On the curvature and torsion effects in one and twodimensional waveguides

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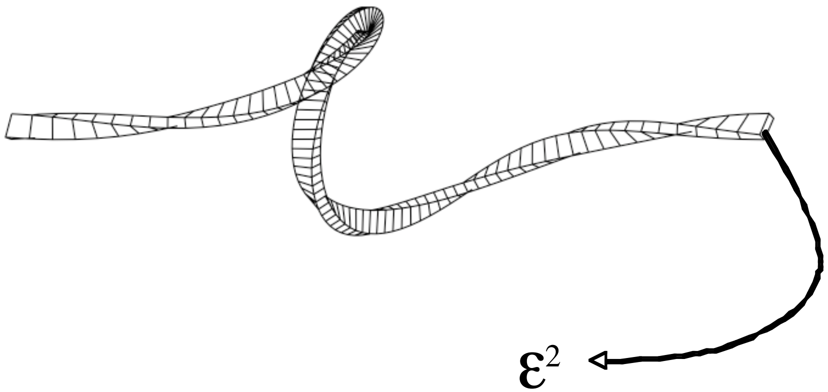
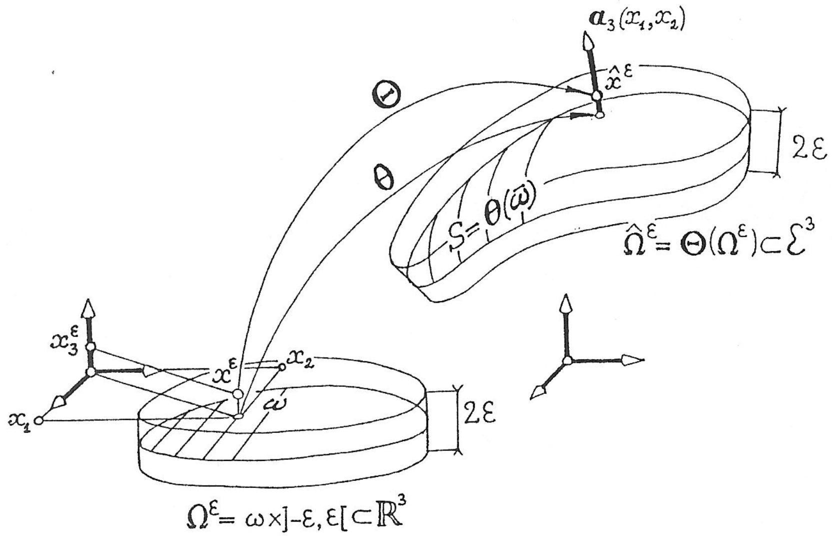
The problem under study :

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon \; u_\varepsilon, & \Omega_\varepsilon \\ u_\varepsilon = 0, & \partial\Omega_\varepsilon \end{cases}$$

$$\lim_{\varepsilon \to 0} (\lambda_\varepsilon, u_\varepsilon) = ?$$





The Physical Motivation

Schrödinger's equation for the time dependent wave function $\overline{\Psi}$ associated to a particle :

$$i\,\hbar \frac{\partial \overline{\Psi}}{\partial t} = \overline{H}\ \overline{\Psi},$$

$\hbar = h/2\pi,$

h – Plank's constant ($h = 6.6262 \times 10^{-27}$ erg s $= 6.6262 \times 10^{-34}$ J s)

$\overline{H}$ is the Hamiltonian operator

For a threedimensional problem one has :

$$\overline{H}\ \overline{\Psi} = -\frac{\hbar^2}{2m}\ \Delta\overline{\Psi} + V\ \overline{\Psi},$$

$-\frac{\hbar^2}{2m}\ \Delta$ – Kinetic energy operator

V – potential operator

m – the mass of the particle

The Physical Motivation (cont.)

H1 : V is independent of time

$\overline{\Psi}(x,t) = \Psi(x)\ T(t)$

$$T(t) = e^{-i(E/\hbar)t}$$

E – Energy of the system

$$-\frac{\hbar^2}{2m}\ \Delta\Psi + V\ \Psi = E\ \Psi$$

Time independent Scrödinger's equation

The Physical Motivation (cont.)

H2 :

$$V = \begin{cases} +\infty & \text{if } x \notin \Omega_\varepsilon, \\ 0 & \text{if } x \in \Omega_\varepsilon, \end{cases}$$

$$\begin{cases} -\Delta \Psi_\varepsilon = \frac{2m}{\hbar^2}\ E\ \Psi_\varepsilon, & \Omega_\varepsilon \\ \Psi_\varepsilon = 0, & \partial\Omega_\varepsilon \end{cases}$$

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon\ u_\varepsilon, & \Omega_\varepsilon \\ u_\varepsilon = 0, & \partial\Omega_\varepsilon \end{cases}$$

Since Ω^ε is bounded one has a discrete spectrum :

$$\sigma^\varepsilon = \{\lambda_i^\varepsilon : i \in \mathbb{N}\}, \quad 0 < \lambda_0^\varepsilon \leq \lambda_1^\varepsilon \leq \cdots \leq \lambda_i^\varepsilon \leq \lambda_{i+1}^\varepsilon \cdots$$

The Result

$$\lambda_i^\varepsilon = \frac{\lambda_0}{\varepsilon^2} + \left(\frac{\lambda_1}{\varepsilon}\right) + \mu_i^\varepsilon, \quad \mu_i^\varepsilon \longrightarrow \mu_i$$

$$-w'' + q(s)\ w = \mu\ w, \quad w \in H_0^1(0, L).$$

Geometry of the domain (1D waveguide)

The "axis" of the domain :

$r : s \in [0, L] \to r(s) \in \mathbb{R}^3$ – a curve in $\mathbb{R}^3$ $\quad$ s – arc length parameter

The "cross section" of the domain :

$\omega \subset \mathbb{R}^2$ – an open bounded, simply connected subset of $\mathbb{R}^2$

Geometry of the domain (1D waveguide) – The Frenet System

$$
\begin{aligned}
T &= \frac{dr}{ds} = r', \quad \|r'\|_{\mathbb{R}^3} = 1, \\
N &= T'/\|T'\|_{\mathbb{R}^3}, \quad T'(s) \neq 0, \\
B &= T \times N.
\end{aligned}
$$

$k : s \in [0, L] \longrightarrow k(s) \in \mathbb{R}$ – curvature function

$\tau : s \in [0, L] \longrightarrow \tau(s) \in \mathbb{R}$ – torsion function

$$
\begin{aligned}
T' &= k\ N, \\
N' &= -k\ T + \tau\ B, \\
B' &= -\tau\ N.
\end{aligned}
$$

Geometry of the domain (1D waveguide) – Ω^F

$$\Omega^F = \{x \in \mathbb{R}^3 : x = r(s) + y_1\ N(s) + y_2\ B(s),\ s \in [0, L],\ y = (y_1, y_2) \in \omega\}$$

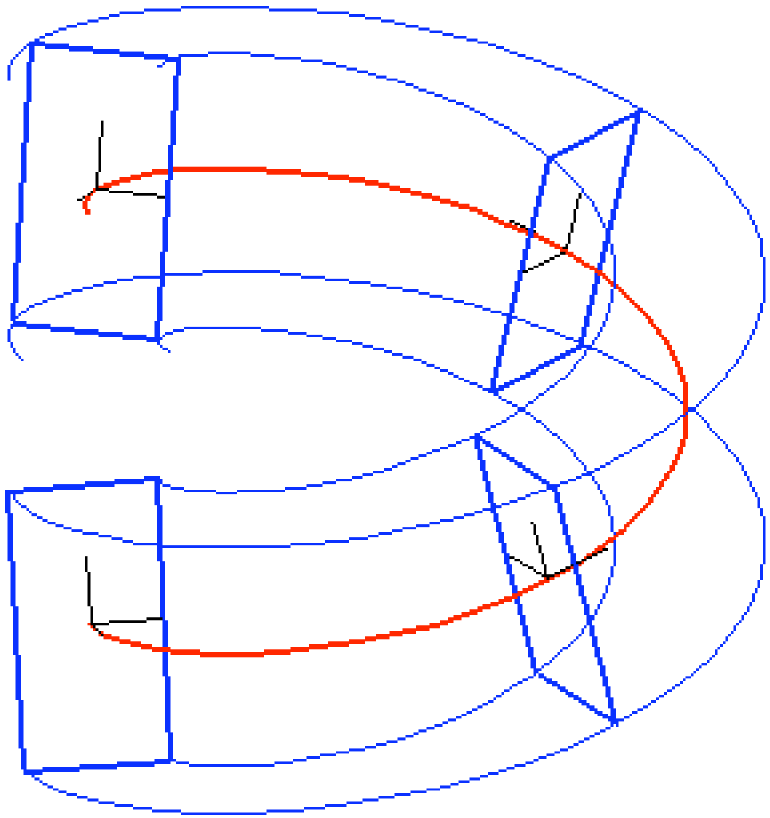


Figure 1.1 - Reference domain associated with Frenet's system

Geometry of the domain (1D waveguide) – Ω^F (cont.)

i) The Frenet system my not be defined for all $s \in [0, L]$ for one may have points for which $T' = 0$.

ii) In each point $s \in [0, L]$, the cross section of the domain Ω^F has a prescribed rotation with respect to curve r, given by the value of the torsion function τ at that point.

$$X' = \lambda\, T,$$
$$Y' = \mu T,$$
$$T' = -\lambda\, X - \mu\, Y,$$

where λ and μ are functions of the arclength parameter s.

For each $s \in [0, L]$ Tang's reference system is such that (X, Y) can be seen as a two dimensional basis, in ω, rotated from (N, B), around T, of an angle $\alpha = \alpha(s)$. In fact if :

$$X = \cos\alpha\ N + \sin\alpha\ B = N_\alpha,$$
$$Y = -\sin\alpha\ N + \cos\alpha\ B = B_\alpha,$$

using Frenet's formulas, one obtains :

$$\alpha' = -\tau,$$
$$\lambda = -k\ \cos\alpha,$$
$$\mu = k\ \sin\alpha,$$

Geometry of the domain (1D waveguide) – Ω^T

$$\Omega^T = \{x \in \mathbb{R}^3 : x = r(s) + y_1\ X(s) + y_2\ Y(s),\ \ s \in [0, L],\ \ y = (y_1, y_2) \in \omega\}$$

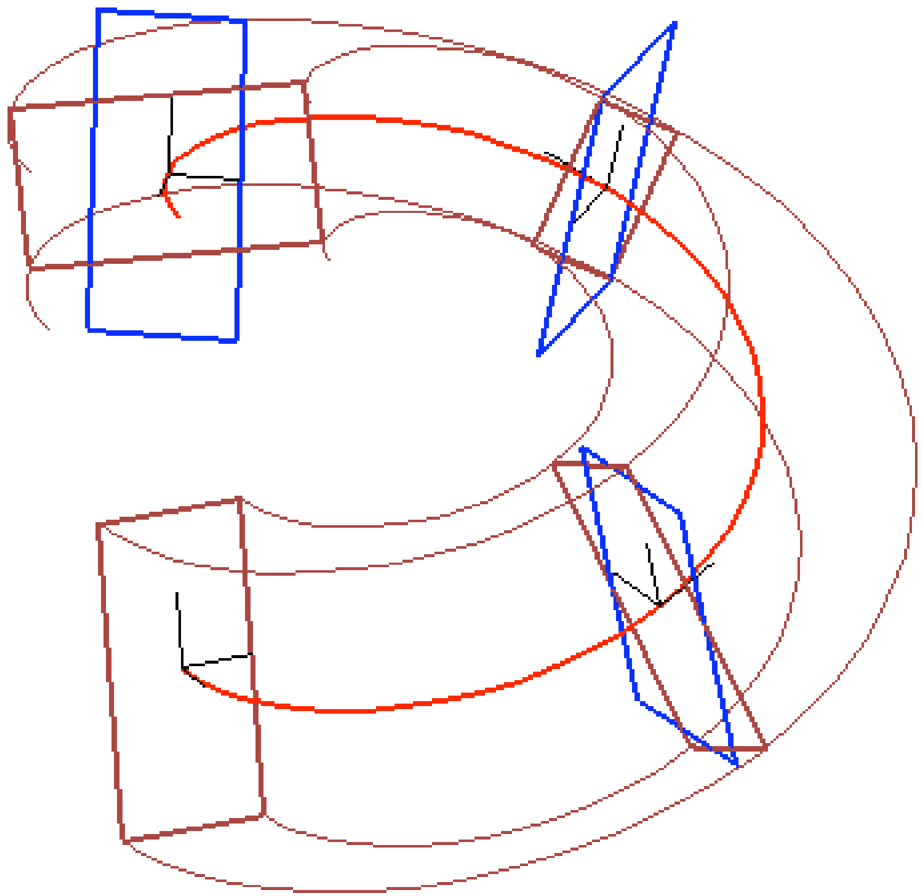


Figure 1.2 - Reference domain associated with Tang's system

Geometry of the domain (1D waveguide) – Ω^α

We are then faced with three possible choices for the reference set, namely :

i) We may follow Tang's reference system and obtain a domain Ω^T, without torsion with respect to the central axis r ;

ii) We may follow Frenet's reference system and obtain a domain Ω^F, rotated of the same amount as Frenet's system (τ), with respect to the central axis r ;

iii) We may follow yet another reference system (T, N_α, B_α), and obtain a generic domain Ω^α defined through :

$$\Omega^\alpha = \{x \in \mathbb{R}^3 : x = r(s) + y_1\, N_\alpha(s) + y_2\, B_\alpha(s),\ s \in [0, L],\ y = (y_1, y_2) \in \omega\}, \quad (1.1)$$

whose cross section presents an arbitrary rotation of an angle α with respect Frenet's domain.

If for every $s \in [0, L]$, $\alpha = 0$ then $\Omega^\alpha \equiv \Omega^F$ and if α is such that $\alpha' = -\tau$, then $\Omega^\alpha = \Omega^T$.

Geometry of the domain (1D waveguide) – Ω_ϵ^α

We are interested in the eigenvalue problem posed in a domain for which the diameter of the cross section ω is much smaller than its length L. Specifically, we consider a real parameter $\varepsilon > 0$ and a cross section, obtained from the reference one, by an homothety of ratio ε. That is we define the thin domain :

$$\Omega_\varepsilon^\alpha := \{x \in \mathbb{R}^3 : x = r(s) + \varepsilon\, y_1\, N_\alpha + \varepsilon\, y_2\, B_\alpha,\ s \in [0, L],\ y = (y_1, y_2) \in \omega\}, \quad (1.2)$$

and study the behavior of the eigensolution $(\lambda_\varepsilon, u\varepsilon)$, associated with problem

$$\begin{cases} -\Delta u_\varepsilon = \lambda_\varepsilon u_\varepsilon, \\ u_\varepsilon \in H_0^1(\Omega_\varepsilon^\alpha). \end{cases}$$

as ε goes to zero, and hope to see the influence of the curvature $(k(s))$ and torsion $(\tau(s))$ functions in the limit problem.

If for every $s \in [0, L]$, $\alpha = 0$ then $\Omega_\varepsilon^\alpha \equiv \Omega_\varepsilon^F$ and if α is such that $\alpha' = -\tau$, then $\Omega_\varepsilon^\alpha = \Omega_\varepsilon^T$.

Variational Formulation and change of variable

$$F_\varepsilon(w) := \int_{\Omega_\varepsilon^\alpha} \left(|\nabla w|^2 - \lambda_\varepsilon |w|^2 \right) dx.$$

Consider, then, the following transformation, for each $\varepsilon > 0$,

$$\begin{aligned} &\psi : [0, L] \times \omega \longrightarrow \Omega_\varepsilon^\alpha \\ &(s, (y_1, y_2)) \mapsto x = r(s) + \varepsilon\ y_1 N_\alpha + \varepsilon\ y_2 B_\alpha \end{aligned}$$

and define, for each $w \in H_0^1(\vee_\varepsilon^\alpha)$, $v(s, (y_1, y_2)) := w(\psi(s, (y_1, y_2)))$.

Recalling that

$$\begin{aligned} N_\alpha &:= \cos\alpha(s) N(s) + \sin\alpha(s) B(s) \\ B_\alpha &:= -\sin\alpha(s) N(s) + \cos\alpha(s) B(s), \end{aligned}$$

we obtain, in the Frénet system (T, N, B) :

Variational Formulation and change of variable (cont.)

$$\nabla\psi = \begin{pmatrix} \beta_\varepsilon & 0 & 0 \\ -\varepsilon(\tau+\alpha')(z_\alpha^\perp\cdot y) & \varepsilon\cos\alpha & -\varepsilon\sin\alpha \\ \varepsilon(\tau+\alpha')(z_\alpha\cdot y) & \varepsilon\sin\alpha & \varepsilon\cos\alpha \end{pmatrix} \ , \quad \det\nabla\psi = \varepsilon^2\beta_\varepsilon \ ,$$

where :

$$z_\alpha := (\cos\alpha, -\sin\alpha) \ , \quad z_\alpha^\perp := (\sin\alpha, \cos\alpha) \ , \quad \beta_\varepsilon := 1 - \varepsilon k(z_\alpha\cdot y)$$

Then

$$\nabla\psi^{-1} = \begin{pmatrix} \frac{1}{\beta_\varepsilon} & 0 & 0 \\ \frac{(\tau+\alpha')y_2}{\beta_\varepsilon} & \frac{\cos\alpha}{\varepsilon} & \frac{\sin\alpha}{\varepsilon} \\ \frac{-(\tau+\alpha')y_1}{\beta_\varepsilon} & \frac{-\sin\alpha}{\varepsilon} & \frac{\cos\alpha}{\varepsilon} \end{pmatrix}$$

Variational Formulation in the fixed domain

$$G_\varepsilon(v) := \frac{1}{\varepsilon^2} F_\varepsilon(w) = \int_0^L \int_\omega \left\{ \frac{1}{\beta_\varepsilon} \Big| v' + \nabla_y v \cdot R\, y\, (\tau + \alpha') \Big|^2 + \right.$$

$$\left. + \frac{\beta_\varepsilon}{\varepsilon^2} \left(|\nabla_y v|^2 - \varepsilon^2 \lambda_\varepsilon |v|^2 \right) \right\} \, dy \, ds,$$

where

$(\ )'$ – derivative of $(\ )$ with respect to s,

$\nabla_y v$ – the derivative of v with respect to y,

R – rotation matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow R\, y = \begin{Bmatrix} y_2 \\ -\, y_1 \end{Bmatrix}.$

The main result

The sequence $\{G_\varepsilon\}$ of functionals defined in $H_0^1((0,L)\times\omega)$ Γ-converges, to the functional G, defined by

$$G(v) := \begin{cases} G_0(w) & \text{if } v(s,y) = w(s)\ u_0(y) \\ +\infty & \text{if not} \end{cases}$$

$$G_0(w) := \int_0^L \left\{ |w'(s)|^2 + \left[\left(\tau(s) + \alpha'(s)\right)^2\ C(\omega) - \frac{k^2(s)}{4} \right] |w(s)|^2 \right\} ds,$$

where $C(\omega) := \int_\omega\ |\nabla_y u_0 \cdot R\ y|^2\ dy$,

$$R \text{ – rotation matrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \Longrightarrow R\ y = \begin{Bmatrix} y_2 \\ -\ y_1 \end{Bmatrix}$$

and where u_0 is the normalized eigenfuntion corresponding to the first eigenvalue of problem

$$-\Delta u = \gamma u\ , \quad u \in H_0^1(\omega). \qquad \longrightarrow \qquad (\lambda_0, u_0)$$

i) The infimum for λ_2 is always attained and it corresponds to the first eigenvalue of the following Sturm-Liouville problem :

$$-\varphi'' + q\,\varphi = \mu\,\varphi, \quad \varphi \in H_0^1(0,L), \quad q(s) := (\tau(s) + \alpha'(s))^2\, C(\omega) - \frac{(k(s))^2}{4}.$$

ii) It is possible to prove that μ_1 coincides with the second order term (λ_2) of the asymptotic expansion

$$\varepsilon^2 \lambda_\varepsilon = \lambda_0 + \varepsilon\lambda_1 + \varepsilon^2\lambda_2 + \cdots,$$

where λ_0 is the first eigenvalue of the eigenvalue problem in ω and λ_1 is zero.

iii) It is clear that if q is constant, then $\lambda_2 = \dfrac{\pi^2}{L^2} + q$ and, consequently,

$$G_0(w) := \int_0^L \left(|w'|^2 - \frac{\pi^2}{L^2}\,|w|^2 \right)\, ds.$$

iv) Due to the definition of λ_2, $G_0(w) \geq 0$ for all $w \in H_0^1(0,L)$ and the minimizers of G_0 coincide, up to a multiplying constant, with the minimizers of λ_2.

Some remarks on the main result (cont.)

i) The Euler-Lagrange equation associated $G_0(w)$ is of the form :

$$-w'' + \underbrace{\left[(\tau + \alpha')^2\, C - \frac{k^2}{4}\right]}_{q}\, w - \lambda_2 \;\; w = 0.$$

As mentioned before, this is a problem of the Sturm-Liouville type and it is exactly the same problem for λ_2. The only difference being that the minimum for w is zero and for λ_2 is, obviously, λ_2.

ii) This equation may be interpreted as a onedimensional problem for the spatial wave equation with :

$$\frac{2m}{\hbar^2}\,(V - E) = \left[(\tau + \alpha')^2\, C - \frac{k^2}{4} - \lambda_2\right],$$

that is, although we have started from a threedimensional problem without a potential in the interior of the domain under consideration, in the limit, in a onedimensional curved waveguide, the particle sees the curvature, the torsion and the influence of the cross section as a (nonhomogeneous) potential function in an equivalent straight waveguide of the same total length.

Some remarks on the main result (cont.)

$k(s)$ – influence of the curvature,

$\tau(s) + \alpha'(s)$ – influence of the torsion,

C – the influence of the shape of the cross section.

iii) If, from the start, we have a straight waveguide then $k \equiv 0$, $\tau \equiv 0$, $\alpha' \equiv 0$ and one obtains the classical onedimensional result :

$$w(x) = \varphi_0(x) = \sqrt{\frac{2}{L}}\ \sin\frac{\pi x}{L}, \qquad \frac{2m}{\hbar^2}\ E = \left(\frac{\pi}{L}\right)^2$$

.

iv) If k and $\tau + \alpha'$ are constants then, once again, one obtains the classical onedimensional result.

Some remarks on the main result (cont.)

v) For a circular cross section of radius R, the ground state (u_0) is radial, associated with the eigenproblem $-\Delta u = \gamma u$ and of the form :

$$u_0(r) = \frac{\sqrt{2}}{R J_1(\sqrt{\gamma_0} R)} J_0(\sqrt{\gamma_0}\, r), \quad \gamma_0 = \left(\frac{r_n}{R}\right)^2, \quad n \in \mathbb{N},$$

where,

r – radial direction,

J_0 and J_1 – first and second Bessel functions of the first kind,

r_0 – first zero of J_0.

Since u_0 is a radial function, its gradient is lso radial and, therefore, orthogonal to the direction defined by $R\, y = (y_2, -y_1)$.

Consequentely, for the circular cross section, $C \equiv 0$.

Some ideas about the proof

$$-\Delta_y \ u_0 = \lambda_0 \ u_0, \quad u_0 \in H_0^1(\omega)$$

$$-\Delta_y \ u_1 - \lambda_0 \ u_1 = -k \ (z_\alpha \cdot \nabla_y \ u_0), \quad u_1 \in H_0^1(\omega), \quad (s \text{ fixed})$$

Fredholm orthogonality condition

$$k \int_\omega (z_\alpha \cdot \nabla_y \ u_0) \ u_0 \ dy = 0,$$

ensuring the existence of a solution u_1.

Some properties of u_0 and u_1, for example :

$$\int_\omega \left(|\nabla_y u_0|^2 - \lambda_0 |u_0|^2 \right) \ dy = 0, \quad \int_\omega (z_\alpha \cdot y) \left(|\nabla_y u_0|^2 - \lambda_0 |u_0|^2 \right) \ dy = 0$$

Some ideas about the proof (cont.)

Lemma. *Let*

$$\gamma_2 := \inf_{v \in H_0^1(\omega)} \int_\omega \left[|\nabla_y v|^2 - \lambda_0 |v|^2 + 2\ k\ (z_\alpha \cdot \nabla_y u_0) v \right] dy.$$

Then, the infimum is attained in u_1 and

$$\gamma_2 = -\frac{k^2}{4}.$$

Proof : Use the properties of u_0, u_1 and integration by parts sucessively.

Some ideas about the proof (cont.)

$$G_\varepsilon(v) := \frac{1}{\varepsilon^2} F_\varepsilon(w) = \int_0^L \int_\omega \left\{ \frac{1}{\beta_\varepsilon} \Big| v' + \nabla_y v \cdot R\, y\, (\tau + \alpha') \Big|^2 + \right.$$

$$\left. + \frac{\beta_\varepsilon}{\varepsilon^2} \left(|\nabla_y v|^2 - \varepsilon^2 \lambda_\varepsilon |v|^2 \right) \right\} dy\, ds,$$

Lemma. *Let* γ_ε *be given by*

$$\gamma_\varepsilon := \inf_{\substack{v \in H_0^1(\omega) \\ v \not\equiv 0}} \frac{\int_\omega \beta_\varepsilon |\nabla_y v|^2\, dy}{\int_\omega \beta_\varepsilon |v|^2\, dy}.$$

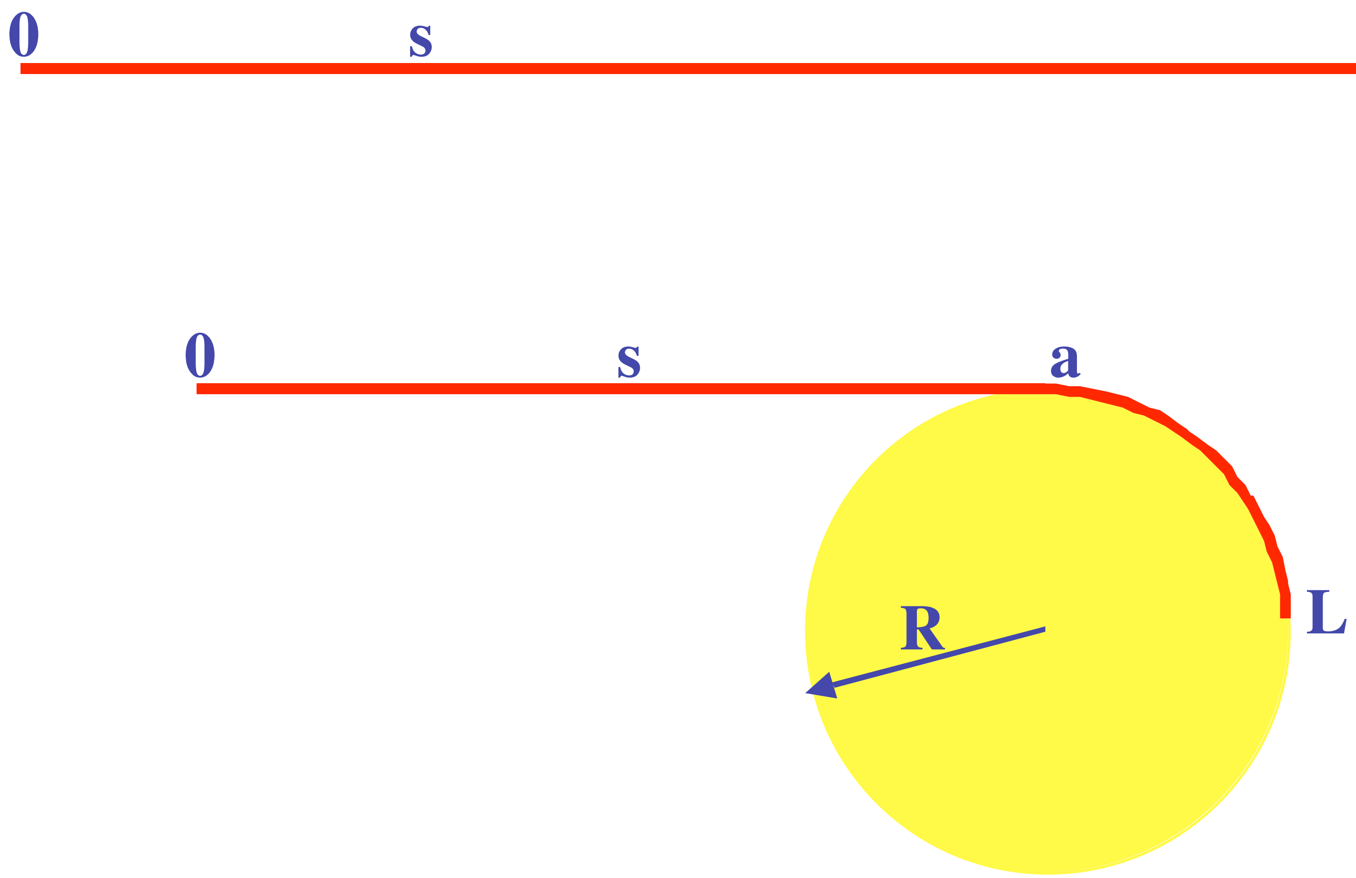
Then

$$\gamma_2(s) = \lim_{\varepsilon \to 0} \frac{\gamma_\varepsilon - \lambda_0}{\varepsilon^2} = -\frac{k^2(s)}{4}, \quad \text{uniformly in } [0, L]$$

Lemma. *Let λ_ε be the first eigenvalue of the problem under study and recall the definition of λ_2, introduced in the theorem, then the following convergence holds*

$$\lim_{\varepsilon \to 0} \frac{\varepsilon^2 \lambda_\varepsilon - \lambda_0}{\varepsilon^2} = \lambda_2.$$

Example



$(\tau + \alpha')^2 \ C \equiv 0$

k is constant in a certain interval $[a, L] \subset [0, L]$ and zero in $[0, a[$.

Example (cont.)

$w \in H^1(0, L)$, such that :

$$w(s) = \begin{cases} w_1(s) & \text{if } 0 \leq s \leq a, \\ w_2(s) & \text{if } a \leq s \leq L, \end{cases}$$

solving :

$$\begin{cases} -w_1'' - \lambda_2 \, w_1 = 0, & \text{if } 0 \leq s \leq a, \\ -w_2'' - (\lambda_2 - q) \, w_2 = 0 & \text{if } a \leq s \leq L, \end{cases} \tag{1.3}$$

subjected to the boundary conditions :

$$w_1(0) = 0, \quad w_2(L) = 0,$$

and to the compatibility conditions :

$$w_1(a) = w_2(a), \quad w_1'(a) = w_2'(a). \tag{1.4}$$

Example (cont.)

Let $k_1 = \sqrt{\lambda_2}$ and $k_2 = \sqrt{\lambda_2 - q}$, therefore :

$$\left(e^{ik_1 a} - e^{-ik_1 a}\right)\left[e^{ik_2(L-a)} + e^{-ik_2(L-a)}\right] + \\ + \frac{k_1}{k_2}\left(e^{ik_1 a} + e^{-ik_1 a}\right)\left[e^{ik_2(L-a)} - e^{-ik_2(L-a)}\right] = 0 \quad \Longrightarrow \quad \lambda_2 = \cdots$$

In the present case, solving this equation is equivalent to solving :

$$\sinh(\overline{k}_1 a)\ \cos[k_2(L-a)] + \frac{\overline{k}_1}{k_2}\ \cosh(\overline{k}_1 a)\ \sin[k_2(L-a)] = 0,$$
$$\overline{k}_1 = \sqrt{-\lambda_2}, \quad k_2 = \sqrt{\lambda_2 - q}, \quad \text{if } q < \lambda_2 < 0,$$

or

$$\sin(k_1 a)\ \cos[k_2(L-a)] + \frac{k_1}{k_2}\ \cos(k_1 a)\ \sin[k_2(L-a)] = 0,$$
$$k_1 = \sqrt{\lambda_2}, \quad k_2 = \sqrt{\lambda_2 - q}, \quad \text{if } q < 0 < \lambda_2.$$

Example (cont.)

$q = -6$ and $L = 2$

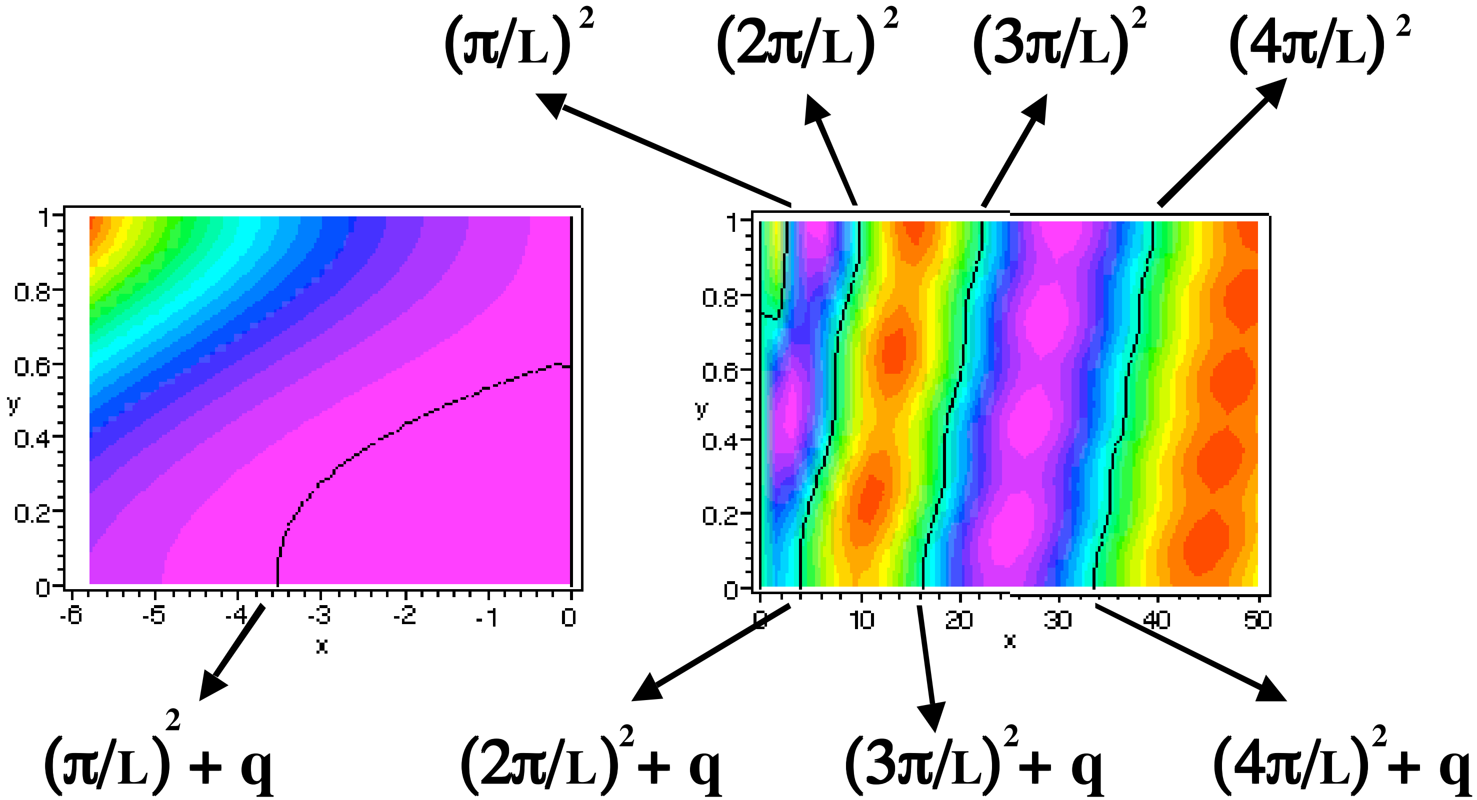


Figure 1.3 - λ_n vs. a/l for $q = -6$ and $L = 2$.

Example (cont.)

$$a/L = 1 \quad \Longrightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n \in \mathbb{N}.$$

$$a/L = 0 \quad \Longrightarrow \quad \lambda_n = \left(\frac{n\pi}{L}\right)^2 + q, \quad n \in \mathbb{N}.$$

$$q = -6 = -k^2/4, \quad a = 1, \quad L = 2, \quad \Longrightarrow \quad \lambda_2 \approx -1.363855334$$

Example (cont.)

$P(s) = w^*(s)w(s)$ becomes :

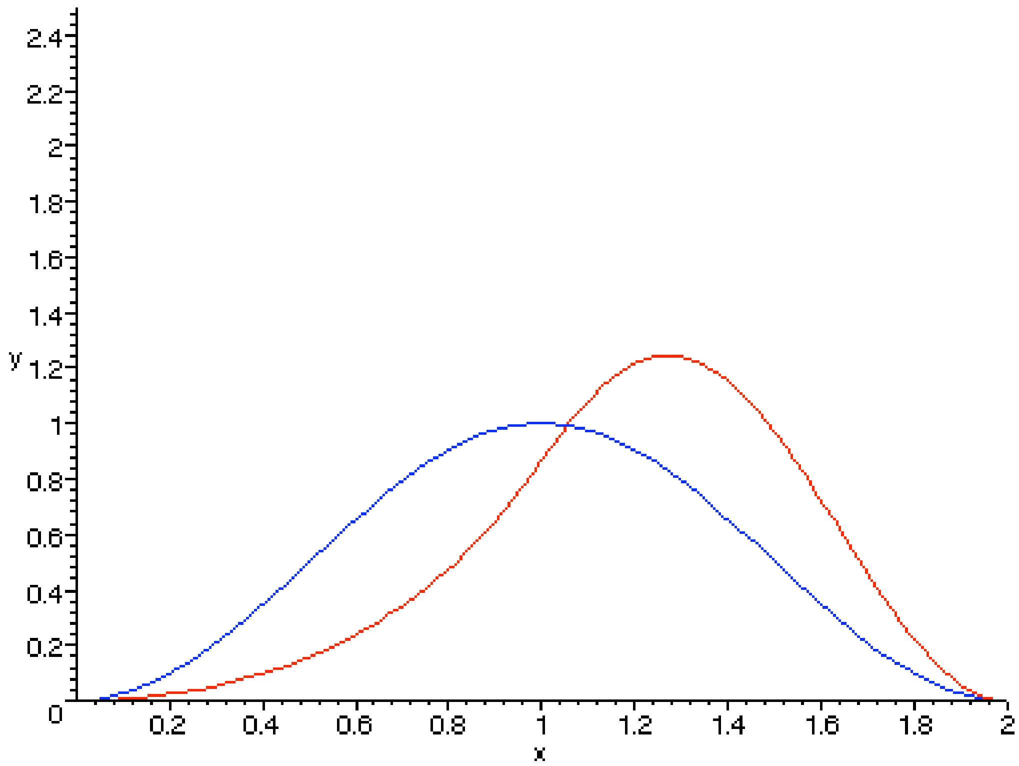


Figure 1.4 - Probability density function (in red) and for the classical case (in blue) ($q = -6 = -k^2/4$, $a = 1$ and $L = 2$)

Example (cont.)

$q = -80 = -k^2/4, \quad a = 1.8, \quad L = 2.$

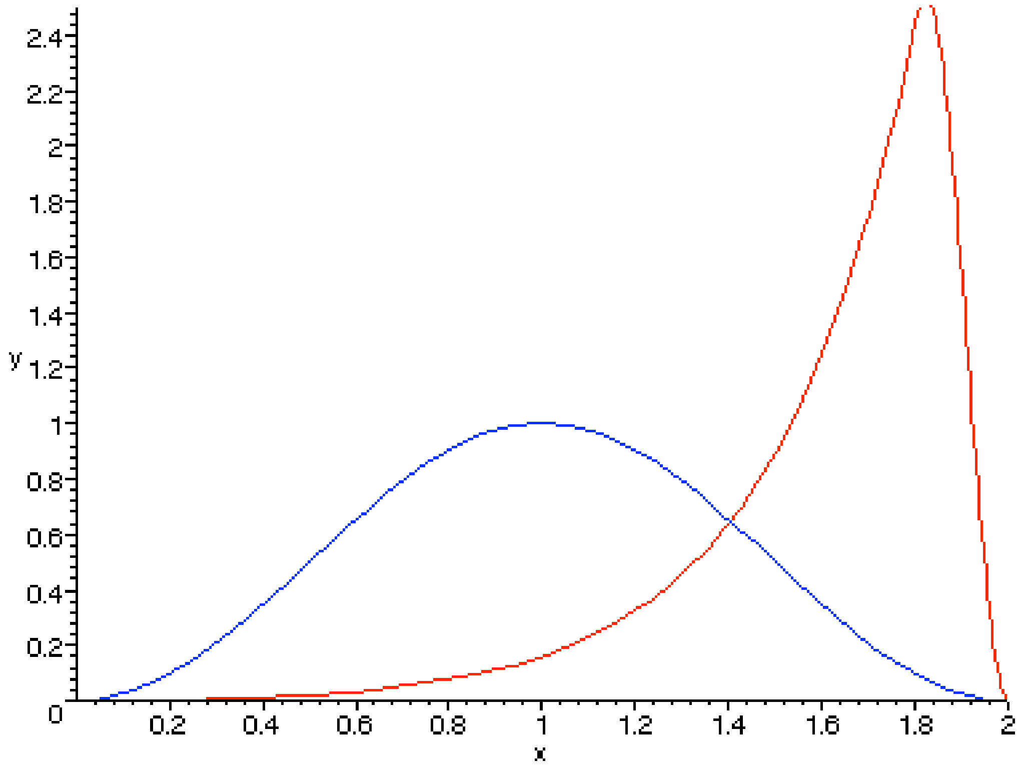
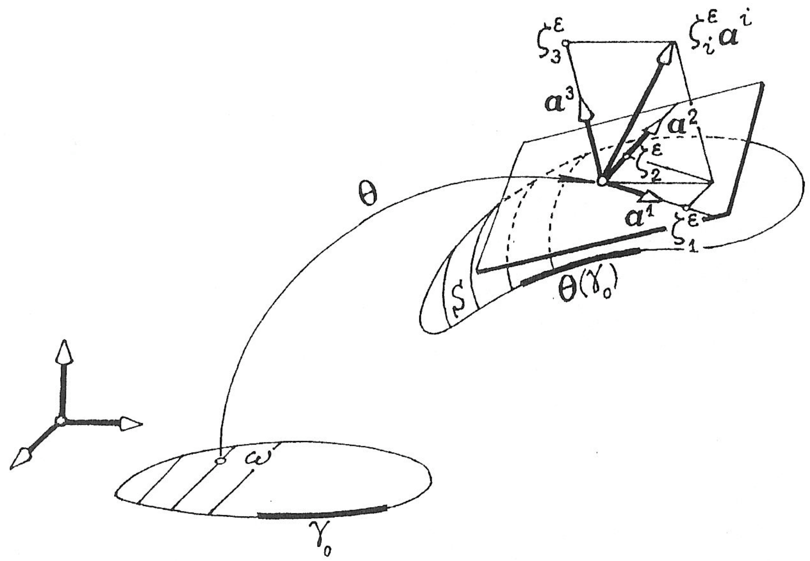


Figure 1.5 - Probability density function (in red) and for the classical case (in blue) ($q = -80 = -k^2/4$, $a = 1.8$ and $L = 2$)

Geometry of the domain (2D waveguide)

The "reference middle surface" – a surface in $\mathbb{R}^3$:

$$\tilde{\omega} = \{\ \tilde{x} = (x_1, x_2, \theta(x_1, x_2)) \in \mathbb{R}^3 :\ (x_1, x_2) \in \omega \subset \mathbb{R}^2, \quad \theta \in C^3(\overline{\omega})\}$$

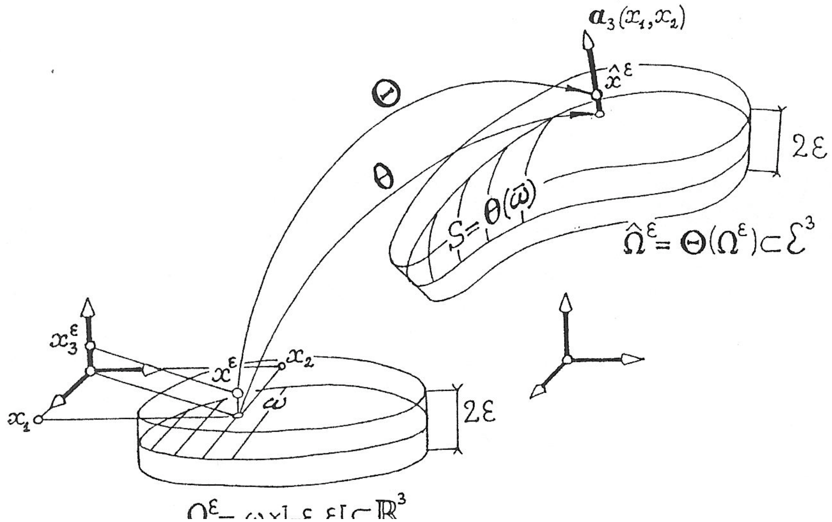


Geometry of the domain (2D waveguide)

The curvilinear reference system :

$$a_\alpha = \frac{\partial \tilde{x}}{\partial x_\alpha}, \qquad a_1 = (1, 0, \partial_1 \theta), \quad a_2 = (0, 1, \partial_2 \theta)$$

$$a_3 = n = \frac{a_1 \times a_2}{|a_1 \times a_2|} = \frac{1}{\sqrt{\alpha}} (-\partial_1 \theta, -\partial_2 \theta, 1), \quad \alpha = 1 + |\partial_1 \theta|^2 + |\partial_2 \theta|^2$$



Geometry of the domain (2D waveguide)

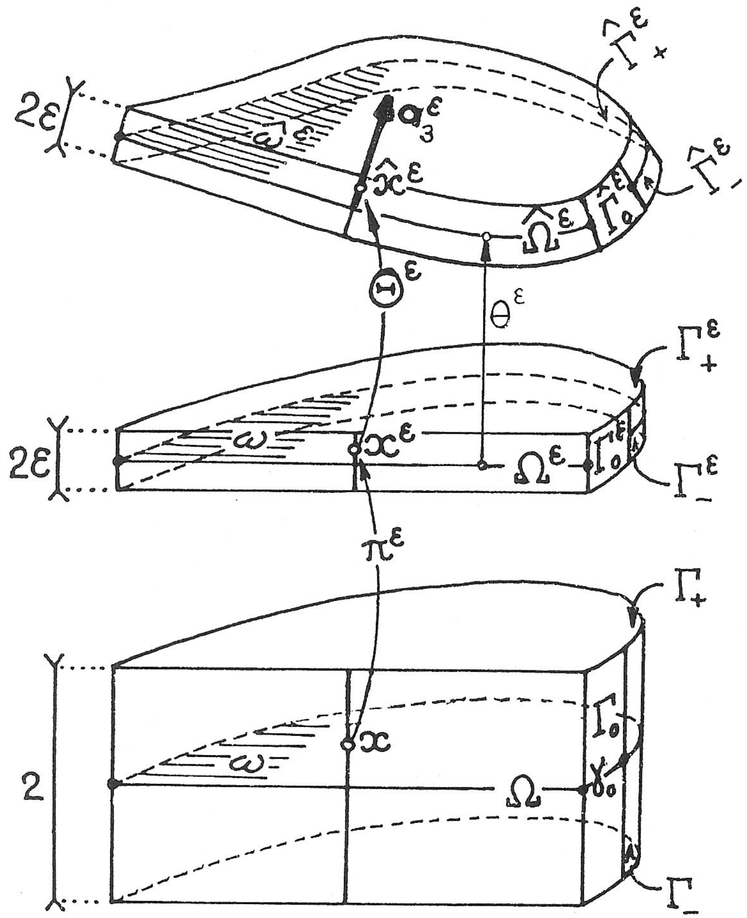
The “shell” :

$$\tilde{\Omega}^{\varepsilon} = \{ \ \tilde{x}^{\varepsilon} = ((x_1, x_2, \theta(x_1, x_2)) + x_3^{\varepsilon} \ n(x_1, x_2)) \in \mathbb{R}^3 : \ (x_1, x_2) \in \omega \subset \mathbb{R}^2 \}$$

The thickness of the shell : $2 \ \varepsilon, \quad \varepsilon > 0$

The thickness variable : $x_3^{\varepsilon} = \varepsilon \ x_3$

Geometry of the domain (2D waveguide)



Fundamental Forms (2D waveguide)

The first fundamental form matrix $[a] = (a_{\alpha\beta})$, $\quad a_{\alpha\beta} = a_\alpha \cdot a_\beta$:

$$a_{11} = 1 + |\partial_1 \theta|^2, \quad a_{22} = 1 + |\partial_2 \theta|^2, \quad a_{12} = a_{21} = \partial_1 \theta \; \partial_2 \theta$$

The second fundamental form matrix $[b] = (b_{\alpha\beta})$, $\quad b_{\alpha\beta} = -\, n \cdot a_{\alpha,\beta}$:

$$b_{11} = -\frac{\partial_{11}\theta}{\sqrt{\alpha}}, \quad b_{22} = -\frac{\partial_{22}\theta}{\sqrt{\alpha}}, \quad b_{12} = b_{21} - \frac{\partial_{12}\theta}{\sqrt{\alpha}}$$

Remark : $|a| = \alpha$

Curvature Functions (2D waveguide)

The Mean Curvature function H is given by :

$$H = \frac{b_{11}a_{22} + b_{22}a_{11} - 2b_{12}a_{12}}{2|a|}$$

The Gaussian Curvature function K is given by :

$$K = \frac{|b|}{|a|}$$

Remark : $\det[a] = \alpha$

Variational Formulation

$$\tilde{F}_\varepsilon(\tilde{w}^\varepsilon) := \int_{\tilde{\Omega}^\varepsilon} \left(|\nabla \tilde{w}^\varepsilon|^2 - \lambda_\varepsilon |\tilde{w}^\varepsilon|^2 \right) \, d\tilde{x}^\varepsilon .$$

The Limit (eigenvalue) Problem

$$-\partial_\beta \left(\frac{A_{\alpha\beta}}{\sqrt{|a|}} \, \partial_\alpha w \right) + (K - H^2) \, w \sqrt{|a|} = \lambda_2 \, w \, \sqrt{|a|}.$$

Remarks :

$$K = k_1 k_2, \quad H = \frac{(k_1 + k_2)}{2} \quad \Longrightarrow \quad K - H^2 = -\frac{(k_1 - k_2)^2}{4}$$

If $k_1 \equiv 0$ or $k_2 \equiv 0$ then $K - H^2 = -k^2/4$ as in the 1D case !

The first term represents the Laplacian written in the curviliner coordinates. In fact from the variational formulation of this limit problem one has :

$$\lambda_2 = \inf \frac{\displaystyle\int_\omega \frac{A_{\alpha\beta}}{\sqrt{|a|}}\ \partial_\alpha w\ \partial_\beta w\ dx_1 dx_2 + (K - H^2)\ w\sqrt{|a|}\ dx_1 dx_2}{\int_\omega\ w\ \sqrt{|a|}\ dx_1 dx_2}$$

but :

$$\frac{A_{\alpha\beta}}{|a|}\ \partial_\alpha w\ \partial_\beta w = |\partial_\tau w|^2 \quad \text{and} \quad \sqrt{|a|}\ dx_1 dx_2 = ds_1 ds_2$$

therefore, in curvilinear coordinates, the limit problem is :

$$-\partial_{\tau\tau} w + (K - H^2)\ w = \lambda_2 w$$

$$K = k_1 k_2, \quad H = \frac{(k_1 + k_2)}{2} \quad \Longrightarrow \quad K - H^2 = -\frac{(k_1 - k_2)^2}{4}$$

$$\begin{cases} -div\ (a(y)\ \nabla u_\varepsilon) = \lambda_\varepsilon\ u_\varepsilon, & \Omega_\varepsilon \\ u_\varepsilon = 0, & \partial\Omega_\varepsilon \end{cases}$$

$$\gamma_\varepsilon(s) = \gamma_0(s) + \varepsilon\ \gamma_1(s) + \varepsilon^2\ \gamma_2(s) + \cdots$$

$$\lambda_\varepsilon = \frac{1}{\varepsilon^2}\ \lambda_0 + \frac{1}{\varepsilon}\ \lambda_1 + \lambda_2 + \cdots$$

But now

$$\gamma_1(s) \neq 0, \quad \lambda_1 \neq 0, \quad \gamma_2(s) \neq -k^2(s)/4, \quad etc.$$

In fact, if :

$$-div\ (a(y)\ \nabla u_0) - \lambda_0\ u_0 = 0, \quad u_0 \in H_0^1(\omega)$$

$$-div\ (a(y)\ \nabla u_1) - \lambda_0\ u_1 = k \int_\omega a(y)\ (z \cdot \nabla u_0)\ u_0\ dy - a(y)\ (z \cdot \nabla u_0), \quad u_1 \in H_0^1(\omega)$$

Under study (cont.)

Then :

$$\gamma_0(s) = \lambda_0,$$

$$\gamma_1(s) = k(s) \int_\omega a(y)\ (z \cdot \nabla u_0)\ u_0\ dy \neq 0,$$

$$\gamma_2(s) = k^2(s)\ \left[\int_\omega a(y)\ (z \cdot \nabla u_1)\ u_0\ dy - \frac{1}{2} \int_\omega a(y)\ |u_0|^2\ dy\right] \neq -\frac{k^2(s)}{4},$$

$$\lambda_1 = \inf_{\substack{\varphi \in H_0^1(0,L) \\ \|\varphi\|_{L^2(0,L)}=1}} \int_0^L k(s)\,[a(y)\ (z \cdot \nabla u_0)\ u_0]\ |\varphi|^2\ ds \neq 0, \quad etc.$$

Under study (cont.)

$$\begin{cases} -div\ (A(y)\ \nabla u_\varepsilon) = \lambda_\varepsilon\ u_\varepsilon, & \Omega_\varepsilon \\ u_\varepsilon = 0, & \partial\Omega_\varepsilon \end{cases}$$

Neumann boundary conditions, *etc.*

Elasticity operator, *etc.*

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