# Maxwell's equations and elastic waves with a pressure term: Simultaneous controllability

by

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# 1. Introduction

– In a bounded region $\Omega \subseteq \mathbb{R}^3$ with smooth boundary $\partial\Omega$ we consider two different hyperbolic models. One of them is the system of Maxwell equations and the second one is a vector wave equation with a pressure term.

– Under suitable geometric conditions on $\Omega$ we obtain for each one of the above models a boundary observability inequality

– Our main result says that we can collect the above information together with some new identities and suitable relation on the parameters of the models to obtain "simultaneous" exact boundary control for both systems.

– "Simultaneous" exact control for wave equations, Maxwell equations and other hyperbolic systems of second order started with the pioneer work of D. Russell and J.L. Lions in the middle 80's.

– In the absence of dissipations, almost all authors considered two models which *differed only on the boundary conditions* in order to get "simultaneous" exact controllability.

– B. Kapitonov

Two systems of elastic waves (Siberian Math. J., 1994).

Two systems of Maxwell equations (Comp. Appl. Math., 1996)

– B. Kapitonov + G. Perla Menzala

Two quasi-electrostatic piezoelectric systems (Acta Appl. Mathematicae, 2006).

– B. Kapitonov + M.A. Raupp

Two piezoelectric systems in multilayered media (Comp. Appl. Math., 2003).

– There are several articles considering some dissipative effects on the above models or coupled systems obtaining exact controllability through Russell's "controlability via stabilizability" principle.

## Description of the problem

Let $u = u(x,t) = (u_1(x,t), u_2(x,t), u_3(x,t))$ be the displacement vector

$p = p(x,t)$ scalar function, pressure

$E = E(x,t) = (E_1(x,t), E_2(x,t), E_3(x,t))$ be the electric field

$H = H(x,t) = (H_1(x,t), H_2(x,t)), H_3(x,t))$ be the magnetic field

$\mathcal{E}_0$, $\mu_0$, $\rho$ and $\alpha$ are *strictly positive constants* which represent the permittivity, permeability, scalar density and elastic property of the material respectively.

## Maxwell equations

$$\begin{cases} \mathcal{E}_0 E_t = curl H \\ \mu_0 H_t = -curl E \\ \text{div } E = 0 \qquad\qquad\qquad\qquad \text{in } \Omega \times (0,T) \\ \text{div } H = 0 \\ \eta \times E = R(x,t) \text{ on } \partial\Omega \times (0,T) \\ E(x,0) = E_0(x),\ H(x,0) = H_0(x) \text{ in } \Omega \end{cases} \tag{1}$$

## Vector wave equation

$$\begin{cases} \rho u_{tt} - \alpha \Delta u + \text{grad } p = 0 \\ \text{div } u = 0 \qquad\qquad\qquad\qquad \text{in } \Omega \times (0,T) \\ u = S(x,t) \text{ on } \partial\Omega \times (0,T) \\ u(x,0) = u_0(x), \; u_t(x,0) = u_1(x) \text{ in } \Omega \end{cases} \tag{2}$$

## Remark 1

Instead of model (2) we can also treat

$$\rho u_{tt} - \sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial u}{\partial x_i}\right) + \text{grad } p = 0$$

where $A_{ij}$ are $3 \times 3$ matrices given by $A_{ij} = [C^{ij}_{kh}]_{3\times 3}$ where

$$C^{ij}_{kh} = (1 - \delta_{ih}\delta_{jk})a_{ikjh} + \delta_{ik}\delta_{jh}a_{ihjk}$$

with the symmetry

$$a_{ijkh} = a_{jikh} = a_{khij}.$$

The isotropic case will be if

$$a_{ijkh} = \lambda\delta_{ij}\delta_{kh} + \alpha(\delta_{jk}\delta_{jh} + \delta_{ih}\delta_{jk})$$

where $\lambda$ and $\alpha$ are Lame's constants. In this case the term

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial x_j}\left(A_{ij}\frac{\partial u}{\partial x_j}\right)$$

reduces to $\alpha\Delta u + (\lambda + \alpha)\text{grad}(\text{div } u)$. In order to simplify calculations we chose $\lambda + \alpha = \mathbf{0}$ to obtain (2).

## The problem

Given initial states $(E_0, H_0)$, $(u_0, u_1)$, a time $T > 0$ and desired terminal states $(\varphi_0, \varphi_1)$, $(\psi_0, \psi_1)$ we want to find a vector valued function $S = S(x,t)$ such that the solution $\{E, H, u, u_t\}$ of (1), (2) satisfies

$$(E, H)|_{t=T} = (\varphi_0, \varphi_1), \quad (u, u_t)|_{t=T} = (\psi_0, \psi_1)$$

$S$ serving as a control function for (2) while the function $R = \mu_0 \eta \times (\eta \times S_t)$ is a control function for (1).

As we describe below the answer is YES as long as we assume a geometric condition on $\Omega$ and a suitable relation between $\mathcal{E}_0$, $\mu_0$, $\rho$ and $\alpha$.

## Remark 2.

**1)** The techniques we use may allow us to consider variable coeffcients $\mathcal{E}_0(x), \mu_0(x), \rho(x)$ and $\alpha(x)$ smooth and bounded below by strictly positive constants.

**2)** We do not want to reduce the Maxwell equations (1) to a second order vector wave equation (which is usually done in the isotropic case) because we want eventually to extend our discussion to the "anisotropic" Maxwell equations. In this case $\mathcal{E}_0(x)$ and $\mu_0(x)$ are $3 \times 3$ symmetric matrices, positive defined. It is well known that the above reduction can not be done in the anisotropic case.

## Function spaces

Consider Maxwell's equations (1) with $R \equiv 0$. Let

$$\mathcal{H} = [L^2(\Omega)]^3 \times [L^2(\Omega)]^3$$

$$H(\text{curl}, \Omega) = \{w \in [L^2(\Omega)]^3;\ \text{curl}\ w \in [L^2(\Omega)]^3\}$$

with inner products

$$\langle v, w \rangle_{\mathcal{H}} = \int_\Omega \{\mathcal{E}_0 v_1 \cdot w_1 + \mu_0 v_2 \cdot w_2\}\, dx$$

$\forall\ v(v_1, v_2),\ w = (w_1, w_2) \in \mathcal{H}$

and

$$\langle v_1, v_2 \rangle_{H(\text{curl},\Omega)} = \int_\Omega \{v_1 \cdot v_2 + \text{curl}\ v_1 \cdot\ \text{curl}\ v_2\}\, dx.$$

Finally

$$\mathcal{H}_0 = H(\text{curl}, \Omega) \times H(\text{curl}, \Omega)$$

with

$$\langle v, w\rangle_{\mathcal{H}_0} = \int_\Omega \{\mathcal{E}_0 v_1 \cdot w_1 + \mu_0 v_2 \cdot w_2 + \text{curl } v_1 \cdot \text{curl } w_1$$
$$+\text{curl } v_2 \cdot \text{curl } w_2\}\, dx$$

Consider the closed subspace

$$\mathcal{H}_1 = \{w = (w_1, w_2) \in \mathcal{H}_0;\ \eta \times w_1 = 0 \text{ on } \partial\Omega\}.$$

Define

$$\mathcal{A} \colon \mathcal{D}(\mathcal{A}) = \mathcal{H}_1 \mapsto \mathcal{H}$$

Then, $\mathcal{A}$ is skew-selfadjoint. By Stone's theorem $\mathcal{A}$ generates a one parameter group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}$. Remains to Prove that the components of $U(t)f$ are divergente free. Here $U(t)f = (w_1, w_2) = (E, H)$.

Observe that the condition

$$\text{div } w_1 = 0 \qquad \text{div } w_2 = 0$$

(in the sense of distributions) means to say that $w = (w_1, w_2) \in M_1 = M^\perp$ where

$$M = \{(\text{grad } \varphi_1, \text{grad } \varphi_2) \quad \text{with} \quad \varphi_1, \varphi_2 \in C_0^\infty(\Omega)\}.$$

We can prove that $U(t)$ takes $M_1 \cap \mathcal{D}(\mathcal{A})$ into itself. Therefore, problem (1) (with $R \equiv 0$) is globally well posed for any initial data in $M_1 \cap \mathcal{D}(\mathcal{A})$.

**Remark 3.** We can check that any element $v = (v_1, v_2) \in M_1 \cap \mathcal{D}(\mathcal{A})$ satisfies

$$\eta \cdot v_2 = 0 \quad \text{on} \quad \partial\Omega$$

(in the sense of distributions).

Concerning problem (2) (with $S \equiv 0$) we can use Galerkin method to find $u$ and $p$ (defined up to a constant). This is well known by choosing

$$V = \{\varphi \in [C_0^\infty(\Omega)]^3,\ \text{div}\ \varphi = 0\}$$

$$V = \text{the closure of } V \text{ with respect to the norm of } [H_0^1(\Omega)]^3$$

and

$$W = V \cap [H^2(\Omega)]^3.$$

Considering $u_0 \in W$, $u_1 \in V$ we obtain a unique solution $\{u, p\}$ of problem (2) with $p$ unique up to an additive constant.

An alternative would be to use R. Farwing + J. Sohr

(J. Math. Soc. Japan 46, 1994, 607–643) and write

$$[L^2(\Omega)]^3 = \overline{\{v \in [C_0^\infty(\Omega)]^3,\ \text{div}\ v = 0 \text{ in } \Omega\}} \oplus$$

$$\{\text{grad}\ p \in [L^2(\Omega)]^3 \quad \text{with } p \in L^2(\Omega)\}$$

$$= \overset{\circ}{Y}(\Omega) \oplus G(\Omega)$$

the closure is in the norm of $[L^2(\Omega)]^3$.

Let $\mathbb{P}$ the continuous projection from $[L^2(\Omega)]^3$ to $\overset{\circ}{Y}(\Omega)$ and the Stokes operator $\mathbb{A} = -\mathbb{P}\Delta$ with domain

$$\mathcal{D}(\mathbb{A}) = \{w \in \overset{\circ}{Y}(\Omega) \cap [H^2(\Omega)]^3;\ w|_{\partial\Omega} = 0\}$$

Let

$$\mathcal{H} = \{u = (u_1, u_2),\ u_1 \in [H^1(\Omega)]^3,\ \text{div}\ u_1 = 0,\ u_2 \in \overset{\circ}{Y}(\Omega)\}$$

with inner product

$$\langle u, w\rangle_{\mathcal{H}} = \int_\Omega \left\{ \rho u_2 \cdot w_2 + \alpha \sum_{j=1}^{3} \frac{\partial u_1}{\partial x_j} \cdot \frac{\partial w_1}{\partial x_j} \right\} dx$$

whenever $u = (u_1, u_2)$, $w = (w_1, w_2) \in \mathcal{H}$. In $\mathcal{H}$ we define the operator $\widetilde{A}$

$$\widetilde{A}u = \widetilde{A}(u_1, u_2) = (u_2, -\rho^{-1}\alpha\mathbb{A}u_1)$$

with domain

$$\mathcal{D}(\widetilde{A}) = \{u = (u_1, u_2) \in \mathcal{H},\ u_1 \in [H^2(\Omega)]^3 \cap \overset{\circ}{Y}(\Omega),$$
$$u_1 = 0 \text{ on } \partial\Omega,\ u_2 \in \overset{\circ}{Y}(\Omega)\}.$$

Using results in the above article we deduce that $\widetilde{A}$ generates a one-parameter group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ on $\mathcal{H}$.

**Observation.** In the standard way we could obtain more regular solutions of either problem (1) or (2).

## Boundary observability

Let $h = h(x)$ smooth scalar function on $\overline{\Omega}$

$$M_1 = M_1(E, H) = tE + \mu_0 \nabla h \times H$$
$$M_2 = M_2(E, H) = tH - \mathcal{E}_0 \nabla h \times E.$$

If $\{E, H\}$ regular solution of problem (1) (with $R \equiv 0$). Then

$$0 = 2M_1 \cdot \{\mathcal{E}_0 E_t - \text{curl } H\} + 2M_2 \cdot \{\mu_0 H_t + \text{curl } E\}$$
$$+2\mathcal{E}_0(\nabla h \cdot E)\text{div } E + 2\mu_0(\nabla h \cdot H)\text{div } H.$$

Rearranging terms in the identity to obtain

$$\boxed{\frac{\partial A}{\partial t} = \operatorname{div} \vec{B} + D} \tag{3}$$

(Fundamental Identity)

where

$$\boxed{A = t(\mathcal{E}_0|E|^2 + \mu_0|H|^2) + 2\mathcal{E}_0\mu_0\nabla h \cdot (H \times E)}$$

$$\vec{B} = 2tH \times E + \nabla h\{\mathcal{E}_0|E|^2 + \mu_0|H|^2\}$$
$$- 2\mathcal{E}_0 E(E \cdot \nabla h) - 2\mu_0 H(H \cdot \nabla h)$$

and

$$D = 2\sum_{i,j=1}^{3} \frac{\partial^2 h}{\partial x_i \partial x_j}\{\mathcal{E}_0 E_i E_j + \mu_0 H_i H_j\}$$
$$- (\Delta h - 1)\{\mathcal{E}_0|E|^2 + \mu_0|H|^2\}.$$

Similarly, let $\{u, p\}$ regular solution of problem (2) (with

$S \equiv 0$) and consider

$$M_3 = M_3(u) = tu_t + (\nabla h \cdot \nabla)u + u$$
$$M_4 = M_4(p) = tp\frac{\partial}{\partial t} + p(\nabla h \cdot \nabla) + p$$

then

$$0 = 2M_3 \cdot \{\rho u_{tt} - \alpha \Delta u + \nabla p\} + 2M_4(p)\text{div } u.$$

Rearranging terms in the above identity we obtain

$$\boxed{\frac{\partial A_1}{\partial t} = \text{ div } \vec{G} + D_1} \quad (4)$$

(Fundamental Identity)

where

$$A_1 = t\{\rho|u_t|^2 + \alpha \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^2\} + 2\rho u_t \cdot [(\nabla h \cdot \nabla)u + u]$$

$$\vec{G} = (G_1, G_2, G_3) + (-2\rho[tu_t + (\nabla h \cdot \nabla)u + u])$$
$$G_i = 2[tu_t + (\nabla h \cdot \nabla)u + u] \cdot \alpha\frac{\partial u}{\partial x_i}$$
$$+ \frac{\partial h}{\partial x_i}\left(\rho|u_t|^2 - \alpha \sum_{k=1}^{3} \left|\frac{\partial u}{\partial x_k}\right|^2\right)$$

and

$$D_1 = (3-\Delta h)\rho|u_t|^2 + (\Delta h - 1)\alpha \sum_{k=1}^{3} \left|\frac{\partial u}{\partial x_k}\right|^2$$
$$- 2\alpha \sum_{i,q=1}^{3} \frac{\partial^2 h}{\partial x_q \partial x_i}\left(\frac{\partial u}{\partial x_i}\cdot\frac{\partial u}{\partial x_q}\right) + 2p \sum_{i,k=1}^{3} \frac{\partial^2 h}{\partial x_k \partial x_i}\frac{\partial u_k}{\partial x_i}.$$

Integration over $\Omega \times (0,T)$ of identity (3) give us

$$T\int_\Omega \{\mathcal{E}_0|E|^2 + \mu_0|H|^2\}\,dx + 2\mathcal{E}_0\mu_0\int_\Omega \nabla h\cdot(H\times E)\,dx\Big|_{t=0}^{t=T}$$
$$= \int_0^T\int_{\partial\Omega} J(E,H,h)\,d\Gamma\,dt| \int_0^T\int_\Omega D\,dxdt \tag{5}$$

where

$$J = 2t\eta\cdot(H\times E) + \frac{\partial h}{\partial \eta}(\mathcal{E}_0|E|^2 + \mu_0|H|^2)$$
$$- 2\mathcal{E}_0(E\cdot\eta)(E\cdot\nabla h) - 2\mu_0(H\cdot\eta)(H\cdot\nabla h)$$

We use the boundary condition of problem (1) (with $R\equiv 0$) i.e. $\eta\times E = 0$ on $\partial\Omega\times(0,T)$ and obtain

$$\boxed{J = \frac{\partial h}{\partial \eta}\{\mu_0|H\times\eta|^2 - \mathcal{E}_0(E\cdot\eta)^2\}}$$

Next, we want to find appropiate bounds for $\int_0^T\int_\Omega D\,dxdt$.

Consider the problem

$$\begin{cases} \Delta\Phi = 1 \text{ in } \Omega \\ \dfrac{\partial\Phi}{\partial\eta} = \dfrac{\text{Vol}(\Omega)}{\text{Area}(\partial\Omega)} \text{ on } \partial\Omega \end{cases}$$

which admits solution $\Phi \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

Let $0 < \delta < 1$ and define

$$\boxed{h(x) = \delta\Phi(x) + \frac{1}{2}|x - x_0|^2}$$

for some $x_0 \in \mathbb{R}^3$.

Direct calculations proves that

$$D = 2\delta \sum_{i,j=1}^{3} \frac{\partial^2\Phi}{\partial x_i \partial x_j}(\mathcal{E}_0 E_i E_j + \mu_0 H_i H_j) - \delta(\mathcal{E}_0|E|^2 + \mu_0|H|^2)$$

Let $C = C(\Phi)$ be

$$C(\Phi) = \max_{\substack{x\in\overline{\Omega} \\ i,j=1,2,3}} \left|\frac{\partial^2\Phi(x)}{\partial x_i \partial x_j}\right|.$$

We can verify that $C(\Phi) \geq \frac{1}{3}$ and obtain the bound

$$|D| \leq \delta\{6C(\Phi) - 1\}\{\mathcal{E}_0|E|^2 + \mu_0|H|^2\}$$

which give us the estimate

$$\int_0^T \int_\Omega D\,dxdt \le \delta(6C(\Phi)-1)T\int_\Omega (\mathcal{E}_0|E|^2+\mu_0|H|^2)\,dx. \tag{6}$$

Finally we want to get bounds for the term

$$2\mathcal{E}_0\mu_0\int_\Omega \nabla h\cdot(H\times E)\,dx\Big|_{t=0}^{t=T}$$

in (5). Let

$$C_1(\Phi)=\max_{x\in\overline{\Omega}}\{|\nabla\Phi|+|x-x_0|\}.$$

Then we can verify that

$$\begin{aligned} &2\int_\Omega \mathcal{E}_0\mu_0\nabla h\cdot(H\times E)\,dx \\ &\qquad \le 4\sqrt{\mathcal{E}_0\mu_0}C_1(\Phi)\int_\Omega\{\mathcal{E}_0|E|^2+\mu_0|H|^2\}. \end{aligned} \tag{7}$$

Hence, we obtain the estimate

$$\begin{aligned} &[1-\delta(6C(\Phi)-1)](T-T_0)\int_\Omega\{\mathcal{E}_0|E|^2+\mu_0|H|^2\}\,dx \\ &\qquad \le \int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\{\mu_0|H\times\eta|^2-\mathcal{E}_0(E\cdot\eta)^2\}\,d\Gamma \end{aligned} \tag{8}$$

where

$$T_0=\frac{4\sqrt{\mathcal{E}_0\mu_0}C_1(\Phi)}{1-\delta(6C(\Phi)-1)}$$

In the same line of ideas, using identity using (4), we find that the solution of problem (2) (with $S \equiv 0$) satisfies

$$
\begin{aligned}
&[1-\delta\tilde{c}_1](T-\tilde{T}_0)\int_\Omega \left\{\rho|u_t|^2 + \alpha\sum_{i=1}^{3}\left|\frac{\partial u}{\partial x_i}\right|^2\right\} dx \\
&\quad \le \int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\alpha\left|\eta\times\frac{\partial u}{\partial\eta}\right|^2 d\Gamma\, dt
\end{aligned}
\tag{9}
$$

for some $\tilde{c}_1 > 0$, $\tilde{T}_0 > 0$ and $T > \tilde{T}_0$.

To use conveniently inequalities (8) and (9) we will choose $\delta = \delta_1 > 0$ such that

$$
1-\delta_1(6C(\Phi)-1) > 0, \quad 1-\delta_1\tilde{c}_1 > 0
$$

and a geometric condition on $\Omega$:

## Hipothesis

**There exists $x_0 \in \Omega$ such that**

$$
\delta_1\frac{\mathbf{Vol}(\Omega)}{\mathbf{Area}(\partial\Omega)} + (x-x_0)\cdot\eta > 0 \qquad \textbf{for all } x \in \partial\Omega.
$$

Observe that since $h(x) = \delta_1\Phi(x) + \frac{1}{2}|x-x_0|^2$ then

$$
\frac{\partial h}{\partial\eta}(x) = \delta_1\frac{\partial\Phi}{\partial\eta} + (x-x_0)\cdot\eta = \delta_1\frac{\mathrm{Vol}(\Omega)}{\mathrm{Area}(\partial\Omega)} + (x-x_0)\cdot\eta
$$

for any $x \in \partial\Omega$.

From (8) and (9) we deduce

$$(1-\delta_1 c_2(T-T_1)\int_\Omega \Big\{\mathcal{E}_0|E|^2+\mu_0|H|^2 + \rho|u_t|^2+\alpha\sum_{i=1}^3\Big|\frac{\partial u}{\partial x_i}\Big|^2\Big\}\,dx \tag{10}$$
$$\le \int_0^T\int_{\partial\Omega}\frac{\partial h}{\partial\eta}\Big\{\alpha\Big|\eta\times\frac{\partial u}{\partial\eta}\Big|^2+\mu_0|H\times\eta|^2-\mathcal{E}_0(E\cdot\eta)^2\Big\}\,d\Gamma$$

where $c_2=\max\{6c(\Phi)-1,\tilde c_1\}$ and $T_1=\max\{T_0,\tilde T_0\}$

We need *additional identities*:

Let $\{E,H,u,u_t\}$ solution of (1), (2). We have

$$\begin{aligned}
&\mu_0 H\cdot\{\rho u_{tt}-\alpha\Delta u+\text{grad }p\}\\
&\quad+\rho\mathcal{E}_0^{-1}\text{ curl }u\cdot\{\mathcal{E}_0E_t-\text{curl }H\}\\
&\quad+\rho u_t\cdot\{\mu_0H_t+\text{curl }E\}+(\mu_0p-\alpha\mu_0\text{div }u)\text{div }H\\
&\quad+(\rho\mathcal{E}_0^{-1}-\alpha\mu_0)\text{curl }u\cdot\text{curl }H \qquad\qquad (11)\\
&=\frac{\partial}{\partial t}[\rho u_t\cdot\mathcal{E}_0H+\rho\text{curl }u\cdot E]\\
&\quad-\ \text{div}[\rho u_t\times E+\alpha\mu_0(\text{div }u)H\\
&\qquad\qquad+\alpha\mu_0H\times\text{curl }u-\mu_0pH]
\end{aligned}$$

Observe that identity (11) represents a conservation law for the Maxwell system and the hyperbolic system with pressure term if $\rho\mathcal{E}_0^{-1} = \alpha\mu_0$.

Assume $\rho\mathcal{E}_0^{-1} = \alpha\mu_0$. Integration of identity (11) in $\Omega \times (0,T)$ give us

$$\begin{aligned}
&\int_\Omega \{\rho u_t \cdot \mathcal{E}_0 H + \rho \text{ curl } u \cdot E\}\, dx\Big|_{t=0}^{t=T} \\
&= \int_0^T \int_{\partial\Omega} [\rho(u_t \times E)\cdot \eta \\
&\qquad\qquad + \alpha\mu_0(H \times \text{curl } u)\cdot\eta - \mu_0 p H \cdot \eta]\, d\Gamma\, dt \\
&= -\alpha\mu_0 \int_0^T \int_{\partial\Omega} (H \times \eta)\cdot \text{curl } u\, d\Gamma\, dt
\end{aligned} \tag{12}$$

due to the boundary condition $\eta \times E = 0$ and the fact that $H \cdot \eta = 0$ on $\partial\Omega \times (0,T)$ as we saw in the function space framework.

We use the identity

$$|\mu_0(H \times \eta) - \alpha \text{ curl } u|^2$$

$$= \mu_0^2 |H \times \eta|^2 - 2\alpha\mu_0(H \times \eta)\cdot \text{curl } u + \alpha^2 |\text{curl } u|^2$$

in (12) to obtain

$$\int_\Omega \{\rho u_t \cdot \mathcal{E}_0 H + \rho \text{ curl } u \cdot E\}\, dx\Big|_{t=0}^{t=T}$$
$$= \int_0^T \int_{\partial\Omega} \Big\{\frac{1}{2}|\mu_0(H \times \eta) - \alpha \text{ curl } u|^2 - \frac{1}{2}\mu_0^2 |H \times \eta|^2$$
$$- \frac{\alpha^2}{2}\Big|\eta \times \frac{\partial u}{\partial \eta}\Big|^2\Big\}\, d\Gamma\, dt \qquad (13)$$

because $u\big|_{\partial\Omega\times(0,T)} = 0$ tell us that

$$\frac{\partial u_i}{\partial x_j} = \eta_j \frac{\partial u_i}{\partial \eta}, \qquad \text{curl } u = \eta \times \frac{\partial u}{\partial \eta} \text{ on } \partial\Omega \times (0,T)$$

We multiply identity (13) by a convenient positive constant $C_3$ and add te resulting identity with (10) to obtain

$$(1 - \delta_1 C_2)(T - T_1) \int_\Omega \Big\{\mathcal{E}_0|E|^2 + \mu_0|H|^2 + \rho|u_t|^2$$
$$+ \alpha \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^2\Big\}\, dx$$
$$+ C_3 \int_\Omega \{\rho u_t \cdot \mathcal{E}_0 H + \rho \text{ curl } u \cdot E\}\, dx\Big|_{t=0}^{t=T} \qquad (14)$$
$$\leq \int_0^T \int_{\partial\Omega} \Big\{\frac{1}{2} C_3 |\mu_0(H \times \eta) - \alpha \text{ curl } u|^2$$
$$- \frac{\partial h}{\partial \eta}\mathcal{E}_0 (E \cdot \eta)^2\Big\}\, d\Gamma\, dt$$

We obtain a lower bound for the left hand side of (14) to write

$$
\begin{aligned}
&(1-\delta_1 C_2)(T-T_2)\int_\Omega \Big\{\mathcal{E}_0|E|^2+\mu_0|H|^2+\rho|u_t|^2 \\
&\qquad + \alpha\sum_{i=1}^{3}|\frac{\partial u}{\partial x_i}|^2\Big\}\,dx \\
&\le \int_0^T\int_{\partial\Omega}\Big\{C_4|\mu_0(H\times\eta)-\alpha \text{ curl } u|^2 \\
&\qquad -\frac{\partial h}{\partial \eta}\mathcal{E}_0(E\cdot\eta)^2\Big\}\,d\Gamma\,dt
\end{aligned}
\tag{15}
$$

for some $T_2 > 0$ and $T > T_2$. We can choose $T_2 = T_1 + c_3c_4(1-\delta_1 c_2)^{-1}$ where $c_4 = \max\{\mu_0\sqrt{\alpha\mathcal{E}_0}, \frac{\mathcal{E}_0}{2}\sqrt{\alpha\mathcal{E}_0}\}$.

We claim that the term $|\mu_0(H\times\eta)-\alpha \text{ curl } u|$ on the right hand side of (15) *equals* to

$$\left|\alpha\frac{\partial u}{\partial \eta}+\mu_0 H\right| \quad \text{for any} \quad (x,t)\in\partial\Omega\times(0,T)$$

In fact, using the boundary conditions we konw that $\text{curl } u = \eta\times\frac{\partial u}{\partial \eta} = -\frac{\partial u}{\partial \eta}\times\eta$. Thus

$$|\mu_0 H\times\eta-\alpha \text{ curl } u| = \left|\mu_0 H\times\eta+\alpha(\frac{\partial u}{\partial\eta}\times\eta)\right|.$$

Since $H \cdot \eta = 0$ and $\frac{\partial u}{\partial \eta} \cdot \eta = 0$ on $\partial\Omega \times (0,T)$ we have that

$$\left|\left(\mu_0 H + \alpha \frac{\partial u}{\partial \eta}\right) \times \eta\right|^2 + \left(\mu_0 H + \alpha \frac{\partial u}{\partial \eta}\right) \cdot \eta\Big|^2 = \left|\alpha \frac{\partial u}{\partial \eta} + \mu_0 H\right|^2$$

where we used the identity $|v \times \eta|^2 + (v \cdot \eta)^2 = |v|^2$. This proves our claim. Therefore (15) can be written as

$$(1 - \delta_1 C_2)(T - T_2) \int_\Omega \Big\{ \mathcal{E}_0 |E|^2 + \mu_0 |H|^2 + \rho |u_t|^2 + \alpha \sum_{i=1}^{3} |\frac{\partial u}{\partial x_i}|^2 \Big\}\, dx \qquad (16)$$

$$\leq \int_0^T \int_{\partial\Omega} \Big\{ \frac{1}{2} C_3 |\mu_0 H + \alpha \frac{\partial u}{\partial \eta}|^2 - \frac{\partial h}{\partial \eta} \mathcal{E}_0 (E \cdot \eta)^2 \Big\}\, d\Gamma\, dt$$

We have proved the following

**Theorem.** Let $\{E, H, u, u_t\}$ be the solution of problems (1) and (2) with zero boundary conditions. Assume the geometric condition on $\Omega$ given above and $\rho = \mathcal{E}_0 \mu_0 \alpha$. If the condition

$$\mu_0 H + \alpha \frac{\partial u}{\partial \eta} = 0 \quad \text{on} \quad \partial\Omega \times (0,T)$$

holds, then, for any $T > T_2$ we will have

$$E \equiv H \equiv u \equiv 0 \quad \text{in} \quad \Omega \times (0,T)$$

It follows by the above theorem that for $T > T_2$ the expression

$$||(f,g)||_{\mathcal{F}} = \Big(\int_0^T \int_{\partial\Omega} \Big|\mu_0 H + \alpha \frac{\partial u}{\partial \eta}\Big|^2 d\Gamma dt\Big)^{1/2} \qquad (17)$$

defines a norm on the set of initial data $f = (\varphi_0, \varphi_1)$ and $g = (\psi_0, \psi_1)$ of problems (1) and (2) with zero boundary conditins. We denote by $\mathcal{F}$ the Hilbert space obtained by completing $M_1 \cap \mathcal{D}(\mathcal{A}) \times \mathcal{D}(\widetilde{\mathcal{A}})$ with respect to the norm (17). If we denote by

$$\int_\Omega \Big\{\mathcal{E}_0|E|^2 + \mu_0|H|^2 + \rho|u_t|^2 + \alpha \sum_{j=1}^{3} \Big|\frac{\partial u}{\partial x_j}\Big|^2\Big\} dx = ||(f,g)||_Y^2.$$

Then, we have

$$\mathcal{F} \subseteq Y \quad \text{ and } \quad ||(f,g)||_Y^2 \le C||(f,g)||_{\mathcal{F}}^2.$$

Let us denote by $\mathcal{F}'$ the dual space of $\mathcal{F}$ with respect to $Y$.

We consider $P(x,t) \in [L^2(\partial\Omega \times (0,T))]^3$ and $(f,g) \in \mathcal{F}'$.

Let $\{E, H\}$ be the solution of problem (1) with boundary condition

$$\eta \times E = \mu_0 \eta \times (\eta \times P) \quad \text{on } \partial\Omega \times (0, T) \tag{18}$$

and $\{u, u_t\}$ be the solution of problem (2) with boundary condition

$$u_t = P \quad \text{on } \partial\Omega \times (0, T) \tag{19}$$

**Definition.** We say that

$$(E(\cdot, t), H(\cdot, t), u(\cdot, t), u_t(\cdot, t)) \in L^\infty(0, T; \mathcal{F}')$$

is a solution of problems (1) and (2) with boundary conditions (18) and (19) respectively if the identity

$$\begin{aligned} &\Big\langle (E(t), H(t), u(t), u_t(t)), (\tilde{E}(t), \tilde{H}(t), \tilde{u}(t), \tilde{u}_t(t)) \Big\rangle_Y \\ &\quad = \Big\langle (\varphi_0, \varphi_1, \psi_0, \psi_1), (\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\psi}_0, \tilde{\psi}_1) \Big\rangle_Y \\ &\quad + \int_0^t \int_{\partial\Omega} P \cdot \left( \mu_0 \tilde{H} + \alpha \frac{\partial \tilde{u}}{\partial \eta} - \tilde{p}\eta \right) d\Gamma \, d\tau \end{aligned} \tag{20}$$

holds for any $(\tilde{f}, \tilde{g}) \in \mathcal{F}$ and $t \in (0, T)$.

In (20),

$$\Big\langle (\varphi_0, \varphi_1, \psi_0, \psi_1), (\tilde{\varphi}_0, \tilde{\varphi}_1, \tilde{\psi}_0, \tilde{\psi}_1) \Big\rangle_Y$$

$$\int_\Omega \left\{ \mathcal{E}_0 \varphi_0 \cdot \tilde{\varphi}_0 + \mu_0 \varphi_1 \cdot \tilde{\varphi}_0 + \alpha \sum_{i=1}^{3} \frac{\partial \psi_0}{\partial x_i} \cdot \frac{\partial \tilde{\psi}_0}{\partial x_i} + \rho \psi_1 \cdot \tilde{\psi}_1 \right\} dx$$

and $(\tilde{E}, \tilde{H}, \tilde{u}, \tilde{u}_t)$ is the solution of problems (1) and (2) with zero boundary conditions. Also, $\tilde{p}$ denotes the pressure term for the solution $\tilde{u}$ (of problem (2)) with zero boundary conditions

**Definition.** We say that

$$(E(t), H(t), u(t), u_t(t)) \in L^\infty(0, T; \mathcal{F}')$$

is a solution of problem (1) and (2) with boundary conditions (18) and (19) respectively *with zero initial data at $t = T$* if

$$\Big\langle (E(t), H(t), u(t), u_t(t)), (\tilde{E}(t), \tilde{H}(t), \tilde{u}(t), \tilde{u}_t(t)) \Big\rangle_Y$$

$$= - \int_t^T \int_{\partial\Omega} P \cdot \Big( \mu_0 \tilde{H} + \alpha \frac{\partial \tilde{u}}{\partial \eta} - \tilde{p}\eta \Big) \, d\Gamma d\tau \qquad (21)$$

for any $(\tilde{f}, \tilde{g}) \in \mathcal{F}$ and $t \in (0, T)$.

**Theorem.** Assume the geometric assumption on the geometry of $\Omega$ and the relation $\rho = \mathcal{E}_0 \mu_0 \alpha$. If $T > T_2$ (with $T_2$ as above), then for any initial data $(f, g) \in \mathcal{F}'$ of problems (1) and (2) there exists a control $P = P(x,t) \in H^1(0, T; [L^2(\Omega))]^3)$ such that the corresponding solution of problem (2) satisfies

$$(u, u_t)\big|_{t=T} = (0, 0)$$

while the vector-valued function

$$Q = \mu_0 \eta \times (\eta \times P_t)$$

drives system (1) to the state of rest at the same time $T$

$$(E, H)\big|_{t=T} = (0, 0)$$

**Idea of Proof.** We use **HUM.**

Let $(h, q) = (h_1, h_2, q_1, q_2)$ be an (arbitrary) element of $\mathcal{F}$ and $(\varphi, \psi, v, v_t)$ the solution of problems (1) and (2) with zero boundary conditions and initial data at $t = 0$ equal to

$$\begin{aligned} (\varphi, \psi)\big|_{t=0} &= (h_1, h_2) \\ (v, v_t)\big|_{t=0} &= (q_1, q_2) \end{aligned} \tag{22}$$

Let $(E, H, u, u_t)$ be the solution of problems (1) and (2) with boundary conditions (18) and (19) with zero initial data at $t = T > T_2$ where $P$ is chosen to be

$$-P = \mu_0 \psi + \alpha \frac{\partial v}{\partial \eta} \quad \text{on} \quad \partial\Omega \times (0, T). \tag{23}$$

We consider the map

$$\Lambda\colon \mathcal{F} \longmapsto \mathcal{F}'$$

given by

$$\Lambda(h,q) = \Lambda(h_1,h_2,q_1,q_2) = (E,H,u,u_t)\Big|_{t=0}$$

**Claim:** $\Lambda$ is an isomorphism from $\mathcal{F}$ onto $\mathcal{F}'$. From (21) (with $t=0$) and (23) it follows

$$\begin{aligned}
&\left\langle \Lambda(h_1,h_2,q_1,q_2),(\tilde{h}_1,\tilde{h}_2,\tilde{q}_1,\tilde{q}_2)\right\rangle_Y \\
&\quad = \int_0^T \int_{\partial\Omega} -P\cdot\left(\alpha\frac{\partial\tilde{u}}{\partial\eta}+\mu_0\tilde{H}-\tilde{p}\eta\right)d\Gamma d\tau \qquad (24)\\
&\quad = \int_0^T \int_{\partial\Omega} \left(\mu_0\psi+\alpha\frac{\partial v}{\partial\eta}\right)\cdot\left(\alpha\frac{\partial\tilde{u}}{\partial\eta}+\mu_0\tilde{H}-\tilde{p}\eta\right)d\Gamma d\tau.
\end{aligned}$$

Observe that $\left(\mu_0\psi+\alpha\frac{\partial v}{\partial\eta}\right)\cdot\tilde{p}\eta = 0$ on $\partial\Omega\times(0,T)$. In fact using the boundary conditions, we know that

$$\psi\cdot\eta = 0 \quad \text{and} \quad \frac{\partial v}{\partial\eta}\cdot\eta = 0 \quad \text{on } \partial\Omega\times(0,T).$$

Hence, (24) can be written as

$$\left\langle \Lambda(h_1,h_2,q_1,q_2),(\tilde h_1,\tilde h_2,\tilde q_1,\tilde q_2)\right\rangle_Y$$
$$\int_0^T\int_{\partial\Omega}\left(\mu_0\psi+\alpha\frac{\partial v}{\partial\eta}\right)\cdot\left(\alpha\frac{\partial\tilde u}{\partial\eta}+\mu_0\tilde H\right)d\Gamma d\tau \qquad (25)$$
$$=\left\langle (h_1,h_2,q_1,q_2),(\tilde h_1,\tilde h_2,\tilde q_1,\tilde q_2)\right\rangle_{\mathcal F}$$

for any $(h,q)=(h_1,h_2,q_1,q_2)\in\mathcal F$.

Clearly (25) implies that $\Lambda$ is an isomorphism from $\mathcal F$ onto the whole $\mathcal F'$. Now, we return to problems (1) and (2) with boundary conditions (18) and (19) respectively.

Suppose that the initial data $(f,g)$ belongs to $\mathcal F'$. Here $f=(f_1,f_2)=(E_0,H_0)$ and $g=(g_1,g_2)=(u_0,u_1)$. We set

$$(h,q)=\Lambda^{-1}(f,g)$$

and

$$P=-\left(\mu_0\psi+\alpha\frac{\partial v}{\partial\eta}\right)$$

where $(\varphi,\psi,v,v_t)$ is a solution of (1)–(2) with zero boundary conditions and initial conditions at $t=0$ as in (22). Using identity (21) with $t=T>T_2$ we obtain

$$\left\langle (E(T), H(T), u_t(T), u_t(T)), (\tilde{E}(T), \tilde{H}(T), \tilde{u}(T), \tilde{u}_t(T)) \right\rangle_Y$$
$$= \left\langle \Lambda(h, q), (\tilde{f}, \tilde{g}) \right\rangle_Y - \left\langle (h, q), \tilde{f}, \tilde{g}) \right\rangle_{\mathcal{F}}$$

for any $(\tilde{f}, \tilde{g}) \in \mathcal{F}$. Using (25) we conclude that the right hand side of the above identity equals to zero. This means that $(E(T), H(T), u(T), u_t(T))$ generates the zero functional em $\mathcal{F}$. Now that conclusion of the Theorem follows because we construct $P$ as in (23) and set

$$S(x, t) = \int_0^t P(x, \tau)\, d\tau + g_1(x)$$

consequently, $u = S$ and $\eta \times E = \mu_0 \eta \times (\eta \times S_t) = R$ on $\partial\Omega \times (0, T)$. In view of the linearity it suffices to consider controls that reduces both systems to the state of rest.