EchoScan: Elastography and Electrical Impedance Tomography

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Report on joint work with

- Yves Capdeboscq (Université de Versailles Saint-Quentin & Oxford University)
- Jérôme Fehrenbach (Université de Versailles Saint-Quentin & Université de Toulouse)
- Frédéric de Gournay (Université de Versailles Saint-Quentin)

Today's Talk

Polarization Tensor
Elastography + EIT

Polarization Tensor

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Here is the result of Y. Capdeboscq & Michael Vogelius (2003) on a representation formula for $u_\varepsilon - u$ on $\partial\Omega$ where

$$\begin{cases} -\mathrm{div}(\gamma_\varepsilon \nabla u_\varepsilon) = 0 & \text{in } \Omega \\ \gamma_\varepsilon \dfrac{\partial u_\varepsilon}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega \end{cases} \qquad \begin{cases} -\mathrm{div}(\gamma \nabla u) = 0 & \text{in } \Omega \\ \gamma \dfrac{\partial u}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

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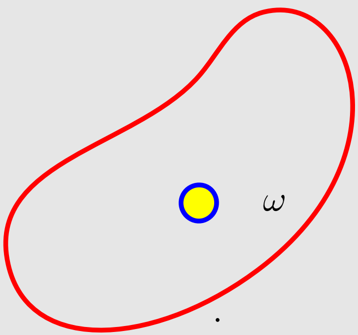


The small domain $\omega(x)$ centered at $x \in \Omega$ is perturbed into $\omega_\varepsilon(x)$ with a volume

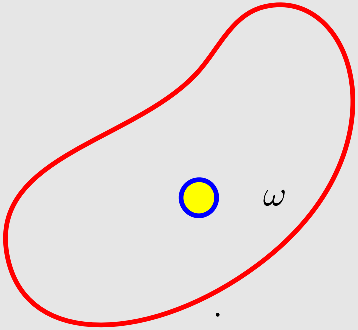
$$|\omega_\varepsilon| \approx (1 + 3r^{-1}\delta r)|\omega|.$$

We assume that *locally* $\gamma(x)$ is constant and that

$$\gamma_\varepsilon(x) = \gamma(x)\nu_\varepsilon(x) \approx \gamma(x)\nu(x),$$

with a known coefficient $\nu(x) = \lim_{\varepsilon\to 0} |\omega_\varepsilon(x)|/|\omega(x)|$.

Polarization Tensor



One has $|\omega_\varepsilon|^{-1} 1_{\omega_\varepsilon} \rightharpoonup \mu$ in $M(\overline{\Omega})$, and for $y \in \partial\Omega$ let $N(x, y)$ be the Green function

$$\begin{cases} -\mathrm{div}(\gamma(x)\nabla_x N(x, y)) = 0 & \text{in } \Omega \\ \gamma(\sigma)\dfrac{\partial}{\partial \mathbf{n}_x} N(\sigma, y) = -\delta_y + |\partial\Omega|^{-1} & \text{on } \partial\Omega \end{cases}$$

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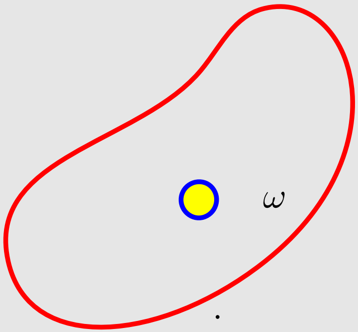

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Theorem. *Assume that* $\gamma_\varepsilon(x) = \gamma(x) + [\widetilde{\gamma}(x) - \gamma(x)]\, 1_{\omega_\varepsilon}(x)$. *Then there exists a positive definite* $M \in (L^2(\Omega, d\mu))^{N\times N}$ *such that for* $y \in \Omega$ *we have*

$$u_\varepsilon(y) - u(y) = |\omega_\varepsilon| \int_\Omega (\widetilde{\gamma}(x) - \gamma(x)) M(x) \nabla u(x) \cdot \nabla_x (x,y) d\mu(x) + o(|\omega_\varepsilon|).$$

Polarization Tensor

- As a matter of fact (Y. Capdeboscq & M. Vogelius, 2007), the polarization tensor M may be be characterized by the following identity: for all $\xi \in \mathbb{R}^N$ and $v \in C(\overline{\Omega})$

$$\int_\Omega (\widetilde{\gamma} - \gamma) M(x)\xi \cdot \xi \, v(x) dx = \frac{1}{|\omega_\varepsilon|} \min_{w \in H^1_{\text{per}}} \int_\Omega \gamma_\varepsilon \left| \nabla w + \frac{\widetilde{\gamma} - \gamma}{\widetilde{\gamma}} 1_{\omega_\varepsilon} \xi \right|^2 v(x)\, dx$$

$$+ \quad \frac{|\xi|^2}{|\omega_\varepsilon|} \int_{\omega_\varepsilon} (\widetilde{\gamma} - \gamma) \frac{\gamma}{\widetilde{\gamma}} v(x)\, dx + o(1)$$

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- For some simple geometries such as disks, M is well known: if ω_ε is a disk of radius ε centered at $z \in \Omega$

$$\int_{\partial\Omega} (u_\varepsilon - u)\varphi(\sigma) d\sigma = \int_{\omega_\varepsilon} \gamma(x) \frac{\nu(x) - 1}{\nu(x) + 1} \nabla u(x) \cdot \nabla u(x) dx + O(|\omega_\varepsilon|^{1+\alpha})$$
$$\approx |\nabla u(z)|^2 \gamma(z) \int_{\omega_\varepsilon} \frac{\nu(x) - 1}{\nu(x) + 1} dx + O(|\omega_\varepsilon|^{1+\alpha})$$

Elastography + EIT

Elastography + EIT

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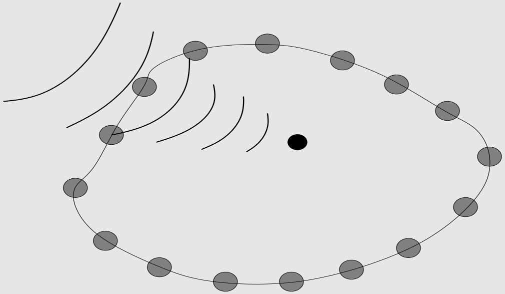

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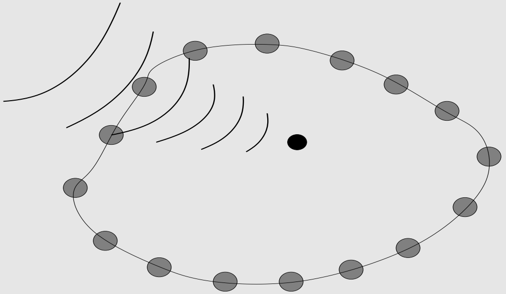

- This implies a contraction and a dilation of a small area $B := B(x, \varepsilon)$ around x, inducing a change in the conductivity $\gamma \mapsto \gamma_\varepsilon$ (with a known factor ν)

$$\gamma_\varepsilon(x) := (1 + (\nu - 1)1_B)\gamma(x)$$

Elastography + EIT

- So one has an asymptotic formula for the perturbed electrical potential u_ε

$$\int_{\partial\Omega}(u_\varepsilon - u)\varphi d\sigma = |B| \int_{\Omega}(\gamma_\varepsilon - \gamma)M_B \nabla u \cdot \nabla u dx + o(|B|)$$

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- Here M_B is the polarization tensor which depends only on the geomtery of B. In the case where B is a ball one has

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- Finally for ω_ε a ball centered at $z \in \Omega$ we obtain $\gamma(z)|\nabla u(z)|^2$ which is the local electrical energy density

$$\gamma(z)|\nabla u(z)|^2 \approx \left(\int_{\omega_\varepsilon} \frac{\nu(x) - 1}{\nu(x) + 1} dx\right)^{-1} \int_{\partial\Omega}(u_\varepsilon - u)\varphi d\sigma.$$

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- Hence for each current density φ on $\partial\Omega$ we know $S(x) := \gamma(x)|\nabla u(x)|^2$, the corresponding local electrical energy density.

Elastography + EIT

- One can now study the nonlinear equation

$$
\begin{cases} -\operatorname{div}\left(S(x)\dfrac{\nabla u}{|\nabla u|^2}\right) = 0 \\ \dfrac{S}{|\nabla u|^2}\,\dfrac{\partial u}{\partial \mathbf{n}} = \varphi \end{cases} \tag{2.1}
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- Indeed several difficulties arise: we need a current φ on the boundary to ensure that $|\nabla u| \neq 0$,
- solving **(2.1)** is not easy since its solutions correspond to critical points of

$$J(u) := \int_\Omega S(x)\log(|\nabla u(x)|^2)dx - 2\int_{\partial\Omega}\varphi(\sigma)d\sigma.$$

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- Another approach is to set $v := e^u$ and $\gamma := e^a$ and one finds that v satisfies (here we may assume that u is also known on th eboundary)

$$\begin{cases} -\mathrm{div}\left(e^{a(x)}\,\nabla v(x)\right) + S(x)v = 0 & \text{in } \Omega \\ v = \psi & \text{on } \partial\Omega. \end{cases}$$

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- Then one seeks $a \in L^{\infty}(\Omega)$ such that

$$\mathrm{e}^{a}\,|\nabla v|^{2} = S|v|^{2}.$$

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- Let $K := \{\gamma \in L^\infty(\Omega) \; ; \; \gamma \geq \varepsilon_0 > 0\}$ and consider the functional

$$F : K \longrightarrow L^1(\Omega), \qquad F(\gamma) := \gamma |\nabla u|^2$$

where u satisfies

$$\left\{ \begin{aligned} -\mathrm{div}(\gamma \nabla u) &= 0 \\ \gamma \frac{\partial u}{\partial \mathbf{n}} &= \varphi \end{aligned} \right. \tag{2.2}$$

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$$\text{(2.2)} \qquad \begin{cases} -\mathrm{div}(\gamma\nabla u) = 0 \\ \gamma \dfrac{\partial u}{\partial \mathbf{n}} = \varphi \end{cases}$$

- $\gamma \mapsto F(\gamma)$ is an analytic operator. and one checks easily that

$$F'(\gamma)\delta = \delta|\nabla u|^2 + 2\gamma\nabla u \cdot \nabla v,$$

where v satisfies

$$\text{(2.3)} \qquad \begin{cases} -\mathrm{div}(\gamma\nabla v) = \mathrm{div}(\delta\nabla \mathrm{u}) & \text{in } \Omega \\ \gamma \dfrac{\partial v}{\partial \mathbf{n}} = 0 & \text{on } \partial\Omega. \end{cases}$$

Elastography + EIT

- An observed data $F_{\mathrm{obs}} := S_{\mathrm{obs}}$ being given, we try to find γ^* such that $F(\gamma^*) = F_{\mathrm{obs}}$, by minimizing a cost functional depending on $F(\gamma^*) - F_{\mathrm{obs}}$.

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Elastography + EIT

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- Multigrid approach

$$J_1(\gamma) := \sum_{1 \le k \le m} \left(\int_{\omega_k} F(\gamma) dx - \int_{\omega_k} S_{\text{obs}}(x) dx \right)^2$$

Elastography + EIT

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- A classical quadratic functional such as

$$J_2(\gamma) := \int_{\Omega} \left(F(\gamma)(x) - S_{\mathrm{obs}}(x) \right)^2 dx$$

has been considered.

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- Also we have considered a slightly different functional (with $\gamma = \mathrm{e}^{a}$)

$$J_3(a) := \int_\Omega \left| \mathrm{e}^{a(x)/2} |\nabla u(x)| - S_{\mathrm{obs}}(x)^{1/2} \right|^2 dx$$

gives quite robust results,

Elastography + EIT

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- and one can show that if $S_{\text{obs}} = \gamma^* |\nabla u^*|^2$ for some admissible γ^*, then the functionals J_2 and J_3 are strictly convex in a neighbourhood of γ^*.

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- and one can show that if $S_{obs} = \gamma^* |\nabla u^*|^2$ for some admissible γ^*, then the functionals J_2 and J_3 are strictly convex in a neighbourhood of γ^*.

- If one assumes that S_{obs} is known only in a subdomain $\Omega_0 \subset\subset \Omega$, then the functionals J_1, J_2, J_3 may be defined only on Ω_0 and numerically one obtains quite good results.

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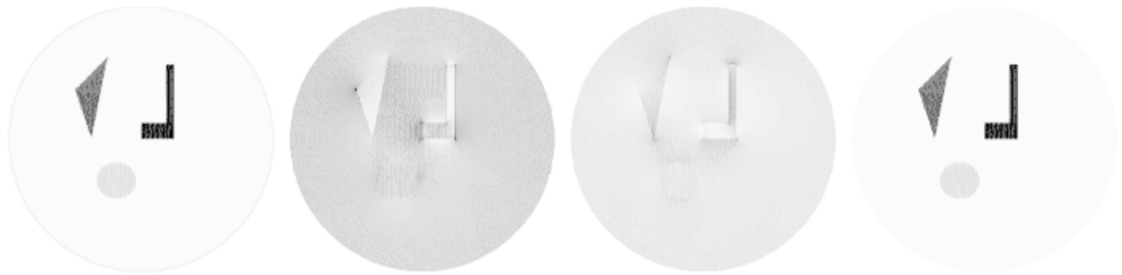
Test case : background at 0.5, triangle at 2, ellipse at 0.75, and “L” at 2.55.



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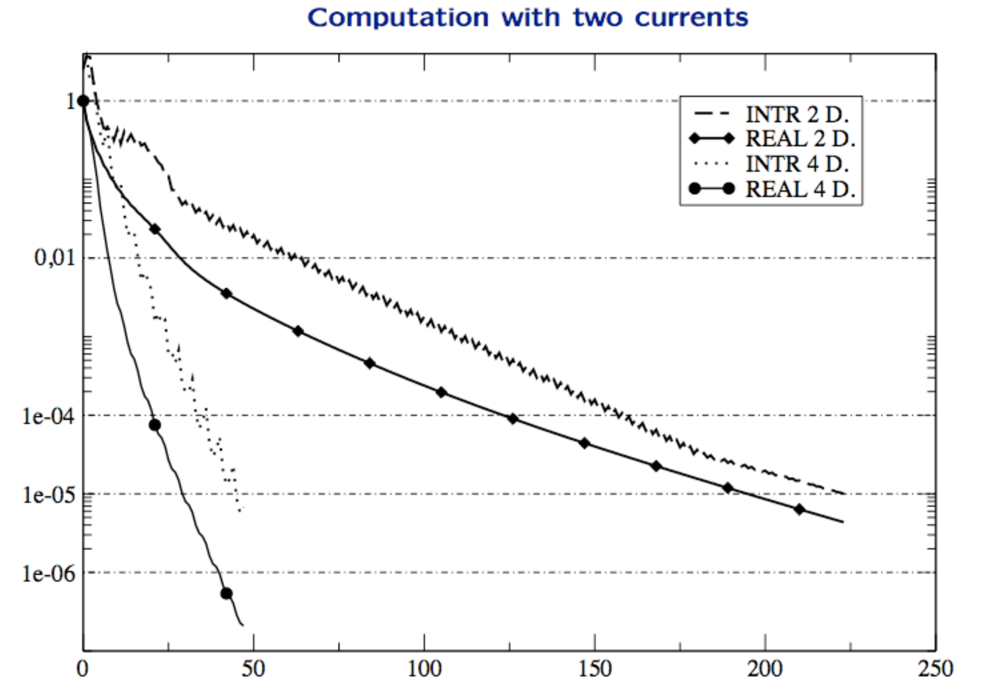

The four directions correspond to two currents, $x/|x|$ and $y/|y|$.

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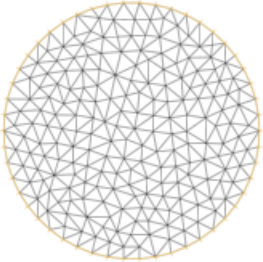
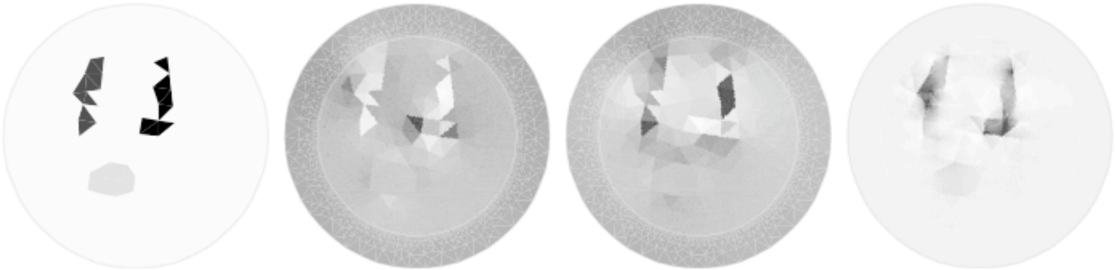
Reconstruction test

Coarse mesh: few measurement points (50 bdy points).





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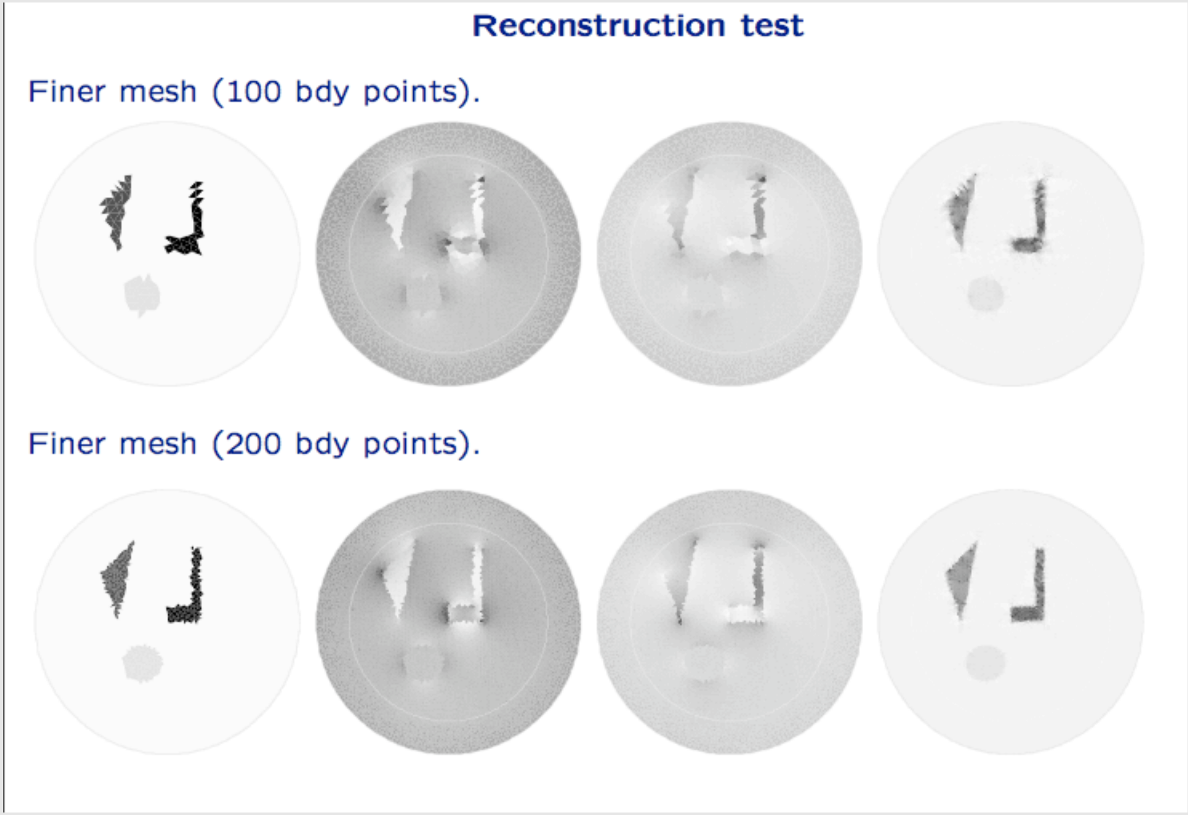

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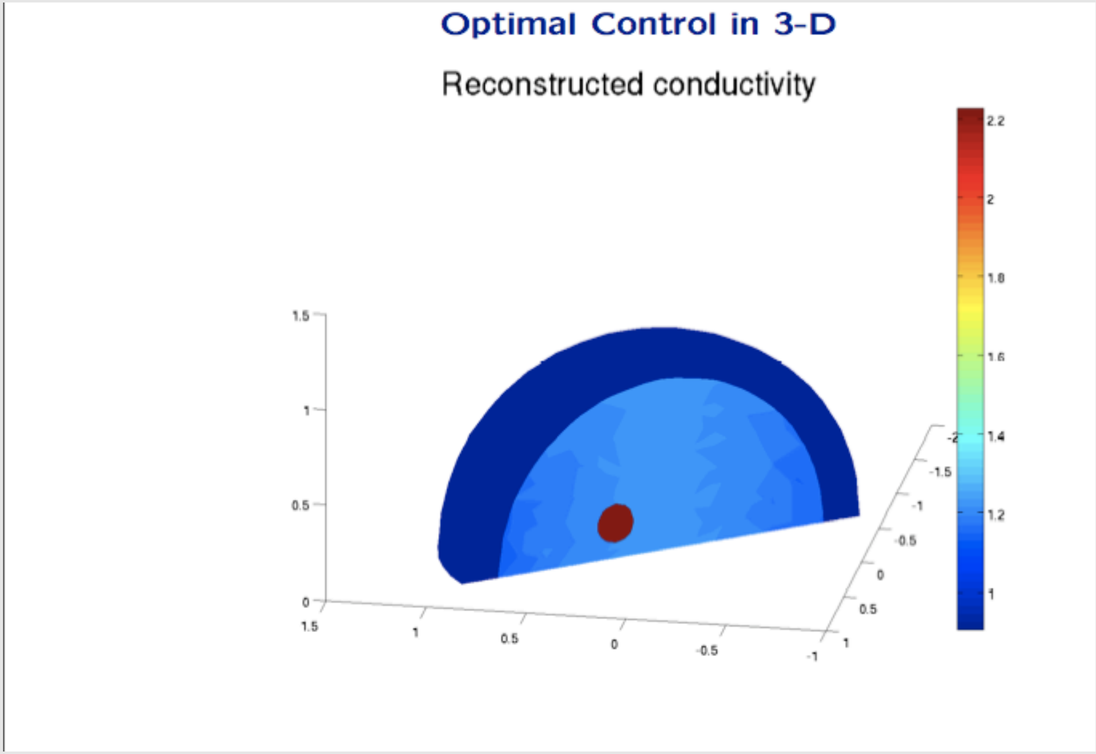

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