambda \geq s] \supseteq K'$. Thus, $[u_\lambda \geq s]$ is a nonempty convex solution of $(P)_\mu$. Now, if G is any other solution of $(P)_\mu$, then by Lemma (2.1).(i) we have

$$[u_\lambda > s] = \cup_{\epsilon > 0} [u_\lambda \geq s + \epsilon] \subseteq G \subseteq \cap_{\epsilon > 0} [u_\lambda \geq s - \epsilon] = [u_\lambda \geq s]. \tag{10}$$

Since u_λ is concave in $[u_\lambda > 0]$, we have that $G = [u_\lambda > s] = [u_\lambda \geq s]$ modulo a null set. Thus, the solution of $(P)_\mu$ is unique and convex.

By Lemma 2.1.(ii), the set $K = \cap_{\mu > h_C} C_\mu$ is a convex solution of $(P)_{h_C}$. Notice that $P(K) - h_C|K| \leq P(K') - h_C|K'| = 0$. Hence K is a Cheeger set. Notice that, by Lemma 2.1.(i), any Cheeger set is contained in K.

The construction of K, together with the argument in (10) proves that the map $\mu \in [h_C, \infty) \to C_\mu$ is continuous. By Remark 2.3 (Lemma 3 in [1]), we know that $u_\lambda \to \chi_C$ as $\lambda \to \infty$, and this implies that $C_\mu \to C$ as $\mu \to \infty$. □

Remark 2.4. Thanks to Proposition 3 , we may repeat the construction of $H_C(x)$ in the proof of Proposition 2.2 to conclude that $u_\lambda(x) = (1 + \lambda H_C(x))^+ \chi_C$ is the solution of $(\mathrm{Q})_{\lambda,C}$ for any $\lambda > 0$. Moreover, the set $[u_\lambda = \|u_\lambda\|_\infty] = K$.

Remark 2.5. As in [1], we can prove that for any $V \in [|K|, |C|]$ there is a unique solution of the isoperimetric problem with fixed volume

$$\min_{F \subseteq C, |F| = V} P(F). \tag{11}$$

Moreover, this solution is convex.

Proposition 2.6. *The maximal Cheeger set K is $C^{1,1}$.*

Proof. Since K is a solution of $(P)_{h_C}$, classical computations (see, for instance, [29]) it follows that $0 \leq \mathbf{H}_K \leq h_C$. Since K is convex, it follows that K is $C^{1,1}$ (see, for instance, [3, Proposition 1.3] for a more general statement). □

Remark 2.7. As we proved in [1], as a consequence of Theorem 2, if $C \subseteq \mathbb{R}^N$ is non-trivial convex body of class $C^{1,1}$, then C is calibrable if and only if $(N-1)\mathbf{H}_C \leq \lambda_C$. Notice that the proof of Proposition 2.6 implies that if C is non-trivial convex body of $\mathbb{R}^N$, then C is calibrable if and only if C is of class $C^{1,1}$ and $(N-1)\mathbf{H}_C \leq \lambda_C$.

3 Some linear algebra inequalities

We begin with some classical inequalities inside the cone of symmetric positive definite matrices.

Proposition 3.1. *The inversion $A \mapsto A^{-1}$ is strictly convex in $S_N^{++}(\mathbb{R})$, the set of real symmetric positive definite matrices, i.e.* $\forall A, B \in S_N^{++}(\mathbb{R}), A \neq B, \forall \lambda \in (0,1)$, *we have*

$$(\lambda A + (1-\lambda)B)^{-1} - \lambda A^{-1} - (1-\lambda)B^{-1} \in S_N^{++}(\mathbb{R}). \tag{12}$$

Proof. From a classical result on the simultaneous diagonalization of two quadratic forms [11], we know that there exists an invertible matrix P and a diagonal matrix $D = \mathrm{diag}(d_i)_{i\in\{1,\ldots,N\}}$ such that $A = {}^tPP$ and $B = {}^tPDP$, where ${}^tP$ denotes the tranpose of P. Using this, we can write

$$\begin{aligned} & (\lambda A + (1-\lambda)B)^{-1} - \lambda A^{-1} - (1-\lambda)B^{-1} \\ = \; & P^{-1}\left((\lambda I_N + (1-\lambda)D)^{-1} - \lambda I_N - (1-\lambda)D^{-1}\right)({}^tP)^{-1} \end{aligned}$$

where I_N denotes the $N \times N$ identity matrix. Now, the result follows by observing that, since $x \mapsto \frac{1}{x}$ is strictly convex for $x > 0$, each diagonal element of $(\lambda I_N + (1-\lambda)D)^{-1} - \lambda I_N - (1-\lambda)D^{-1}$ is non-negative. □

Since $\mathrm{Tr}(A) > 0$ for any $A \in S_N^{++}(\mathbb{R})$, we get the following useful consequence.

Corollary 3.2. $A \mapsto \mathrm{Tr}(A^{-1})$ *is strictly convex in* $S_N^{++}(\mathbb{R})$.

Proposition 3.3. *Let A and $B \in \mathcal{S}_N^{++}(\mathbb{R})$. Then*

$$\frac{1}{\mathrm{Tr}((A+B)^{-1})} \geq \frac{1}{\mathrm{Tr}(A^{-1})} + \frac{1}{\mathrm{Tr}(B^{-1})}. \tag{13}$$

Moreover, the equality holds if and only iff A and B are homothetic, i.e. it exists $\lambda > 0$ with $A = \lambda B$.

Proof. Observe that we can rewrite the inequality (13) as

$$\mathrm{Tr}((A+B)^{-1})\mathrm{Tr}(A^{-1}+B^{-1}) - \mathrm{Tr}(A^{-1})\mathrm{Tr}(B^{-1}) \leq 0. \tag{14}$$

Let P and D be as in the proof of Proposition 3.1. We may write

$$\begin{aligned}
\mathrm{Tr}((A+B)^{-1}) &= \mathrm{Tr}(({}^tP(I_N+D)P)^{-1}) = \mathrm{Tr}((I_N+D)^{-1}({}^tP)^{-1}P^{-1}) \\
\mathrm{Tr}(A^{-1}) &= \mathrm{Tr}(P^{-1}({}^tP)^{-1}) = \mathrm{Tr}(({}^tP)^{-1}P^{-1}) \\
\mathrm{Tr}(B^{-1}) &= \mathrm{Tr}(P^{-1}D^{-1}({}^tP)^{-1}) = \mathrm{Tr}(D^{-1}({}^tP)^{-1}P^{-1}).
\end{aligned}$$

Let us write $\widetilde{C} = (c_{ij})_{i,j=1}^N := ({}^tP)^{-1}P^{-1} \in \mathcal{S}_N^{++}(I\!R)$. Using the above identities, proving (14) is equivalent to prove that

$$\mathrm{Tr}((I_n+D)^{-1}\widetilde{C})\mathrm{Tr}(C+D^{-1}\widetilde{C}) - \mathrm{Tr}(\widetilde{C})\mathrm{Tr}(D^{-1}\widetilde{C}) \leq 0.$$

Since $c_{ii} > 0$ for all $i = 1, \ldots, N$, the result follows from next elementary computations

$$\begin{aligned}
&\mathrm{Tr}((I_N+D)^{-1}\widetilde{C})\mathrm{Tr}(\widetilde{C}+D^{-1}\widetilde{C}) - \mathrm{Tr}(\widetilde{C})\mathrm{Tr}(D^{-1}\widetilde{C}) \\
= &\sum_{i=1}^N \frac{c_{ii}}{1+d_i} \sum_{j=1}^N c_{jj}(1+\frac{1}{d_j}) - \sum_{i=1}^N c_{ii} \sum_{j=1}^N \frac{c_{jj}}{d_i} \\
= &\sum_{i=1}^N \sum_{j=1}^N c_{ii}c_{jj} \frac{d_i(d_j+1) - d_j(d_i+1)}{d_i d_j (1+d_i)} = \sum_{i=1}^N \sum_{j=1}^N c_{ii}c_{jj} \frac{d_i - d_j}{d_i d_j (1+d_i)} \\
= &\sum_{1 \leq i < j \leq N} c_{ii}c_{jj} \left(\frac{d_i - d_j}{d_i d_j (1+d_i)} + \frac{d_j - d_i}{d_i d_j (1+d_j)} \right) \\
= &\sum_{1 \leq i < j \leq N} c_{ii}c_{jj} \frac{(d_i - d_j)(1+d_j) + (d_j - d_i)(1+d_i)}{d_i d_j (1+d_i)(1+d_j)} \\
= &\sum_{1 \leq i < j \leq N} c_{ii}c_{jj} \frac{-(d_i - d_j)^2}{d_i d_j (1+d_i)(1+d_j)} \leq 0
\end{aligned}$$

the last inequality being an equality if and only if $D = d_1 I_N$, that is, when A and B are homothetic. $\square$

4 Some convexity properties of the mean curvature

In this section, we apply the inequalities proved in last Section to study the behavior of the mean curvature of the boundary of the convex combination of two smooth convex and strictly convex sets.

We denote by $X \oplus Y$ the Minkowski's addition of two convex sets $X, Y \subseteq I\!R^N$, .e., $X \oplus Y := \{x + y : x \in X,\, y \in Y\}$.

In this Section K and L will be two non-empty open bounded convex sets in $I\!R^N$. For all $t \in [0,1]$, let

$$K_t := (1-t)K \oplus tL = \{(1-t)x + ty : (x,y) \in K \times L\}.$$

Notice that K_t is also an open bounded convex set.

Lemma 4.1. *Assume that $\nu \in S^{N-1}$ is a normal to ∂K at x and to ∂L at y, and let $x_t = (1-t)x + ty$. Then $x_t \in \partial K_t$ and ν is normal to ∂K_t at x_t.*

Proof. Recall that ν is normal to ∂K_t at x_t if $K_t \subset H^-_{x_t,\nu} := \{z \in I\!R^N : \langle z, \nu\rangle < \langle x_t, \nu\rangle\}$ with $x_t \in \overline{K_t}$. Observe that, since $x \in \overline{K}$ and $y \in \overline{K}$, by continuity of the addition we have $x_t \in \overline{K_t}$. Now, as ν is normal to ∂K at x and to ∂L at y, we have that $K \subset H^-_{x,\nu}$ and $L \subset H^-_{y,\nu}$. It follows that $K_t = (1-t)K \oplus tL \subset (1-t)H^-_{x,\nu} \oplus H^-_{y,\nu} = H^-_{x_t,\nu}$. □

When K is of class C^1, we denote by $\nu^K(x)$ the outer unit normal to $x \in \partial K$, so that $\nu^K : \partial K \to S^{N-1}$ is the spherical image map. We say that K is C^2 and strictly convex near $x \in \partial K$ if ∂K is C^2 and ν^K is a diffeomorphism in a neighborhood of x.

The following result is an application of the linear algebra inequalities of the previous section.

Theorem 4. *Suppose that K and L are C^2 and strictly convex near x and y, respectively, and $\nu \in S^{N-1}$ is normal to ∂K at x and to ∂L at y. Let $x_t = (1-t)x + ty$. Then K_t is C^2 and strictly convex near x_t and the functions $t \in [0,1] \to \mathbf{H}_{K_t}(x_t) \in (0,\infty)$ and $t \in [0,1] \to \frac{1}{\mathbf{H}_{K_t}(x_t)}$ are convex and concave in t, respectively.*

Proof. Recall that the support function of a convex body $B \subset I\!R^N$ is defined by $h_B(u) = \sup\limits_{x \in B}\langle x, u\rangle$, $\forall u \in I\!R^N$. It is a sublinear function in u and is additive with respect to the Minkowski sum (in particular, we have $h_{K_t} = (1-t)h_K + th_L$). It is also well-known that if the convex body B is smooth, the eigenvalues of its Hessian matrix at $\nu^B(x)$ are 0 (with eigenvector $\nu^B(x)$) and the principal radii of curvature $r_1, \ldots, r_{N-1}$ of ∂B at x [26, Corollary 2.5.2, p. 109].

First, we observe that our assumptions imply that K_t remains C^2 and strictly convex near x_t because this property is equivalent to have a C^2 support function with bounded positive radii of curvature locally around x_t.

Let $\nu = \nu^K(x)$ and let $(e_1, ..., e_{N-1}, \nu)$ be an orthonormal basis of $I\!R^N$. Let A, B be the Hessian matrices of h_K and h_L restricted to $\nu^\perp$, i.e.,

$$A = \left(\frac{\partial^2 h_K(\nu)}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq N-1} \quad \text{and } B = \left(\frac{\partial^2 h_L(\nu)}{\partial x_i \partial x_j}\right)_{1\leq i,j\leq N-1}.$$

Then $A, B \in \mathcal{S}^{++}_{N-1}(I\!R)$ because all radii of curvature are positive. The mean-curvature $\mathbf{H}_{K_t}(x_t)$ is given by

$$\mathbf{H}_{K_t}(x_t) \quad = \quad \frac{\mathrm{Tr}(((1-t)A + tB)^{-1})}{N-1}.$$

Now, Corollary 3.2 shows that $t \mapsto \mathbf{H}_{K_t}(x_t)$ is convex, with strict convexity if $A \neq B$, and Proposition 3.3 shows that

$$\frac{1}{\mathbf{H}_{K_t}(x_t)} \geq \frac{1-t}{\mathbf{H}_K(x)} + \frac{t}{\mathbf{H}_L(y)}. \tag{15}$$

This proves the concavity of the function $t \mapsto \mathbf{H}_{K_t}(x_t)^{-1}$. □

Corollary 4.2. *Let K, L be two nonempty open bounded convex sets in $I\!R^N$ of class $C^{1,1}$. Then K_t is $C^{1,1}$ and, if $H(t) = \operatorname{ess\,sup}_{x\in\partial K_t} \mathbf{H}_{K_t}(x)$, then the functions $t \in [0,1] \mapsto H(t)$ and $t \in [0,1] \mapsto \frac{1}{H(t)}$ are convex and concave, respectively.*

Proof. If K and L are C^2_+ (i.e. C^2 and strictly convex), this is a straightforward consequence of the previous theorem as the supremum of convex functions is convex, and the infimum of concave functions is also concave.

The general case is a consequence of the previous case and the following convergence and approximation result concerning $C^{1,1}$ convex sets.

Lemma 4.3. *(i) Convergence: If $(K_n)_{n\in I\!N}$ a sequence of $C^{1,1}$ convex bodies in $I\!R^N$ with $\operatorname{ess\,sup}_{x\in\partial K_n} \mathbf{H}_{K_n}(x) \leq H$ for all $n \in I\!N$, and $K_n \to K$ in the Hausdorff sense, then K is $C^{1,1}$ and $\operatorname{ess\,sup}_{x\in\partial K} \mathbf{H}_K(x) \leq H$.*
(ii) Approximation: Let K be a $C^{1,1}$ convex body in $I\!R^N$ with $\operatorname{ess\,sup}_{x\in\partial K} \mathbf{H}_K(x) \leq H$. Then there exists a sequence $K_n \in C^2_+$ with $K_n \to K$ in the Hausdorff sense and $\max_{x\in\partial K_n} \mathbf{H}_{K_n}(x) \leq H_n$ with $H_n \to H$.

Proof. (i) It is a straightforward application of the Blaschke's Rolling Theorem [26, Theorem 3.2.9] extended in [3, Corollary 1.13] for $C^{1,1}$ convex sets (see also [6] where such extension is also derived in the general context of smooth anisotropic norms).

Almost everywhere in ∂K_n, the principal curvatures are bounded by $(N-1)H$, because $\operatorname{ess\,sup}_{x\in\partial K_n} \mathbf{H}_{K_n}(x) \leq H$. Using [3, Corollary 1.13], we deduce that a ball $B(r)$ of radius $r = \frac{1}{(N-1)H} > 0$ "rolls freely" inside K_n, i.e. there exists a convex body K'_n such that $K_n = K'_n \oplus B(r)$. In particular, we have $h_{K_n} = h_{K'_n} + h_{B(r)}$.

As h_{K_n} converge uniformly to h_K, $h_{K'_n} = h_{K_n} - h_{B(r)}$ are sublinear convex functions uniformly convergent to $h_K - h_{B(r)}$. We deduce that $h_K - h_{B(r)}$ is a sublinear convex function, so there exists a convex body K' in $I\!R^N$ such that $h_{K'} = h_K - h_{B(r)}$ and $K = K' \oplus B(r)$. Hence, K is a $C^{1,1}$ convex body.

The fact that the mean curvature remains bounded above by H is a consequence of the well-known property that the curvature measures of K_n weakly converge to the curvature measures of K [26].

(ii) We approximate K by $K(t)$ where $K(t)$ is the motion by mean curvature of K at time $t > 0$. By the results in [9, 10], for any initial convex set, and in particular for K, there is a generalized motion by mean curvature $K(t)$ such that $K(t) \to K$ as $t \to 0^+$ in the Hausdorff sense, $K(t)$ is smooth (C^∞) for any $t \in (0, T]$, for some $T > 0$, and satisfies

$$X_t = -\mathbf{H}_{K(t)}\nu^{K(t)}$$

where X is a parameterization of $K(t)$ and $\nu^{K(t)}$ the outer unit normal to $K(t)$.

Now, the results in [6] for smooth anisotropies prove that if K is $C^{1,1}$, then the generalized motion is also $C^{1,1}$ with a uniform bound for mean curvature for some time, say again for $t \in [0, T]$, $T > 0$. Notice that, in the present case, the uniform bound in the curvature can be characterized by a uniform norm in the Laplacian of the signed distance function to $K(t)$. Using these results and passing to the limit in formula (59) in [6], we obtain that $\|\mathbf{H}_{K(t)}\|_\infty \leq \|\mathbf{H}_K\|_\infty e^{Ct}$ for some constant $C > 0$, and this proves that $\|\mathbf{H}_{K(t)}\|_\infty \to \|\mathbf{H}_K\|_\infty$ as $t \to 0+$. Now, by the result is [18, 8], the sets $K(t)$ are C^2_+. □

Remark 4.4. We have derived the approximation Lemma 4.3.(ii) as a consequence of the estimates in [6] though it could also be derived with some additional work from the estimates in [18]. At this point, let us first proceed formally to explain the argument. Using the formulas in [18], $\mathbf{H}_{K(t)}$ satisfies the PDE

$$\frac{\partial \mathbf{H}_{K(t)}}{\partial t} = \Delta \mathbf{H}_{K(t)} + |A(t)|^2 \mathbf{H}_{K(t)} \tag{16}$$

where $A(t) = (h_{ij}(t))$ is the second fundamental form of $\partial K(t)$, $g^{ij}(t)$ is the inverse of the metric $g_{ij}(t)$ of $\partial K(t)$, and $|A(t)|^2 = g^{ij}(t)g^{kl}(t)h_{ik}(t)h_{jl}(t)$. Notice that $\mathbf{H}_{K(t)}$ is a weak solution of (16). Since $|A(t)|^2 \le (\mathbf{H}_{K(t)})^2$, $\mathbf{H}_{K(t)}$ is a subsolution of

$$Q_t = \Delta Q + Q^3$$

with initial condition

$$Q(0) = \|\mathbf{H}_K\|_\infty.$$

If $\mathbf{H}_{K(t)}$ is bounded in L^∞, then after some standard computations we obtain that

$$\frac{d}{dt}\int((\mathbf{H}_{K(t)} - Q(t))^+)^2 \le C\int((\mathbf{H}_{K(t)} - Q(t))^+)^2,$$

for some constant $C > 0$, and this implies that

$$\|\mathbf{H}_{K(t)}\|_\infty \le \left(\frac{\|\mathbf{H}_K\|_\infty^2}{1 - 2\|\mathbf{H}_K\|_\infty^2 t}\right)^{1/2}, \qquad (17)$$

if we know that

$$\int_{K(t)}((\mathbf{H}_{K(t)} - Q(t))^+)^2\, d\mathcal{H}^{N-1} \to 0 \qquad (18)$$

as $t \to 0$. Notice then that (17) implies that $\|\mathbf{H}_{K(t)}\|_\infty \to \|\mathbf{H}_K\|_\infty$ as $t \to 0^+$.

At this point, we only know that as $K(t) \to K$ as $t \to 0^+$ in the Hausdorff sense, we have that $\mathbf{H}_{K(t)} \to \mathbf{H}_K$ weakly* as measures, but this is not enough. To prove (18), as in [10], we observe that we may write locally the evolving convex sets as the graph of a function $u(t,x)$ satisfying

$$u_t = \Delta u - \frac{D^2u(Du, Du)}{1 + |Du|^2}. \qquad (19)$$

Then, by convexity, we have a uniform bound on Du (the bound is also proved in [10]). Then, by standard local estimates for uniformly parabolic equations [25], Ch. IV, Thm 9.1 and Ch. VI, Thm. 1.1, we have that $u(t,x)$ is uniformly bounded in $W^{2,p}_{loc}$ for any $p \in [1,\infty)$. After some computations and using the results in [25] one can prove that $\mathbf{H}_{K(t)} \in C([0,T], L^2)$ and the comparison argument leading to (17) can be justified.

In the statement of next theorem we use the notation of Theorem 4.

Theorem 5. *Let $\Omega \subset \partial K$, $\Omega' \subset \partial L$ be open and connected subsets where K and L are C^2 and strictly convex, respectively, and suppose that $\Omega' = \nu_L^{-1} \circ \nu_K(\Omega)$. If $\mathbf{H}_K(x) = \mathbf{H}_L(\nu_L^{-1} \circ \nu_K(x))$, $\forall x \in \Omega$, and the convexity (resp. concavity) in t of the mean curvature (resp. inverse mean curvature) function $\mathbf{H}_{K_t}(x_t)$ is not strict, then Ω' is a translate of Ω, i.e. there exists $z \in \mathbb{R}^N$ with $\Omega' = z + \Omega$.*

Proof. Let $x \in \Omega$ and $y = \nu_L^{-1} \circ \nu_K(x)$. We use the same notation as in the proof of Theorem 4. From Corollary 3.2 and Proposition 3.3, the equality in (15) arises if and only if there is $\lambda > 0$ such that $A = \lambda B$. Since $\text{Tr}(A^{-1}) = \mathbf{H}_K(x) = \mathbf{H}_L(y) = \text{Tr}(B^{-1})$, the equality in (15) arises if and only if $A = B$.

Thus, we have that $d^2 h_K(\nu) = d^2 h_L(\nu)$, $\forall \nu \in \nu_K(\Omega) = \nu_L(\Omega')$. As h_K and h_L are positively homogeneous (of degree 1), this equation extends to a neighborhood $U \subset \mathbb{R}^N$ of $\nu_K(\Omega)$ which can be chosen connected because $\nu_K(\Omega)$ is connected. This shows that there exist $z \in \mathbb{R}^N$ and $\alpha \in \mathbb{R}$ such that

$$h_L(u) = h_K(u) + \langle z, u \rangle + \alpha, \quad \forall u \in U.$$

Since $h_K(0) = h_L(0) = 0$, we deduce that $\alpha = 0$. As the support function describes the convex set locally, we get that $\Omega' = z + \Omega$. □

5 Strict convexity of the free boundary of an isoperimetric region

In order to prove Proposition 5.2 we state without proof the following known result about convex sets.

Lemma 5.1. *Let $K \subseteq \mathbb{R}^N$ be a convex set. Let $x, y \in \partial K$ and $\nu \in S^{N-1}$ be such that ν is normal to ∂K at x, y. Then the segment $[x, y] \subseteq \partial K$ and ν is also normal to ∂K at the points of $[x, y]$.*

Proposition 5.2. *Let K a non-trivial convex body of class C^1, and $C \subset K$ an isoperimetric region inside K which is convex, then $\partial C \backslash \partial K$ is C^∞ and strictly convex.*

Recall that we say that $C \subset K$ an isoperimetric region inside K if C minimizes perimeter with a volume constraint among all sets contained in K which satisfy the constraint.

Proof. As C are isoperimetric regions inside K, we know that the boundary $\Sigma = \partial C \backslash \partial K$ satisfies [15, 16, 30, 13]:

1. There is a closed singular set $\Sigma_s \subset \Sigma$ of Hausdorff dimension less than or equal to $N - 8$ such that $\Sigma_r = \Sigma \backslash \Sigma_s$ is a smooth embedded hypersurface;

2. ∂C is of class C^1 on a neighborhood of $\partial K \cap \partial C$;

3. At every point $x \in \Sigma_s$, there is a tangent minimal cone C_x different from an hyperplane. The square sum $|\sigma|^2 = k_1^2 + \cdots + k_{N-1}^2$ of the principal curvatures of Σ tends to ∞ when we approach x from Σ;

4. Σ_r has constant mean curvature with respect to the inner normal.

But, in our case, as C is a convex set, the tangent minimal cone is included in an half-space, but the only kind of such minimal cone is the hyperplane [27], so $\Sigma_s = \emptyset$, which implies that Σ is a C^∞ constant mean curvature surface.

In order to prove the strict convexity of Σ, by the result of [17, Theorem 3, p.297], we know that a constant mean curvature hypersurface with non-negative sectional curvatures follows a strong minimal principle for its Gaussian curvature $\mathbf{K}$. When applied to our case in $\Sigma_\epsilon := \{x \in \Sigma : \mathrm{dist}(x, \partial K) \geq \epsilon\}$, we have

$$\min_{x \in \Sigma_\epsilon} \mathbf{K}_C(x) = \min_{x \in \partial \Sigma_\epsilon} \mathbf{K}_C(x)$$

where $\mathbf{K}_C$ has no interior minimum except if it is constant. So, if there exists $a \in \Sigma_\epsilon$ with $\mathbf{K}_C(a) = 0$, then $\mathbf{K}_C(a) = 0 \ \forall a \in \Sigma_\epsilon$, so Σ_ϵ is a part of a cylinder. Thus, either our statement is true or Σ is part of a cylinder. The last possibility cannot happen. Indeed, let L be a maximal segment contained in Σ. Notice that its extrema points $x, y \in \partial C \cap \partial K$. Since $\nu^K(x) = \nu^K(y)$, by Lemma 5.1 we deduce that $L \subset \partial K$ and this is a contradiction since $L \subset \partial C \setminus \partial K$. Hence, C is strictly convex in Σ. □

6 Uniqueness of the Cheeger set inside a $C^{1,1}$ convex body

In this section, we prove the following result, which (in view of Proposition 2.6) implies Theorem 1.

Theorem 6. *Let C a $C^{1,1}$ convex body in $I\!R^N$. Then we have a unique Cheeger set inside C.*

Let C be a convex body in $I\!R^N$ of class $C^{1,1}$. By the results in [7] we know that there exist two convex sets C_* and C^* which are the minimal and maximal (with respect to inclusion) Cheeger sets of C. Both are solutions of $\min_{E \subseteq C} P(E) - h_C|E|$ [1, 7]. Thus, for both of them we know that $(N-1)\mathbf{H} \leq h_C$ with equality inside C (see Proposition 2.6). Since they are convex, we have that they are of class $C^{1,1}$. The uniqueness of Cheeger sets inside C is implied if we prove that $C_* = C^*$. This was done in [7] when C is of class C^2 and uniformly convex. We are going to remove both assumptions.

Thus, in the rest of this section, we suppose that $C^* \neq C_*$, and write $h_C = \frac{P(C_*)}{|C_*|} = \frac{P(C^*)}{|C^*|}$ the Cheeger constant.

Proposition 6.1. *For any $t \in [0,1]$, $C_t := (1-t)C_* \oplus tC^*$ is a Cheeger set.*

Proof: As C^* and C_* are $C^{1,1}$ convex Cheeger sets with

$$\operatorname*{ess\,sup}_{x\in\partial C^*} \mathbf{H}_{C^*}(x) \leq \frac{h_C}{N-1} \text{ and } \operatorname*{ess\,sup}_{x\in\partial C_*} \mathbf{H}_{C_*}(x) \leq \frac{h_C}{N-1},$$

from Corollary 4.2, we obtain that C_t is $C^{1,1}$ and

$$\operatorname*{ess\,sup}_{x\in\partial C_t} \mathbf{H}_{C_t}(x) \quad \leq \quad \frac{h_C}{N-1}. \tag{20}$$

Observe that $h_C \leq \frac{P(C_t)}{|C_t|}$, since $C_t \subset C^*$. With the inequality (20) and the characterization of calibrable sets proved in [1], this shows that C_t is calibrable. In other words, C_t minimizes

$$\min_{E\subseteq C_t} P(E) - \lambda_{C_t}|E| \qquad \text{where } \lambda_{C_t} = \frac{P(C_t)}{|C_t|}.$$

But $C_* \subset C_t$, and this implies that $\frac{P(C_t)}{|C_t|} \leq \frac{P(C_*)}{|C_*|} = h_C$. We conclude that C_t is a Cheeger set. □

Proposition 6.2. *For any $t \in [0,1]$ the sets C_* and C_t are equivalent by telescoping, more precisely, $\exists \overline{z} \in I\!R^N$ such as C_t is a translate of $C_* \oplus [0,t]\overline{z}$.*

Proof. In the context of this proof we assume that C_* and C^* are open sets. Since the result is obviously true for $t = 0$ (take $z = 0$) and follows for $t = 1$ by passing to the limit as $t \to 1-$, we may assume that $t \in (0,1)$.

Step 1. Let Ω be a connected component of $\partial C_* \setminus \partial C^*$ and let

$$\Omega_t := (\nu^{C_t})^{-1} \circ \nu^{C_*}(\Omega) \subset \partial C_t.$$

Then both Ω and $\Omega^t \subset \partial C_t \setminus \partial C^*$ are open, connected, C^2, and strictly convex. Moreover ν^{C_*} and ν^{C_t} are diffeomorphism from Ω, resp. Ω_t, onto $\nu^{C_*}(\Omega)$.

As $\partial C_* \setminus \partial C^*$ is an open set of ∂C_*, Ω is also an open set of ∂C_*. By Proposition 5.2 we know that Ω is C^2 and strictly convex. This implies that $\nu^{C_*}|_\Omega$ is a diffeomorphism onto its image. Then, by definition of Ω_t, we know that Ω_t is an open set. Let us prove that $\Omega_t \cap \partial C^* = \emptyset$. Indeed, if $p \in \Omega_t \cap \partial C^*$, then there is $\overline{x} \in \Omega$ such that $\nu^{C_*}(\overline{x}) = \nu^{C_t}(p) = \nu^{C^*}(\overline{p})$. Then, by Lemma 4.1, $p_t := (1-t)\overline{x} + tp \in \partial C_t$, $\nu^{C_t}(p_t) = \nu^{C_t}(p)$ and

$p_t \neq p$, a contradiction. Since $\Omega_t \subset \partial C_t \setminus \partial C^*$ and C_t is a Cheeger set, by Proposition 5.2, we know that Ω_t is C^2 and strictly convex. Then ν^{C_t} is a diffeomorphism from Ω_t onto $\nu^{C_*}(\Omega)$. In particular, Ω_t is connected.

Observe that we may identify Ω_t by a point x_t in the following way. Let $x \in \Omega$, $y \in \partial C^*$ with $\nu^{C_*}(x) = \nu^{C^*}(y)$. As $x \in \overline{C^*} \setminus \partial C^* = C^*$, $y \in \partial C^*$, and C^* is a convex non-empty open set, it is straightforward to show that $x_t := (1-t)x + ty \in C^*$ and, by Lemma 4.1, $x_t \in \partial C_t$, and $\nu^{C_t}(x_t) = \nu^{C_*}(x)$. Thus $x_t \in \Omega_t$.

Step 2. Let us prove that there exists $z \in \mathbb{R}^N$, $z \neq 0$, such that $\Omega_t = tz + \Omega$ for all $t \in (0,1)$, and

$$\nu^{C_*}(\Omega) = S_z^+ \ , \quad \text{where } S_z^+ = \{u \in S^{N-1}, \langle u, z\rangle > 0\}. \tag{21}$$

Thus, we conclude that ν^{C_*} and ν^{C_t} are diffeomorphisms from Ω and Ω_t, respectively, onto S_z^+.

To prove the first assertion, we observe that by Step 1 we have that Ω and Ω_t satisfy $\mathbf{H}_{C_*}|_\Omega = \mathbf{H}_{C_t}|_{\Omega_t} = \frac{h_C}{N-1}$ together with the other assumptions of Theorem 5. Thus, Ω_t is a translation of Ω. By the observation previous to Step 2, we know that all $x_t \in \Omega_t$ with the same normal are collinear. This implies that there exists $z \in \mathbb{R}^N$, $z \neq 0$, with $\Omega_t = tz + \Omega$ where z does not depend on $t \in (0,1)$.

To prove (21) we prove both that

$$\langle \nu^{C_*}(x), z\rangle > 0 \qquad \forall x \in \Omega, \tag{22}$$

and

$$\langle \nu^{C_*}(x), z\rangle = 0 \qquad \forall x \in \partial_{\partial C_*}\Omega. \tag{23}$$

To prove (22), observe that for any $x \in \Omega$, writing $x_t := x + tz \in \partial C_t$ and knowing that C_t is strictly convex near x_t, we get

$$\langle \nu^{C_*}(x), z\rangle = \langle \nu^{C_t}(x_t), z\rangle = \langle \nu^{C_t}(x_t), \frac{x_t - x}{t}\rangle > 0\,.$$

To prove (23), let $x \in \partial_{\partial C_*}\Omega$. By approximating x by points inside Ω and using (22) we have that $\langle \nu^{C_*}(x), z\rangle \geq 0$. On the other hand, $x \in \partial C^*$ and, by letting $t \to 1-$ in $x_t = x + tz \in \partial C_t$, we also have that $x + z \in \partial C^*$. This implies that

$$\langle \nu^{C_*}(x), z\rangle = \langle \nu^{C^*}(x), x + z - x\rangle \leq 0.$$

Now we observe that (22) and (23) can be written respectively as $\nu^{C_*}(\Omega) \subseteq S_z^+$ and $\nu^{C_*}(\partial_{\partial C_*}\Omega) \subseteq S_z^0 := \{u \in S^{N-1} : u \perp z\}$. On one hand, we know that $\nu^{C_*}(\Omega)$ is open in

S_z^+. On the other hand, since $\overline{\nu^{C_*}(\Omega)} = \nu^{C_*}(\overline{\Omega})$, we also have that $\nu^{C_*}(\Omega)$ is closed in S_z^+. Indeed

$$\overline{\nu^{C_*}(\Omega)} \cap S_z^+ = \nu^{C_*}(\overline{\Omega}) \cap S_z^+ = \left(\nu^{C_*}(\Omega) \cap S_z^+\right) \cup \left(\nu^{C_*}(\partial_{\partial C_*}\Omega) \cap S_z^+\right) = \nu^{C_*}(\Omega) \cap S_z^+.$$

Being nonempty, open and closed in S_z^+, we have (21).

Step 3. Conclusion. If Ω is the only connected component of $\partial C_* \backslash \partial C^*$, then last equality (21) implies that $C_t = C_* \oplus [0,1]tz$. In this case, we take $\overline{z} = z$. If Ω' is another connected component of $\partial C_* \backslash \partial C^*$, by Step 2 we know that there exists $z' \in I\!R^N$, $z' \neq 0$, such that ν^{C_*} and ν^{C_t} are diffeomorphisms from Ω' and $\Omega'_t := (\nu^{C_t})^{-1}(\nu^{C_*}(\Omega'))$, respectively, onto $S_{z'}^+$. Moreover $\Omega'_t = \Omega' + tz'$. Notice that, since $\Omega \cap \Omega' = \emptyset$ and ν^{C_*} is a diffeomorphism restricted to Ω and Ω', we have $S_z^+ \cap S_{z'}^+ = \emptyset$. This implies that there exists $\alpha > 0$ with $z' = -\alpha z$, and we deduce that $C_t = C_* \oplus [0,t]z \oplus [0,t](-\alpha z) = C_* \oplus [0,1]t(1+\alpha)z - t\alpha z$. In this case, we take $\overline{z} = (1+\alpha)z$. □

Proposition 6.3. *For all $t \geq 0$, let $C^t := C_* \oplus [0,t]\overline{z}$, $\overline{z}$ being the vector found in Proposition 6.2, i.e., such that C_t is a translate of C^t for any $t \in [0,1]$. Then C^t is $C^{1,1}$ and calibrable with $\frac{P(C^t)}{|C^t|} = h_C$.*

Proof. As $C^t = C_* \oplus [0,t]z$, we know that $P(C^t)$ and $|C^t|$ are two linear functions of t, i.e., there exists $\alpha, \beta > 0$ such that $P(C^t) = P(C_*) + \alpha t$ and $|C^t| = |C_*| + \beta t$ [26, Theorem 6.7.1, p.379]. As $P(C^t) = P(C_t)$ and $|C^t| = |C_t|$ if $t \in [0,1]$, and $\frac{P(C_t)}{|C_t|} = h_C$, this equality extends to all $t \geq 0$, that is, $\frac{P(C^t)}{|C^t|} = h_C$ for all $t \geq 0$.

As C_* is $C^{1,1}$ and $(N-1)\operatorname{ess\,sup}_{x \in \partial C_*} \mathbf{H}_{C_*}(x) \leq h_C$, it is straightforward to show that C^t is $C^{1,1}$ and

$$(N-1)\operatorname*{ess\,sup}_{x \in \partial C^t} \mathbf{H}_{C^t}(x) \leq h_C = \frac{P(C^t)}{|C^t|}.$$

Hence, by the results in [1], we have that C^t is calibrable. □

Proposition 6.4. *If $D := \mathrm{Proj}_{z^\perp}(C_*)$ (the basis of the cylinder), which is a convex body in $I\!R^{N-1}$, then $\frac{P(D)}{|D|} = h_C$.*

Proof. Let t big enough to have an hyperplane $H_{z,\alpha} := \{x \in I\!R^N : \langle x, z\rangle = \alpha\}$ such that $H_{z,\alpha} \cap C^t = D$. Let $t' > t$, then we have $(t'-t)P(D) = P(C^{t'}) - P(C^t) = h_C(|C^{t'}| - |C^t|) = h_C(t'-t)|D|$. □

Let us choose t big enough and α such that $H_{z,\alpha} \cap C^t = D$, and $C_* \subset C^t \cap H_{z,\alpha}^-$, where $H_{z,\alpha}^- = \{x \in I\!R^N : \langle x, z\rangle < \alpha\}$. Let us consider the convex $S = (C^t \cap H_{z,\alpha}^-) \cup$

$\mathrm{Sym}_{H^0_{z,\alpha}}(C^t \cap H^-_{z,\alpha})$ which is symmetrical and $C^{1,1}$ (we denote by $\mathrm{Sym}_{H^0_{z,\alpha}}(C^t \cap H^-_{z,\alpha})$ the symmetrization with respect to $H_{z,\alpha}$ of $C^t \cap H^-_{z,\alpha}$).

Proposition 6.5. *S is calibrable, with* $\frac{P(S)}{|S|} = h_C$.

Proof. As for t' big enough, we can translate $C^{t'}$ to have $S \subset C^{t'}$, we get that $\frac{P(S)}{|S|} \geq h_C$. Since we have

$$(N-1)\operatorname*{ess\,sup}_{x \in \partial S} \mathbf{H}_S(x) \leq h_C \leq \frac{P(S)}{|S|},$$

by the results in [1] we obtain that S is calibrable. Since we have chosen t big enough to have $C_* \subset S$, then

$$\frac{P(S)}{|S|} \leq \frac{P(C^*)}{|C^*|} = h_C.$$

Thus $\frac{P(S)}{|S|} = h_C$. □

Proof of Theorem 6. We suppose $C_* \neq C^*$, and we take S as above. Observe that there is a function $u : D \to \mathbb{R}$ such that we can write S as

$$S = \{x + tu(x)z_0, x \in D, t \in [-1,1]\}, \qquad \text{where } z_0 = \tfrac{z}{\|z\|}.$$

We have

$$\frac{|S|}{2} = \int_D u,$$

and

$$\frac{P(S)}{2} = \int_D \sqrt{1+|Du|^2} + \int_{\partial D} u.$$

At the same time, u is solution of

$$-\mathrm{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) = h_C \tag{24}$$

and the graph of u is a $C^{1,1}$ surface above D having zero contact angle with $\partial D \times \mathbb{R}$, i.e.

$$\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu^D = -1. \tag{25}$$

Then we compute

$$
\begin{aligned}
\frac{P(S)}{2} &= h_C \frac{|S|}{2} = -\int_D \operatorname{div}\Big(\frac{Du}{\sqrt{1+|Du|^2}}\Big) u \\
&= \int_D \frac{|Du|^2}{\sqrt{1+|Du|^2}} - \int_{\partial D} \left(\frac{Du}{\sqrt{1+|Du|^2}} \cdot \nu^D \right) u \\
&= \int_D \frac{|Du|^2}{\sqrt{1+|Du|^2}} + \int_{\partial D} u \\
&< \int_D \sqrt{1+|Du|^2} + \int_{\partial D} u = \frac{P(S)}{2},
\end{aligned}
$$

and we obtain a contradiction. Our statement is proved. □

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