Uniqueness of the Cheeger set of a convex body

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Abstract

We prove that if $C \subset \mathbb{R}^N$ is of class C^2 and uniformly convex, then the Cheeger set of C is unique. The Cheeger set of C is the set which minimizes, inside C, the ratio perimeter over volume.

1 Introduction

Given an nonempty open bounded subset Ω of $\mathbb{R}^N$, we call Cheeger constant of Ω the quantity

$$h_\Omega \;=\; \min_{K\subseteq\Omega} \frac{P(K)}{|K|}\,. \tag{1}$$

Here $|K|$ denotes de N-dimensional volume of K and $P(K)$ denotes the perimeter of K. The minimum in (1) is taken over all nonempty sets of finite perimeter contained in Ω. A Cheeger set of Ω is any set $G \subseteq \Omega$ which minimizes (1). If Ω minimizes (1), we say that it is Cheeger in itself. We observe that the minimum in (1) is attained at a subset G of Ω such that ∂G intersects $\partial\Omega$: otherwise we would diminish the quotient $P(G)/|G|$ by dilating G.

For any set of finite perimeter K in $\mathbb{R}^N$, let us denote

$$\lambda_K := \frac{P(K)}{|K|}.$$

Notice that for any Cheeger set G of Ω, $\lambda_G = h_G$. Observe also that G is a Cheeger set of Ω if and only if G minimizes

$$\min_{K\subseteq\Omega} P(K) - \lambda_G|K|. \tag{2}$$

We say that a set $\Omega \subset \mathbb{R}^N$ is calibrable if Ω minimizes the problem

$$\min_{K\subseteq\Omega}\; P(K) - \lambda_\Omega|K|. \tag{3}$$

In particular, if G is a Cheeger set of Ω, then G is calibrable. Thus, Ω is a Cheeger set of itself if and only if it is calibrable.

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Finding the Cheeger sets of a given Ω is a difficult task. This task is simplified if Ω is a convex set and $N = 2$. In that case, the Cheeger set in Ω is unique and is identified with the set $\Omega^R \oplus B(0, R)$ where $\Omega^R := \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > R\}$ is such that $|\Omega^R| = \pi R^2$ and $A \oplus B := \{a + b : a \in A,\, b \in B\}$, $A, B \subset \mathbb{R}^2$ [2, 19]. We see in particular that it is convex. Moreover, a convex set $\Omega \subseteq \mathbb{R}^2$ is Cheeger in itself if and only if $\max_{x\in\partial\Omega} \kappa_\Omega(x) \leq \lambda_\Omega$ where $\kappa_\Omega(x)$ denotes the curvature of $\partial\Omega$ at the the point x. This has been proved in [14, 9, 19, 2, 20], though it was stated in terms of calibrability in [9, 2]. The proof in [14] had also a complement result: if Ω is Cheeger in itself then Ω is strictly calibrable, that is, for any set $K \subset \Omega$, $K \neq \Omega$, then

$$0 = P(\Omega) - \lambda_\Omega|\Omega| < P(K) - \lambda_\Omega|K|,$$

and this implies that the capillary problem in absence of gravity (with vertical contact angle at the boundary)

$$\begin{aligned} -\mathrm{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) &= \lambda_\Omega \quad \text{in } \Omega \\ -\frac{Du}{\sqrt{1+|Du|^2}}\cdot \nu^\Omega &= 1 \qquad \text{in } \partial\Omega \end{aligned} \tag{4}$$

has a solution. Indeed, both problems are equivalent [14, 18].

Our purpose in this paper is to extend the above result to $\mathbb{R}^N$, that is, to prove the uniqueness and convexity of the Cheeger set contained in a convex set $\Omega \subset \mathbb{R}^N$. We have to assume, in addition, that Ω is uniformly convex and of class C^2. This regularity assumption is probably too strong, and its removal is the subject of current research [1]. The characterization of a convex set $\Omega \subset \mathbb{R}^N$ of class $C^{1,1}$ which is Cheeger in itself (also called calibrable) in terms of the mean curvature of its boundary was proved in [3]. The precise result states that such a set Ω is Cheeger in itself if and only if $\kappa_\Omega(x) \leq \lambda_\Omega$ for any $x \in \partial\Omega$, where $\kappa_\Omega(x)$ denotes the sum of the principal curvatures of the boundary of Ω, i.e. $(N - 1)$ times the mean curvature of $\partial\Omega$ at x. Moreover, in [3], the authors also proved that for any convex set $\Omega \subset \mathbb{R}^N$ there exists a maximal Cheeger set contained in Ω which is convex. These results were extended to convex sets Ω satisfying a regularity condition and anisotropic norms in $\mathbb{R}^N$ (including the crystalline case) in [12].

In particular, we obtain that $\Omega \subset \mathbb{R}^N$ is the unique Cheeger set of itself, whenever Ω is a C^2, uniformly convex calibrable set. We point out that, by Theorems 1.1 and 4.2 in [14], this uniqueness result is equivalent to the existence of a solution $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ of the capillary problem (4).

Let us explain the plan of the paper. In Section 2 we collect some definitions and recall some results about the mean curvature operator in (4) and the subdifferential of the total variation. In Section 3 we state and prove the uniqueness result.

2 Preliminaries

2.1 BV functions

Let Ω be an open subset of $\mathbb{R}^N$. A function $u \in L^1(\Omega)$ whose gradient Du in the sense of distributions is a (vector valued) Radon measure with finite total variation in Ω is called a function of bounded variation. The class of such functions will be denoted by $BV(\Omega)$. The total variation of Du on Ω turns out to be

$$\sup\left\{\int_\Omega u \operatorname{div} z \, dx : z \in C_0^\infty(\Omega;\mathbb{R}^N), \|z\|_{L^\infty(\Omega)} := \operatorname*{ess\,sup}_{x\in\Omega} |z(x)| \leq 1\right\}, \tag{5}$$

(where for a vector $v = (v_1, \dots, v_N) \in \mathbb{R}^N$ we set $|v|^2 := \sum_{i=1}^N v_i^2$) and will be denoted by $|Du|(\Omega)$ or by $\int_\Omega |Du|$. The map $u \to |Du|(\Omega)$ is $L^1_{\mathrm{loc}}(\Omega)$-lower semicontinuous. $BV(\Omega)$ is a Banach space when endowed with the norm $\int_\Omega |u|\, dx + |Du|(\Omega)$. We recall that $BV(\mathbb{R}^N) \subseteq L^{N/(N-1)}(\mathbb{R}^N)$.

A measurable set $E \subseteq \mathbb{R}^N$ is said to be of finite perimeter in $\mathbb{R}^N$ if (5) is finite when u is substituted with the characteristic function χ_E of E and $\Omega = \mathbb{R}^N$. The perimeter of E is defined as $P(E) := |D\chi_E|(\mathbb{R}^N)$. For a complete monograph on functions of bounded variation we refer to [5].

Finally, let us denote by $\mathcal{H}^{N-1}$ the $(N-1)$-dimensional Hausdorff measure. We recall that when E is a finite-perimeter set with regular boundary (for instance, Lipschitz), its perimeter $P(E)$ also coincides with the more standard definition $\mathcal{H}^{N-1}(\partial E)$.

2.2 A generalized Green's formula

Let Ω be an open subset of $\mathbb{R}^N$. Following [7], let

$$X_2(\Omega) := \{z \in L^\infty(\Omega;\mathbb{R}^N) : \operatorname{div}\ z \in L^2(\Omega)\}.$$

If $z \in X_2(\Omega)$ and $w \in L^2(\Omega) \cap BV(\Omega)$ we define the functional $(z \cdot Dw) : C_0^\infty(\Omega) \to \mathbb{R}$ by the formula

$$< (z \cdot Dw), \varphi >:= -\int_\Omega w\, \varphi \operatorname{div}\ z\, dx - \int_\Omega w\, z \cdot \nabla\varphi\, dx.$$

Then $(z \cdot Dw)$ is a Radon measure in Ω,

$$\int_\Omega (z \cdot Dw) = \int_\Omega z \cdot \nabla w\, dx \qquad \forall w \in L^2(\Omega) \cap W^{1,1}(\Omega).$$

Recall that the outer unit normal to a point $x \in \partial\Omega$ is denoted by $\nu^\Omega(x)$. We recall the following result proved in [7].

Theorem 1. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with Lipschitz boundary. Let $z \in X_2(\Omega)$. Then there exists a function $[z \cdot \nu^\Omega] \in L^\infty(\partial\Omega)$ satisfying $\|[z \cdot \nu^\Omega]\|_{L^\infty(\partial\Omega)} \leq \|z\|_{L^\infty(\Omega;\mathbb{R}^N)}$, and such that for any $u \in BV(\Omega) \cap L^2(\Omega)$ we have*

$$\int_\Omega u \operatorname{div} z\, dx + \int_\Omega (z \cdot Du) = \int_{\partial\Omega} [z \cdot \nu^\Omega] u\, d\mathcal{H}^{N-1}.$$

Moreover, if $\varphi \in C^1(\overline{\Omega})$ then $[(\varphi z) \cdot \nu^\Omega] = \varphi[z \cdot \nu^\Omega]$.

This result is complemented with the following result proved by Anzellotti in [8].

Theorem 2. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open set with a boundary of class C^1. Let $z \in C(\overline{\Omega}; \mathbb{R}^N)$ with* $\operatorname{div} z \in L^2(\Omega)$. *Then*

$$[z \cdot \nu^\Omega](x) = z(x) \cdot \nu^\Omega(x) \qquad \mathcal{H}^{N-1} \text{ a.e. on } \partial\Omega.$$

2.3 Some auxiliary results

Let Ω be an open bounded subset of $\mathbb{R}^N$ with Lipschitz boundary, and let $\varphi \in L^1(\Omega)$. For all $\epsilon > 0$, we let $\Psi_\varphi^\varepsilon : L^2(\Omega) \to (-\infty, +\infty]$ be the functional defined by

$$\Psi_\varphi^\epsilon(u) := \begin{cases} \displaystyle\int_\Omega \sqrt{\epsilon^2 + |Du|^2} + \int_{\partial\Omega} |u - \varphi| & \text{if } \ u \in L^2(\Omega) \cap BV(\Omega) \\ \\ +\infty & \text{if } \ u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{6}$$

As it is proved in [15], if $f \in W^{1,\infty}(\Omega)$, then the minimum $u \in BV(\Omega)$ of the functional

$$\Psi_\varphi^\epsilon(u) + \int_\Omega |u(x) - f(x)|^2 \, dx \tag{7}$$

belongs to $u \in C^{2+\alpha}(\Omega)$, for every $\alpha < 1$. The mimimum u of (7) is a solution of

$$\begin{cases} u - \dfrac{1}{\lambda} \operatorname{div} \dfrac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \; = \; f(x) & \text{in } \Omega \\ u \; = \; \varphi & \text{on } \partial\Omega \end{cases} \tag{8}$$

where the boundary condition is taken in a generalized sense [21], i.e.,

$$\left[\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \cdot \nu^\Omega \right] \in \operatorname{sign}(\varphi - u) \qquad \mathcal{H}^{N-1} \text{ a.e. on } \partial\Omega.$$

Observe that (8) can be written as

$$u + \frac{1}{\lambda} \partial \Psi_\varphi^\epsilon(u) \ni f. \tag{9}$$

We are particularly interested in the case where $\varphi = 0$. As we shall show below (see also [3]) in the case of interest to us we have $u > 0$ on $\partial\Omega$ and, thus, $\left[\frac{Du}{\sqrt{\varepsilon^2 + |Du|^2}} \cdot \nu^\Omega \right] = -1$ $\mathcal{H}^{N-1}$ a.e. on $\partial\Omega$. It follows that u is a solution of the first equation in (8) with vertical contact angle at the boundary.

As $\epsilon \to 0^+$, the solution of (8) converges to the solution of

$$\begin{cases} u + \dfrac{1}{\lambda} \partial \Psi_\varphi(u) \; = \; f(x) & \text{in } \Omega \\ u \; = \; \varphi & \text{on } \partial\Omega. \end{cases} \tag{10}$$

where $\Psi : L^2(\Omega) \to (-\infty, +\infty]$ is given by

$$\Psi_\varphi(u) := \begin{cases} \displaystyle\int_{\mathbb{R}^N} |Du| + \int_{\partial\Omega} |u - \varphi| & \text{if} \quad u \in L^2(\Omega) \cap BV(\Omega) \\ \\ +\infty & \text{if} \quad u \in L^2(\Omega) \setminus BV(\Omega). \end{cases} \tag{11}$$

In this case $\partial\Psi_\varphi$ represents the operator $-\mathrm{div}\,(Du/|Du|)$ with the boundary condition $u = \varphi$ in $\partial\Omega$, and this connection is precisely given by the following Lemma (see [6]).

Lemma 2.1. *The following assertions are equivalent:*

(a) $v \in \partial\Psi_\varphi(u)$*;*

(b) $u \in L^2(\Omega) \cap BV(\Omega)$*,* $v \in L^2(\Omega)$*, and there exists* $z \in X_2(\Omega)$ *with* $\|z\|_\infty \leq 1$*, such that*

$$v = -\mathrm{div}\, z \quad \text{in} \quad \mathcal{D}'(\Omega),$$

$$(z \cdot Du) = |Du|,$$

and

$$[z \cdot \nu^\Omega] \in \mathrm{sign}(\varphi - u) \qquad \mathcal{H}^{N-1} \text{ a.e. on } \partial\Omega.$$

Notice that the solution $u \in L^2(\Omega)$ of (10) minimizes the problem

$$\min_{u \in BV(\Omega)} \int_\Omega |Du| \, + \, \int_{\partial\Omega} |u(x) - \varphi(x)| \, d\mathcal{H}^{N-1}(x) \, + \, \frac{\lambda}{2} \int_\Omega |u(x) - f(x)|^2 \, dx, \tag{12}$$

and the two problems are equivalent.

3 The uniqueness theorem

We now state our main result.

Theorem 3. *Let C be a convex body in $\mathbb{R}^N$. Assume that C is uniformly convex, with boundary of class C^2. Then the Cheeger set of C is convex and unique.*

We do not believe that the regularity and the uniform convexity of C is essential for this result (see [1]).

Let us recall the following result proved in [3] (Theorems 6 and 8 and Proposition 4):

Theorem 4. *Let C be a convex body in $\mathbb{R}^N$ with boundary of class $C^{1,1}$. For any $\lambda, \varepsilon > 0$, there is a unique solution u_ε of the equation:*

$$\begin{cases} u_\varepsilon \; - \; \dfrac{1}{\lambda}\mathrm{div}\, \dfrac{Du_\varepsilon}{\sqrt{\varepsilon^2 + |Du_\varepsilon|^2}} \; = \; 1 & \text{in } C \\ u_\varepsilon \; = \; 0 & \text{on } \partial C, \end{cases} \tag{13}$$

such that $0 \le u_\varepsilon \le 1$. Moreover, there exist λ_0 and ε_0, depending only on ∂C, such that if $\lambda \ge \lambda_0$ and $\varepsilon \le \varepsilon_0$, then u_ε is a concave function such that $u_\varepsilon \ge \alpha > 0$ on ∂C for some $\alpha > 0$. Hence, u_ε satisfies

$$\left[\frac{Du^\epsilon}{\sqrt{\epsilon^2 + |Du^\epsilon|^2}} \cdot \nu^C\right] = \text{sign}(0 - u^\epsilon) = -1 \quad \text{on } \partial C. \tag{14}$$

As $\varepsilon \to 0$, the functions u_ε converge to the concave function u which minimizes the problem

$$\min_{u \in BV(C)} \int_C |Du| + \int_{\partial C} |u(x)| \, d\mathcal{H}^{N-1}(x) + \frac{\lambda}{2} \int_C |u(x) - 1|^2 \, dx \tag{15}$$

or, equivalently, if u is extended with zero out of C, u minimizes

$$\int_{\mathbb{R}^N} |Du| + \frac{\lambda}{2} \int_{\mathbb{R}^N} |u - \chi_C|^2 \, dx.$$

The function u satisfies $0 \le u < 1$. Moreover, the superlevel set $\{u \ge t\}$, $t \in (0,1]$, is contained in C and minimizes the problem

$$\min_{F \subset C} P(F) - \lambda(1-t)|F|. \tag{16}$$

It was proved in [3] (see also [12]) that the set $C^* = \{u = \max_C u\}$ is the maximal Cheeger set contained in C, that is, the maximal set that solves (1). Moreover, one has $u = 1 - h_C/\lambda > 0$ in C^* and $h_C = \lambda_{C^*}$.

If we want to consider what happens inside C^* and, in particular, if there are other Cheeger sets, we have to analyze the level sets of u_ε before passing to the limit as $\epsilon \to 0^+$. In order to do this, let us introduce the following rescaling of u_ε:

$$v_\varepsilon = \frac{u_\varepsilon - m_\varepsilon}{\varepsilon} \le 0,$$

where $m_\varepsilon = \max_C u_\varepsilon \to 1 - h_C/\lambda$ as $\varepsilon \to 0$. The function v_ε is a generalized solution of the equation:

$$\begin{cases} \varepsilon v_\varepsilon - \dfrac{1}{\lambda} \text{div} \dfrac{Dv_\varepsilon}{\sqrt{1 + |Dv_\varepsilon|^2}} = 1 - m_\varepsilon & \text{in } C \\ v_\varepsilon = -m_\varepsilon/\varepsilon & \text{on } \partial C. \end{cases} \tag{17}$$

We let $z_\varepsilon = Du_\varepsilon/\sqrt{\varepsilon^2 + |Du_\varepsilon|^2} = Dv_\varepsilon/\sqrt{1 + |Dv_\varepsilon|^2}$. Notice that z_ε is a vector field in $L^\infty(C)$, with uniformly bounded divergence, such that $|z_\varepsilon| \le 1$ a.e. in C and, by (14), $[z_\varepsilon \cdot \nu_C] = -1$ on ∂C.

Let us study the limit of v_ε and z_ε as $\varepsilon \to 0$. Let us observe that, by concavity of v_ε, for each $\varepsilon > 0$ small enough and each $s \in (0, |C|)$, there exists a (convex) superlevel set C_s^ε of v_ε such that $|C_s^\varepsilon| = s$. We also observe that $\{v_\varepsilon = 0\}$ is a null set. Otherwise, since v_ε is concave, it would be a convex set of positive measure, hence with nonempty interior. We would then have that $v_\varepsilon = \text{div}\, z_\varepsilon = 0$, hence $1 - m_\varepsilon = 0$ in the interior of $\{v_\varepsilon = 0\}$. This is a contradiction with Theorem 4 for $\varepsilon > 0$ small enough. Hence we may take $C_0^\varepsilon := \{v_\varepsilon = 0\}$

and $C^\varepsilon_{|C|} := C$. The boundaries $\partial C^\varepsilon_s \cap C$ define in C a foliation, in the sense that for all $x \in C$, there exists a unique value of $s \in [0, |C|]$ such that $x \in \partial C^\varepsilon_s$.

We observe that a sequence of uniformly bounded convex sets is compact both for the L^1 and Hausdorff topologies. Hence, up to a subsequence, we may assume that C^ε_s converge to convex sets C_s, each of volume s, first for any $s \in \mathbb{Q} \cap (0, |C|)$ and then by continuity for any s. Possibly extracting a further subsequence, we may assume that there exists $s_* \in [0, |C|]$ such that v_ε goes to a concave function v in C_s for any $s < s_*$, and to $-\infty$ outside $C_* := C_{s_*}$. We may also assume that $z_\varepsilon \rightharpoonup z$ weakly* in $L^\infty(C)$, for some vector field z, satisfying $|z| \le 1$ a.e. in C. From (13) we have in the limit

$$-\operatorname{div} z \;=\; \lambda(1-u) \qquad \text{in } \mathcal{D}'(C). \tag{18}$$

Moreover, by the results recalled in Section 2, it holds $-\operatorname{div} z \in \partial\Psi_0(u)$. We see from (18) that

$$-\operatorname{div} z = h_C \qquad \text{in } C^*, \tag{19}$$

while $-\operatorname{div} z > h_C$ a.e. on $C \setminus C^*$. We let $s^* := |C^*|$, so that $C^* = C_{s^*}$. By Theorem 4, for $s \ge s^*$, the set C_s is a minimizer of the variational problem

$$\min_{E \subseteq C} P(E) - \mu_s |E|\,, \tag{20}$$

for some $\mu_s \ge h_C$ (μ_s is equal to the constant value of $-\operatorname{div} z = \lambda(1-u)$ on $\partial C_s \cap C$, see eq. (16)). Notice that μ_s is bounded from above by $P(C)/(|C| - s)$: indeed, for $\varepsilon > 0$, one has

$$-\int_{C \setminus C^\varepsilon_s} \operatorname{div} z_\varepsilon(x)\, dx \;=\; \mathcal{H}^{N-1}(\partial C \setminus \partial C^\varepsilon_s) - \int_{\partial C^\varepsilon_s \cap C} \frac{|Du_\varepsilon|}{\sqrt{1+|Du_\varepsilon|^2}} \;\le\; P(C)$$

(since the inner normal to C^ε_s at $x \in \partial C^\varepsilon_s \cap C$ is $Du_\varepsilon(x)/|Du_\varepsilon(x)|$). On the other hand,

$$-\int_{C \setminus C^\varepsilon_s} \operatorname{div} z_\varepsilon(x)\, dx \;=\; \int_{C \setminus C^\varepsilon_s} \lambda(1 - u_\varepsilon(x))\, dx \;\ge\; \mu^\varepsilon_s(|C| - s)\,,$$

where μ^ε_s is the constant value of $\lambda(1-u_\varepsilon)$ on the level set $\partial C^\varepsilon_s \cap C$, and goes to μ_s as $\varepsilon \to 0$. A more careful analysis would show, in fact, that $\mu_s \le (P(C) - P(C_s))/(|C| - s)$.

For $s > s^*$, we have $\mu_s > h_C$ and the set C_s is the unique minimizer of the variational problem (20). As a consequence (see [3, 12]) for any $s > s^*$ the set C_s is also the unique minimizer of $P(E)$ among all $E \subseteq C$ of volume s.

Lemma 3.1. *We have $s_* > 0$ and the sets C_s are Cheeger sets in C for any $s \in [s_*, s^*]$.*

Proof. Let $s_* < s \le |C|$. If $x \in \partial C^\epsilon_s \setminus \partial C$, then

$$0 - v_\varepsilon(x) \;\le\; Dv_\varepsilon(x) \cdot (\bar{x}_\varepsilon - x)$$

where $v_\varepsilon(\bar{x}_\varepsilon) = \max_C v_\varepsilon$. Hence, $\lim_{\varepsilon\to 0} \inf_{\partial C^\varepsilon_s \setminus \partial C} |Dv_\varepsilon| = +\infty$. Since $[z_\varepsilon \cdot \nu^C] = -1$ on ∂C and $P(C^\varepsilon_s) \to P(C_s)$, we deduce

$$\begin{aligned} &-\int_{\partial C^\varepsilon_s} [z_\varepsilon(x) \cdot \nu^{C^\varepsilon_s}(x)]\, d\mathcal{H}^{N-1}(x) \\ &\qquad = \int_{\partial C^\varepsilon_s \setminus \partial C} \frac{|Dv_\varepsilon(x)|}{\sqrt{1+|Dv_\varepsilon(x)|^2}}\, d\mathcal{H}^{N-1}(x) \;+\; \mathcal{H}^{N-1}(\partial C^\varepsilon_s \cap \partial C) \;\to\; P(C_s) \end{aligned}$$

as $\varepsilon \to 0^+$. Hence,

$$\begin{aligned}\int_{\partial C_s} \left[z \cdot \nu^{C_s}\right] \, d\mathcal{H}^{N-1} &= \int_{C_s} \operatorname{div} z = \lim_{\varepsilon \to 0} \int_{C_s^\varepsilon} \operatorname{div} z_\varepsilon \\ &= \lim_{\varepsilon \to 0} \int_{\partial C_s^\varepsilon} [z_\varepsilon \cdot \nu_{C_s^\varepsilon}] \, d\mathcal{H}^{N-1} = -P(C_s).\end{aligned}$$

Since $|z| \leq 1$ a.e. in C, we deduce that $\left[z \cdot \nu^{C_s}\right] = -1$ on ∂C_s for any $s > s_*$ (in particular, we have $|z| = 1$ a.e. in $C \setminus C_*$). Using this and (19), for all $s_* < s \leq s^*$ we have

$$\frac{P(C_s)}{|C_s|} = h_C. \tag{21}$$

This has two consequences. First, from the isoperimetric inequality, we obtain

$$h_C = \frac{P(C_s)}{|C_s|} \geq \frac{P(B_1)}{|B_1|^{\frac{N-1}{N}} s^{\frac{1}{N}}},$$

if $s \in (s_*, s^*]$, so that $s_* > 0$. Moreover, C_s is a Cheeger set for any $s \in (s_*, s^*]$, and by continuity C_* is also a Cheeger set. □

We point out that, since the sets C_s are convex minimizers of $P(E) - \mu_s|E|$ among all $E \subseteq C$, for $s \geq s_*$, their boundary is of class $C^{1,1}$ [10, 22], with curvature less than or equal to μ_s, and equal to μ_s in the interior of C (note that $\mu_s = h_C$ for $s \in [s_*, s^*]$).

Remark 3.2. Observe that we have either $s_* = s^*$ and therefore $C_* = C^*$, or $s_* < s^*$, and we have $C^* = \bigcup_{s \in (s_*, s^*)} C_s$. In the latter case, the supremum of κ_{C^*} (which denotes the sum of the principal curvatures) on ∂C^* is equal to h_C. Indeed, if it were not the case, by considering $C' \subset \operatorname{int}(C^*)$, with curvature strictly below h_C, and the smallest set C_s, with $s > s_*$, which contains C', we would have $\kappa_{C'}(x) \geq \kappa_{C_s}(x) = h_C$ at all $x \in \partial C' \cap \partial C_s$, a contradiction. In particular, if the supremum of κ_C on ∂C is strictly less than $P(C)/|C|$ (which implies $C = C^*$ by [3]) then $C = C_*$.

From the strong convergence of Dv_ε to Dv (in $L^2(C_s)$ for any $s < s_*$), we deduce that $z = \frac{Dv}{\sqrt{1+|Dv|^2}}$ in C_*. It follows that v satisfies the equation

$$-\operatorname{div} \frac{Dv}{\sqrt{1 + |Dv|^2}} = h_C \quad \text{in } C_*. \tag{22}$$

Integrating both terms of (22) in C_*, we deduce that

$$\left[\frac{Dv}{\sqrt{1 + |Dv|^2}} \cdot \nu^{C_*}\right] = -1 \quad \text{on } \partial C_*.$$

Lemma 3.3. *The set C_* is the* minimal *Cheeger set of C, i.e., any other Cheeger set of C must contain C_*.*

Proof. Let $K \subseteq C^*$ be a Cheeger set in C. We have

$$h_C|K| \;=\; -\int_K \operatorname{div} z \;=\; -\int_{\partial K} [z\cdot\nu^K]\, d\mathcal{H}^{N-1} \;=\; P(K)$$

so that $[z\cdot\nu^K] = -1$ a.e. on ∂K. Let ν^ϵ and ν be the vector fields of unit normals to the sets C_s^ϵ and C_s, $s\in[0,|C|]$, respectively. Observe that, by the Hausdorff convergence of C_s^ϵ to C_s as $\epsilon\to 0^+$ for any $s\in[0,|C|]$, we have that $\nu^\epsilon\to\nu$ a.e. in C. On the other hand, $|z_\epsilon+\nu^\epsilon|\to 0$ locally uniformly in $C\setminus\overline{C_*}$: indeed, we have in C

$$|z_\epsilon+\nu^\epsilon| \;=\; \left|\frac{Dv_\varepsilon}{\sqrt{1+|Dv_\varepsilon|^2}} - \frac{Dv_\varepsilon}{|Dv_\varepsilon|}\right| \;=\; \left|\frac{|Dv_\varepsilon|}{\sqrt{1+|Dv_\varepsilon|^2}} - 1\right|.$$

Since (see the first lines of the proof of Lemma 3.1) $|Dv_\epsilon|\to\infty$ uniformly in any subset of C at positive distance from C_*, it shows the uniform convergence of $|z_\epsilon+\nu^\epsilon|$ to 0 in such subsets.

These two facts imply that $z=-\nu$ a.e. on $C\setminus C_*$. By modifying z in a set of null measure, we may assume that $z=-\nu$ on $C\setminus C_*$. We recall that the sets C_s, $s\geq s_*$ are minimizers of variational problems of the form $\min_{K\subseteq C} P(K)-\mu|K|$, for some values of μ (with $\mu=h_C$ as long as $s\leq s^*$ and $\mu=\mu_s>h_C$ continuously increasing with $s\geq s^*$). Since these sets are convex, with boundary (locally) uniformly of class $C^{1,1}$, and the map $s\to C_s$ is continuous in the Hausdorff topology, we obtain that the normal $\nu(x)$ is a continuous function in $C\setminus\operatorname{int}(C_*)$.

Since $|z|<1$ inside C_* and $[z\cdot\nu^K]=-1$ a.e. on ∂K, by [7, Theorem 1]) we have that the boundary of K must be outside the interior of C_*, hence either $K\supseteq C_*$ or $K\cap C_*=\emptyset$ (modulo a null set). Let us prove that the last situation is impossible. Indeed, assume that $K\cap C_*=\emptyset$ (modulo a null set). Since ∂K is of class C^1 out of a closed set of zero $\mathcal{H}^{N-1}$-measure (see [16]) and z is continuous in $C\setminus\operatorname{int}(C_*)$, by Theorem 2 we have

$$z(x)\cdot\nu^K(x) = -1 \qquad \mathcal{H}^{N-1}\text{–a.e. on } \partial K. \tag{23}$$

Now, since $K\cap C_*=\emptyset$ (modulo a null set), then there is some $s\geq s_*$ and some $x\in\partial C_s\cap\partial K$ such that $\nu^K(x)+\nu(x)=0$. Fix $0<\epsilon<2$. By a slight perturbation, if necessary, we may assume that $x\in\partial C_s\cap\partial K$ with $s>s_*$, (23) holds at x and

$$|\nu^K(x)+\nu(x)|<\epsilon. \tag{24}$$

Since by (23) we have $\nu(x)=-z(x)=\nu^K(x)$ we obtain a contradiction with (24). We deduce that $K\supseteq C_*$. □

Therefore, in order to prove uniqueness of the Cheeger sets of C, it is enough to show that

$$C_* = C^*. \tag{25}$$

Recall that the boundary of both C_* and C^* is of class $C^{1,1}$, and the sum of its principal curvatures is less than or equal h_C, and constantly equal to h_C in the interior of C. We now show that if $C_*\neq C^*$ and under additional assumptions, the sum of the principal curvatures of the boundary of C^* (or of any C_s for $s\in(s_*,s^*]$) must be h_C out of C_*.

Lemma 3.4. *Assume that C has C^2 boundary. Let $s \in (s_*, s^*]$ and $x \in \partial C_s \setminus \partial C_*$. If the sum of the principal curvatures of ∂C_s at x is strictly below h_C, then the Gaussian curvature of ∂C at x is 0.*

Proof. Let $x \in \partial C_s \setminus \partial C_*$ and assume the sum of the principal curvatures of ∂C_s at x is strictly below h_C (assuming x is a Lebesgue point for the curvature on ∂C_s). Necessarily, this implies that $x \in \partial C$. Assume then that the Gauss curvature of ∂C at x is positive: by continuity, in a neigborhood of x, C is uniformly convex and the sum of the principal curvatures is less than h_C. We may assume that near x, ∂C is the graph of a non-negative, C^2 and convex function $f : B \to \mathbb{R}$ where B is an $(N-1)$-dimensional ball centered at x. We may as well assume that ∂C_s is the graph of $f_s : B \to \mathbb{R}$, which is $C^{1,1}$ [10, 22], and also nonnegative and convex. In B, we have $f_s \geq f \geq 0$, and

$$D^2 f \;\geq\; \alpha I \quad \text{and} \quad \operatorname{div} \frac{Df}{\sqrt{1+|Df|^2}} \;=\; h$$

with $h \in C^0(\overline{B})$, $h < h_C$, $\alpha > 0$, while

$$\operatorname{div} \frac{Df_s}{\sqrt{1+|Df_s|^2}} \;=\; h\chi_{\{f=f_s\}} + h_C \chi_{\{f_s > f\}}$$

(where $\chi_{\{f=f_s\}}$ has positive density at x).

We let $g = f_s - f \geq 0$. Introducing now the Lagrangian $\Psi : \mathbb{R}^{N-1} \to [0, +\infty)$ given by $\Psi(p) = \sqrt{1+|p|^2}$, we have that for a.e. $y \in B$

$$\begin{aligned} (h_C - h(y))\chi_{\{g>0\}}(y) \;&=\; \operatorname{div}\left(D\Psi(Df_s(y)) - D\Psi(Df(y))\right) \\ &=\; \operatorname{div}\left(\left(\int_0^1 D^2\Psi(Df(y) + t(Df_s(y) - Df(y)))\,dt\right) Dg(y)\right) \end{aligned}$$

so that, letting $A(y) := \int_0^1 D^2\Psi(Df(y) + tDg(y))\,dt$ (which is a positive definite matrix and Lipschitz continuous inside B), we see that g is the minimizer of the functional

$$w \mapsto \int_B \Big(A(y)Dw(y) \cdot Dw(y) + (h_C - h(y))w(y) \Big)\,dy$$

under the constraint $w \geq 0$ and with boundary condition $w = f_s - f$ on ∂B. Adapting the results in [11] we get that $\{f = f_s\} = \{g = 0\}$ is the closure of a nonempty open set with boundary of zero $\mathcal{H}^{N-1}$-measure.

We therefore have found an open subset $D \subset \partial C \cap \partial C_s$, disjoint from ∂C_*, on which C is uniformly convex, with curvature less than h_C. Let φ be a smooth, nonnegative function with compact support in D. One easily shows that if $\varepsilon > 0$ is small enough, $\partial C_s - \varepsilon\varphi\nu^{C_s}$ is the boundary of a set C'_ϵ which is still convex, with $P(C'_\epsilon)/|C'_\epsilon| > P(C_s)/|C_s| = h_C$ (just differentiate the map $\epsilon \to P(C'_\epsilon)/|C'_\epsilon|$), and the sum of its principal curvatures is less than h_C. This implies that for $\epsilon > 0$ small enough, the set $C' := C'_\epsilon$ is calibrable [3], which in turn implies that $\min_{K \subset C'} P(K)/|K| = P(C')/|C'|$. But this contradicts $C_* \subset C'$, which is true for ε small enough. ☐

Proof of Theorem 3. Assume that C is C^2 and uniformly convex. Let us prove that its Cheeger set is unique. Assume by contradiction that $C^* \neq C_*$. From Lemma 3.4 we have that the sum of the principal curvatures of ∂C^* is h_C outside of C_*.

Let now $\bar{x} \in \partial C^* \cap \partial C_*$ be such that $\partial C^* \cap B_\rho(\bar{x}) \neq \partial C_* \cap B_\rho(\bar{x})$ for all $\rho > 0$ ($\partial C^* \cap \partial C_* \neq \emptyset$ since otherwise both C^* and C_* would be balls, which is impossible). Letting T be the tangent hyperplane to ∂C^* at $\bar{x}$, we can write ∂C^* and ∂C_* as the graph of two positive convex functions v^* and v_*, respectively, over $T \cap B_\rho(\bar{x})$ for $\rho > 0$ small enough. Identifying $T \cap B_\rho(\bar{x})$ with $B_\rho \subset \mathbb{R}^{N-1}$, we have that $v_*, v^* : B_\rho \to \mathbb{R}$ both solve the equation

$$-\operatorname{div} \frac{Dv}{\sqrt{1+|Dv|^2}} \;=\; f, \tag{26}$$

for some function $f \in L^\infty(B_\rho)$. Moreover, it holds $v_* \geq v^*$, $v_*(0) = v^*(0)$ and $v_*(y) > v^*(y)$ for some $y \in B_\rho$. Notice that $f = \lambda_C$ in the (open) set where $v_* > v^*$, in particular both functions are smooth in this set. Let D be an open ball such that $\overline{D} \subset B_\rho$, $v_* > v^*$ on D and $v_*(y) = v^*(y)$ for some $y \in \partial D$. Notice that, since both v^* and v_* belong to $C^\infty(D) \cap C^1(\overline{D})$, the fact that $v_*(y) = v^*(y)$ also implies that $Dv_*(y) = Dv^*(y)$. In D, both functions solve (26) with $f = \lambda_C$. Letting now $w = v_* - v^*$, we have that $w(y) = 0$ and $Dw(y) = 0$, while $w > 0$ inside D. Recalling the function $\Psi(p) = \sqrt{1+|p|^2}$, we have that for any $x \in D$

$$\begin{aligned} 0 \;&=\; \operatorname{div}\,(D\Psi(Dv_*(x)) - D\Psi(Dv^*(x))) \\ &=\; \operatorname{div}\left(\left(\int_0^1 D^2\Psi(Dv^*(x) + t(Dv_*(x) - Dv^*(x)))\,dt\right) Dw(x)\right) \end{aligned}$$

so that w solves a linear, uniformly elliptic equation with smooth coefficients. Then Hopf's lemma [13] implies that $Dw(y) \cdot \nu_D(y) < 0$, a contradiction. Hence $C_* = C^*$. $\square$

Remark 3.5. Notice that, as a consequence of Theorem 3 and the results of Giusti [14], we get that if C is of class C^2 and uniformly convex, equation (22) has a solution on the whole of C, if and only if C is a Cheeger set of itself, i.e. if and only if the the sum of the principal curvatures of ∂C is less than or equal to $P(C)/|C|$.

Remark 3.6. The results of this paper can be easily extended to the anisotropic setting (see [12]) provided the anisotropy is smooth and uniformly elliptic.

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