Blowup stability of solutions of the nonlinear heat equation with a large life span.

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Abstract

We study the Cauchy problem for the nonlinear heat equation $u_t - \triangle u = |u|^{p-1}u$ in $\mathbf{R}^N$. The initial data is of the form $u_0 = \lambda\,\varphi$, where $\varphi \in C_0(\mathbf{R}^N)$ is fixed and $\lambda > 0$. We first take $1 < p < p_f$, where p_f is the Fujita critical exponent, and $\varphi \in C_0(\mathbf{R}^N) \cap L^1(\mathbf{R}^N)$ with nonzero mean. We show that $u(t)$ blows up for λ small, extending the H. Fujita blowup result for sign-changing solutions. Next, we consider $1 < p < p_s$, where p_s is the Sobolev critical exponent, and $\varphi(x)$ decaying as $|x|^{-\alpha}$ at infinity, where $p < 1 + 2/\alpha$. We also prove that $u(t)$ blows up when λ is small, extending a result of T. Lee and W. Ni. For both cases, the solution enjoys some stable blowup properties. For example, there is single point blowup even if φ is not radial.

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1. INTRODUCTION. In this work we consider the semilinear heat equation

$$\begin{cases} u_t - \triangle u = |u|^{p-1}u & \text{in } (0,T) \times \mathbf{R}^N, \\ u(0) = u_0 & \text{in } \mathbf{R}^N, \end{cases} \tag{1.1}$$

where $N \in \mathbb{N}$, $p > 1$ and $u_0 \in C_0(\mathbf{R}^N)$. It is well known that (1.1) has a unique classical solution $u(t)$ defined over a maximal interval $[0,T)$, $T \le +\infty$. When $T < \infty$, $||u(t)||_\infty \longrightarrow \infty$ as $t \longrightarrow T$. More precisely, the set of blowup points $B = \{x \in \mathbf{R}^N,$ there exists $(x_n, t_n) \longrightarrow (x,T)$ such that $|u(x_n,t_n)| \longrightarrow \infty\}$ is nonempty. We then say that $u(t)$ blows up at the blowup time T. Outside B the solution $u(t)$ remains bounded, since there exists a blowup profile $u^* \in C^\infty(\mathbf{R}^N \setminus B)$ such that $u(t) \longrightarrow u^*$ as $t \longrightarrow T$ uniformly on compact sets of $\mathbf{R}^N \setminus B$, see [20]. The solution is called global if $T = \infty$.

It is natural to ask for conditions ensuring blowup. For positive solutions, there are the following sufficient conditions. (They are also necessary, in the sense of Theorem 3.8 of [18].)

- Large initial data: $u_0 = \lambda\,\varphi$, for a fixed $\varphi \ge 0$, $\varphi \ne 0$, and λ is large enough, see [18].

- Sub-critical exponent: $p \leq p_f$, where $p_f = 1 + 2/N$ is the so-called Fujita exponent, see [8], [14], [17].

- Slow decay: there exists $0 < \sigma < \dfrac{2}{p-1}$, $C > 0$, $R > 0$ such that $u_0(x) \geq C|x|^{-\sigma}$, for $|x| \geq R$, see [18].

In [18] Lee and Ni also devised a nice way of depicting the mechanism leading to blowup. They study the blowup time T_λ of the solution u_λ whose initial datum is of the form $u_0 = \lambda\,\varphi$. For all $p > 1$, T_λ decays at the ODE rate $\lambda^{-(p-1)}$ when $\lambda \longrightarrow \infty$, showing that the solution blows up because the initial datum is large. When $p < 1 + \min\{2/\sigma, 2/N\}$ and λ is small, there are three possible situations. If $\sigma < N$ the solution blows up due to the slow decay of the initial datum, while it blows up because p is sub-critical when $N < \sigma$. For $N = \sigma$, both mechanisms interact to accelerate the blowup. In fact, when $\varphi \in L^1(\mathbf{R}^N)$ there exist positive constants D_1, D_2 such that

$$D_1 \leq \lambda^{(\frac{1}{p-1} - \frac{N}{2})^{-1}}\, T_\lambda \leq D_2$$

if $\lambda < 1$. However, suppose there exists $\sigma \geq N$ such that $0 < C_1 \leq |x|^\sigma \varphi(x) \leq C_2 < \infty$ if $|x| > R$, for some positive constants C_1, C_2, R. Then, for $\lambda < 1$ and some positive constants D_1, D_2, D_3, D_4,

$$D_1 \leq \lambda^{(\frac{1}{p-1} - \frac{\sigma}{2})^{-1}}\, T_\lambda \leq D_2,$$

if $\sigma < N$, and

$$D_3 \leq (\lambda \log \lambda^{-1})^{(\frac{1}{p-1} - \frac{N}{2})^{-1}}\, T_\lambda \leq D_4,$$

if $\sigma = N$. A sharp result for $\sigma = 0$ was obtained by Gui and Wang [13]. When $\lim_{|x| \longrightarrow +\infty} \varphi(x) = \varphi_\infty > 0$ then

$$\lim_{\lambda \longrightarrow 0} \lambda^{p-1}\, T_\lambda = \frac{1}{p-1} \varphi_\infty^{-(p-1)}.$$

The unconditional blowup result of Fujita is not valid in general, since (1.1) admits global solutions decaying like $|x|^{-\frac{2}{p-1}}$ at infinity and also global solutions fast decaying for all $1 < p < p_f$, see [16], [30]. These (self-similar) solutions obviously change sign. New Fujita critical exponents for sign-changing solutions were obtained by Mizoguchi and Yanagida [22] in one-dimensional space $N = 1$ and for initial data with fast decay at infinity (see [22] for a precise definition). Given $k \in \mathbb{N}$, they consider the set Σ_k of functions which change sign k times in $\mathbb{R}$. They show that u blows up if $1 < p \leq p(k) = 1 + 2/(k+1)$ and $u_0 \in \Sigma_k$. The result is sharp. When $p > p(k)$ there exists a global solution whose initial datum is in Σ_k. Note that $p(0) = p_f$ for $N = 1$. The authors also study the problem in the half-space $\mathbb{R}^+$. For the Dirichlet condition $u(x, 0) = 0$, they show that the corresponding Fujita critical exponent $p^D(k)$ is equal to $1 + 1/(k+1)$. In [23], the authors show that the fast decay of u_0 is a superfluous assumption.

Our extension of the Fujita classical result for sign-changing solutions in $\mathbf{R}^N$ involves small initial data.

Theorem 1.1. *Let $p < p_f$ and let $\varphi \in C_0(\mathbf{R}^N) \cap L_1(\mathbf{R}^N)$ satisfy $\int_{\mathbf{R}^N} \varphi \neq 0$. Given $\lambda > 0$ let u_λ be the solution of (1.1) corresponding to $u_0 = \lambda\,\varphi$. Then there exists $\overline{\lambda} > 0$ such that u_λ blows up in finite time for all $0 < \lambda < \overline{\lambda}$.*

We remark that the nonzero mean hypothesis of Theorem 1.1 is relevant, at least in the one-dimensional case. Indeed, consider $N = 1$, so that $p_f = 3$. As showed in [23], when $p > p^D(0) = 2$ there exists a global positive solution in $\mathbb{R}^+$ satisfying the homogeneous Dirichlet condition and fast decaying at infinity. Let φ be its initial datum. Given $\lambda < 1$, it follows from the maximum principle that $u_0 = \lambda\varphi$ also gives rise to a global positive solution u_λ in $\mathbb{R}^+$. For each $t > 0$, let $\tilde{u}_\lambda(x,t)$ be the odd extension of $u_\lambda(x,t)$ to $\mathbb{R}$. This gives an example of a family u_λ of global solutions with zero mean.

The slow decay case for sign-changing solutions was discussed by Mizoguchi and Yanagida in [24]. Consider the polar decomposition $x = (r,\omega)$ and set $\Omega = \{x = (r,\omega) \in \mathbf{R}^N, r \geq R \text{ and } |\omega - \omega_0| \leq c\}$, for some $c > 0$, $R > 0$, $|\omega_0| = 1$. Let $0 < \sigma(p-1) < 2$ and suppose that there exist $c_1, c_2 > 0$ such that $|x|^\sigma u_0(x) \geq c_1$ and that $|x|^{\sigma+1}|\nabla u_0|(x) \geq c_2$ in Ω. Then there is finite time blowup. The authors also consider $u_0 = \lambda\varphi$ and study the growth of T_λ when $\lambda \longrightarrow 0$. They show that, given $\varepsilon > 0$, there exists $\lambda_\varepsilon > 0$ such that $T_\lambda \leq \lambda^{-(\frac{1}{p-1}-\frac{\sigma}{2})^{-1}-\varepsilon}$ if $\lambda \leq \lambda_\varepsilon$. They also exhibit a φ satisfying (1.2) for which $T_\lambda \geq \lambda^{-(\frac{1}{p-1}-\frac{\sigma}{2})^{-1}+\varepsilon}$.

We also present a result concerning the blowup of slowly decreasing functions for p sub-critical with respect to the Sobolev exponent p_s, defined as $p_s = \frac{N+2}{N-2}$ if $N > 2$, and $p_s = +\infty$ if $N = 1, 2$.

Theorem 1.2. *Let $p < \min\{1 + \dfrac{2}{\sigma}, p_s\}$ and let $\varphi \in C_0(\mathbf{R}^N)$ satisfy $\lim_{|x| \longrightarrow \infty} |x|^\sigma \varphi(x) = c$, where $c \neq 0$. Given $\lambda > 0$ call u_λ the solution of (1.1) corresponding to $u_0 = \lambda\varphi$. Then there exists $\overline{\lambda} > 0$ such that u_λ blows up in finite time for all $\lambda < \overline{\lambda}$.*

Comparing this result with those of [24] described above, we see that our assumptions on u_0 are more stringent, in one hand, and weaker, on the other. Theorem 1.2 requires $p < p_s$, which seems to be unnecessary for the blowup result. In fact, it can be removed, if $N < 11$, or replaced by $p < 1 + 4((N - 4 - 2(N - 1)^{1/2})^{-1}$, if $N \geq 11$, provided φ is radial and verifies the (S) condition of [19]. Under this assumptions, $(T-t)^{1/(p-1)}||u(t)||_\infty$ is bounded, ensuring blowup time continuity [20]. This is what one needs to prove Theorem 1.2. However, we remark that the way we treat the problem allows us to obtain more precise estimates on the growth of T_λ, as well as a detailed description of the blowup profile u^*_λ and of the blowup set B_λ, as we show below. For these, the restriction $p < p_s$ is crucial.

Our approach is related to some ideas introduced in [13]. We rescale the problem using a new parameter $\mu > 0$ and call v_μ the solution thus obtained. We choose $\mu = \mu(\lambda)$ so that $v_\mu(0)$ has a (weak) limit $\tilde{v}_0$ when $\lambda \longrightarrow 0$. It turns out that $\tilde{v}_0$ is either a Dirac measure (Theorem 1.2) or a homogeneous singular function (Theorem 1.1). We then show that $\tilde{v}$, the solution coming from $\tilde{v}_0$, is well defined and blows up at $t = \tilde{T}$ and at a single point $\tilde{x}$. Using known blowup stability results for the nonlinear heat equation [6], [15] we prove both theorems. It follows also from these arguments that u_λ blows up at a single point x_λ and further information about T_λ, x_λ and u^*_λ can be obtained.

When φ decays like $|x|^{-\sigma}$ and $\sigma < N$ then $U = \tilde{v}$ satisfies

$$\begin{cases} U_t - \triangle U = |U|^{p-1}U \quad \text{in } (0, T_U) \times \mathbf{R}^N, \\ U(0) = |x|^{-\sigma} \quad \text{in } \mathbf{R}^N. \end{cases} \tag{1.3}$$

To treat (1.3), we study (1.1) for initial data $u_0 \in E_{r,s} = L^r(\mathbf{R}^N) + L^s(\mathbf{R}^N)$, where $1 \le r < s < \infty$ ($s = \infty$ is precisely the case treated in [13]). Adapting the ideas of [2], [25], [28] we obtain the existenceof a unique solution, classical for $t > 0$, which depends continuously on initial data, see Theorem 2.8. This, combined with the results of [6], [15], leads to the following.

Theorem 1.3. *Let $0 < \sigma < N$, $p < \min\{1 + \frac{2}{\sigma}, p_s\}$ and suppose $\varphi \in C_0(\mathbf{R}^N)$ satisfies*

$$\lim_{|x| \longrightarrow \infty} |x|^\sigma \varphi(x) = 1. \tag{1.4}$$

Then there exists $\overline{\lambda} > 0$ such that u_λ blows up at a finite time T_λ and at a single point x_λ for all $\lambda \le \overline{\lambda}$. Moreover,

$$\lim_{x \longrightarrow x_\lambda} \left(\frac{|x - x_\lambda|^2}{\log|x - x_\lambda|} \right)^{\frac{1}{p-1}} u_\lambda^*(x,t) = \left(\frac{8p}{(p-1)^2} \right)^{\frac{1}{p-1}} \tag{1.5}$$

and

$$\lim_{\lambda \longrightarrow 0} \lambda^{\left(\frac{1}{p-1} - \frac{\sigma}{2}\right)^{-1}} T_\lambda = T_U, \tag{1.6}$$

where T_U is the blowup time of the unique solution U of (1.3), given by Theorem 2.8.

When $p < p_f$ and $\varphi \in L^1(\mathbf{R}^N)$ then $V = \tilde{v}$ satisfies

$$\begin{cases} V_t - \triangle V = |V|^{p-1}V \quad \text{in } (0, T_V) \times \mathbf{R}^N, \\ V(0) = \delta_0 \quad \text{in } \mathbf{R}^N, \end{cases} \tag{1.7}$$

where δ_0 is the Dirac measure supported at the origin. The well-posedness of (1.7) for initial data in the space of finite measures $\mathcal{M}(\mathbf{R}^N)$ is discussed in Theorem 2.10. In this way we obtain the

Theorem 1.4. *Let $p < p_f$ and suppose $\varphi \in L^1(\mathbf{R}^N) \cap C_0(\mathbf{R}^N)$ verifies $\int_{\mathbf{R}^N} \varphi = 1$. Then there exists $\overline{\lambda} > 0$ such that u_λ blows up at a finite time T_λ, at a single point x_λ for all $\lambda < \overline{\lambda}$. Moreover,*

$$\lim_{x \longrightarrow x_\lambda} \left(\frac{|x - x_\lambda|^2}{\log|x - x_\lambda|} \right)^{\frac{1}{p-1}} u_\lambda^*(x,t) = \left(\frac{8p}{(p-1)^2} \right)^{\frac{1}{p-1}} \tag{1.8}$$

and

$$\lim_{\lambda \longrightarrow 0} \lambda^{\left(\frac{1}{p-1} - \frac{N}{2}\right)^{-1}} T_\lambda = T_V, \tag{1.9}$$

where T_V is the blowup time of the unique solution V of (1.7), given by Theorem 2.10.

The case $\sigma = N$, $p < 1 + 2/N$, corresponds to a hybrid situation.

Theorem 1.5. *Let $p < p_f$, $S_N = \int_{|\omega|=1} 1\, d\omega$. Set $g(\mu) = S_N \mu^{\frac{2}{p-1}-N} \log \mu$ for $\mu > \overline{\mu} = e^{(\frac{2}{p-1}-N)^{-1}}$ and consider its inverse $h = g^{-1}$. Suppose $\varphi \in C_0(\mathbf{R}^N)$ satisfies (1.4) for $\sigma = N$. Then there exists $0 < \overline{\lambda} < \overline{\mu}^{-1}$ such that u_λ blows up at a finite time T_λ and at a single point x_λ for all $\lambda < \overline{\lambda}$. Moreover,*

$$\lim_{x \longrightarrow x_\lambda} \left(\frac{|x - x_\lambda|^2}{\log |x - x_\lambda|} \right)^{\frac{1}{p-1}} u_\lambda^*(x,t) = \left(\frac{8p}{(p-1)^2} \right)^{\frac{1}{p-1}} \tag{1.10}$$

and

$$\lim_{\lambda \longrightarrow 0} (h(\lambda^{-1}))^{-2}\, T_\lambda = T_U, \tag{1.11}$$

where T_U is the blowup time of the solution U of (1.3), given by Theorem 2.8.

The rest of this paper is organized as follows. In Section 2 we discuss the well-posedness of (1.1) for singular data $u_0 \in E_{r,s}$ and for $u_0 \in \mathcal{M}(\mathbf{R}^N)$. Theorem 1.3, Theorem 1.4 and Theorem 1.5 are proved in Section 3. Note that Theorem 1.1 is part of Theorem 1.4, while Theorem 1.2 is contained in Theorem 1.3 and Theorem 1.5.

2. The semilinear heat equation with singular initial data We start this section by discussing the existence, regularity and continuous dependence on initial data of solutions of

$$\begin{cases} u_t - \triangle u = |u|^{p-1} u & \text{in } (0,T) \times \mathbf{R}^N, \\ u(0) = u_0 & \text{in } \mathbf{R}^N, \end{cases} \tag{2.1}$$

for $u_0 \in L^r(\mathbf{R}^N) + L^s(\mathbf{R}^N)$, $1 \le r < s < \infty$. We denote $||.||_r$ the usual Lebesgue norm in $L^r(\mathbf{R}^N)$ and define $E_{r,s} = L^r(\mathbf{R}^N) + L^s(\mathbf{R}^N)$ the Banach space endowed with the standard norm $||u||_{r,s} = \inf\{||u_1||_r + ||u_2||_s, u = u_1 + u_2, u_1 \in L^r(\mathbf{R}^N), u_2 \in L^s(\mathbf{R}^N)\}$.

A certain number of properties of $E_{r,s}$ are presented below, where I_Ω denotes the characteristic function of $\Omega \subset \mathbf{R}^N$.

Lemma 2.1. Let $1 \le r < s$. Given $u \in E_{r,s}$, there exists $u_1 \in L^r(\mathbf{R}^N)$, $u_2 \in L^s(\mathbf{R}^N)$ such that $u = u_1 + u_2$, $||u||_{r,s} = ||u_1||_r + ||u_2||_s$. We have that $u^+ = u_1^+ + u_2^+$, $u^- = u_1^- + u_2^-$. In particular, $u^+, u^-, |u| \in E_{r,s}$. Moreover, $||\,|u|\,||_{r,s} = ||u||_{r,s}$ and

$$2^{1-s}(||u^+||_{r,s} + ||u^-||_{r,s}) \le ||u||_{r,s} \le ||u^+||_{r,s} + ||u^-||_{r,s}. \tag{2.2}$$

Proof. Consider two minimizing sequences $u_1^n \in L^r(\mathbf{R}^N)$, $u_2^n \in L^s(\mathbf{R}^N)$ such that $u = u_1^n + u_2^n$ and $||u_1^n||_r + ||u_2^n||_s \longrightarrow ||u||_{r,s}$. If $r > 1$, taking subsequences we may assume that $u_1^n \rightharpoonup u_1$ weak in $L^r(\mathbf{R}^N)$, $u_2^n \rightharpoonup u_2$ weak in $L^s(\mathbf{R}^N)$. Using the norm lower semicontinuity, it follows that $||u||_{r,s} = ||u_1||_r + ||u_2||_s$. When $r = 1$ we consider $u_1^n \rightharpoonup u_1$ weak-* in $\mathcal{M}(\mathbf{R}^N)$, the space of finite measures of $\mathbf{R}^N$. But then $u_1 = u - u_2 \in L^1_{\text{loc}}(\mathbf{R}^N) \cap \mathcal{M}(\mathbf{R}^N)$, so that $u_1 \in L^1(\mathbf{R}^N)$ and the same conclusion holds.

We now prove that $u^+ = u_1^+ + u_2^+$ and $u^- = u_1^- + u_2^-$. In what follows, whenever A, B are two measurable sets of $\mathbf{R}^N$, $A \subset B$ means $|A \setminus B| = 0$. Also, we write $A = B$ if $|B \setminus A| + |A \setminus B| = 0$.

We argue by contradiction and suppose that $|\Omega| > 0$, where $\Omega = \{x \in \mathbf{R}^N, u(x) \geq 0, u_1(x) < 0\}$. If $x \in \Omega$ then $|u_1(x)| = u_2(x) - u(x) \leq u_2(x)$ and thus $u_1 \mathrm{I}_\Omega \in L^s(\mathbf{R}^N)$. Consider the decomposition $u = (u_1 - u_1 \mathrm{I}_\Omega) + (u_2 + u_1 \mathrm{I}_\Omega)$. Then $||u_1 - u_1 \mathrm{I}_\Omega||_r < ||u_1||_r$, $||u_2 + u_1 \mathrm{I}_\Omega||_s < ||u_2||_s$, which is absurd. This shows that $\{x \in \mathbf{R}^N, u(x) \geq 0\} \subset \{x \in \mathbf{R}^N, u_1(x) \geq 0\}$. Taking $-u$ instead of u, we conclude that $\{x, u(x) \geq 0\} = \{x, u_1(x) \geq 0\}$. Since the same argument holds for u_2, $u^+ = u_1^+ + u_2^+$ and $u^- = u_1^- + u_2^-$. It follows that $|u| = |u_1| + |u_2|$, thus $||u||_{r,s} = ||u_1||_r + ||u_2||_s \geq ||\,|u|\,||_{r,s}$. On the other hand, consider an optimal decomposition of $|u|$, $|u| = w_1 + w_2$ and take $v_1 = w_1 \operatorname{sign} u$, $v_2 = w_2 \operatorname{sign} u$, where $\operatorname{sign} u$ is the sign function. Then $u = v_1 + v_2$ and $||u||_{r,s} \leq ||v_1||_r + ||v_2||_s = ||\,|u|\,||_{r,s}$. Thus $||u||_{r,s} = ||\,|u|\,||_{r,s}$.

Finally, we show (2.2). We clearly have that $||u||_{r,s} \leq ||u^+||_{r,s} + ||u^-||_{r,s}$. Since $||u_1||_r \geq 2^{1-r}(||u_1^+||_r + ||u_1^-||_r)$ and $||u_2||_s \geq 2^{1-s}(||u_2^+||_s + ||u_2^-||_s)$, then $||u||_{r,s} \geq 2^{1-s}(||u^+||_{r,s} + ||u^-||_{r,s})$. ◻

A sum $u = u_1 + u_2$ as in Lemma 2.1 will be called an optimal decomposition of $u \in E_{r,s}$. We remark that the minimizing couple may not be unique. In fact, let $\Omega = (0,1)$ and $u = \mathrm{I}_\Omega$. We leave it to the reader to verify that $||u||_{r,s} = ||\theta u||_r + ||(1-\theta)u||_s = 1$ for all $\theta \in [0,1]$ and all $1 \leq r < s$.

Lemma 2.2. *Let $1 \leq r < s < q$. Then $E_{r,s} \cap E_{s,q} \subset L^s(\mathbf{R}^N)$ and there exists $C > 0$ such that*

$$||u||_s \leq C \max\{||u||_{r,s}, ||u||_{s,q}\}$$

for all $u \in E_{r,s} \cap E_{s,q}$.

Proof. It suffices to show that $||u||_s \leq C \max\{||u||_{r,s}, 1\}$ if $||u||_{s,q} = 1$. Using Lemma 2.1, we may also suppose that $u \geq 0$. Again by Lemma 2.1, we decompose $u = v_1 + v_2 = w_1 + w_2$, where $0 \leq v_1 \in L^r(\mathbf{R}^N)$, $0 \leq v_2, w_1 \in L^s(\mathbf{R}^N)$, $0 \leq w_2 \in L^q(\mathbf{R}^N)$, $||u||_{r,s} = ||v_1||_r + ||v_2||_s$, $1 = ||w_1||_s + ||w_2||_q$. Define $A = \{x \in \mathbf{R}^N, u(x) \geq 1\}$, $B = \mathbf{R}^N \setminus A$, $A_1 = \{x \in \mathbf{R}^N, w_1(x) \geq 1/2\}$, $A_2 = \{x \in \mathbf{R}^N, w_2(x) \geq 1/2\}$. Then $|A_1| \leq 2^s ||w_1||_s^s \leq 2^s$, $|A_2| \leq 2^q ||w_2||_q^q \leq 2^q$. Since $A \subset A_1 \cup A_2$ then $|A| \leq 2^s + 2^q$. Set $w_{1,A} = w_1 \mathrm{I}_A$, $w_{2,A} = w_2 \mathrm{I}_A$, $v_{1,B} = v_1 \mathrm{I}_B$, $v_{2,B} = v_2 \mathrm{I}_B$. Then $w_{2,A}, v_{1,B} \in L^s(\mathbf{R}^N)$ so that

$$u = u \mathrm{I}_A + u \mathrm{I}_B = w_{1,A} + w_{2,A} + v_{1,B} + v_{2,B} \in L^s(\mathbf{R}^N), \tag{2.3}$$

$$||w_{1,A}||_s \leq ||w_1||_s \leq 1, \quad ||w_{2,A}||_s \leq ||w_2||_q |A|^{(q-s)/qs} \leq (2^s + 2^q)^{(q-s)/qs}, \tag{2.4}$$

$$||v_{1,B}||_s \leq ||v_{1,B}||_r^{r/s} \leq \max\{||v_1||_r, 1\}, \quad ||v_{2,B}||_s \leq ||v_2||_s. \tag{2.5}$$

Then (2.3), (2.4), (2.5) yield $||u||_s \leq ||w_{1,A}||_s + ||w_{2,A}||_s + ||v_{1,B}||_s + ||v_{2,B}||_s \leq C \max\{||u||_{r,s}, 1\}$. ◻

Lemma 2.3. *Let $1 \leq r < s$, $p > 1$ and $p' = p/(p-1)$. If $u \in L^{pr}(\mathbf{R}^N)$, $v \in L^{p's}(\mathbf{R}^N)$ then $uv \in E_{r,s}$. There exists $C > 0$ such that for all $u \in L^{pr}(\mathbf{R}^N)$, $v \in L^{p's}(\mathbf{R}^N)$ it holds $||uv||_{r,s} \leq C ||u||_r ||v||_s$.*

Proof. It suffices to show that $uv \in E_{r,s}$ and that $||uv||_{r,s} \le 2$ whenever $||u||_{pr} = 1$, $||v||_{p's} = 1$. Also, using Lemma 2.1, we may assume that $u, v \ge 0$. We then set $A = \{x \in \mathbf{R}^N, u(x) > 1\}$, $B = \mathbf{R}^N \setminus A$, $u_A = u\,\mathrm{I}_A$, $u_B = u\,\mathrm{I}_B$, $v_A = v\,\mathrm{I}_A$, $v_B = v\,\mathrm{I}_B$. Since $|A| \le ||u_A||_{pr}^{pr} \le 1$, $v_A \in L^{p'r}(\mathbf{R}^N)$ and $||v_A||_{p'r} \le ||v_A||_{p's} \le 1$. Moreover, $u_B \in L^{ps}(\mathbf{R}^N)$ and $||u_B||_{ps} \le ||u_B||_{pr}^{r/s} \le 1$. It follows from Hölder's inequality that $u_A v_A \in L^r(\mathbf{R}^N)$, $u_B v_B \in L^s(\mathbf{R}^N)$ and $||u_A v_A||_r \le 1$, $||u_B v_B||_s \le 1$. Hence, $uv = u_A v_A + u_B v_B \in E_{r,s}$ and verifies $||uv||_{r,s} \le ||u_A v_A||_r + ||u_B v_B||_s \le 2$. $\square$

Lemma 2.4. *Let $u \in E_{r,s}$ and consider $\mathbf{R}^N = F \cup G$, where F, G are two disjoint measurable sets. Set $u_F = u\, I_F$ and $u_G = u\, I_G$. Then $u_F, u_G \in E_{r,s}$, $||u_F||_{r,s} \le ||u||_{r,s}$ and*

$$2^{1-s}(||u_F||_{r,s} + ||u_G||_{r,s}) \le ||u||_{r,s} \le ||u_F||_{r,s} + ||u_G||_{r,s}.$$

Proof. Clearly, $u_F, u_G \in E_{r,s}$ and $||u||_{r,s} \le ||u_F||_{r,s} + ||u_G||_{r,s}$. Let $u = u_1 + u_2$ be an optimal decomposition of u. Since $||u_F||_{r,s} \le ||u_1\, I_F||_r + ||u_2\, I_F||_s$,

$$||u||_{r,s} = ||u_1||_r + ||u_2||_s \ge 2^{1-s}(||u_1\, I_F||_r + ||u_1\, I_G||_r + ||u_2\, I_F||_s + ||u_2\, I_G||_s) \ge 2^{1-s}(||u_F||_{r,s} + ||u_G||_{r,s}).$$

$\square$

Lemma 2.5. *There exists $C > 0$ such that*

$$|||u|^{p-1}u - |v|^{p-1}v||_{r,s} \le C(\max\{||u||_{pr,ps}, ||v||_{pr,ps}\})^{p-1}||u-v||_{pr,ps} \tag{2.6}$$

for all $u, v \in E_{pr,ps}$.

Proof. Applying Lemma 2.4 to $F = \{x \in \mathbf{R}^N, v(x) \le u(x)\}$, we may suppose that $v \le u$. Set $A = \{x \in \mathbf{R}^N, 0 \le v(x) \le u(x)\}$ and let $u_A = u\,\mathrm{I}_A$, $v_A = v\,\mathrm{I}_A$, $w_A = u_A - v_A$. Consider the optimal decompositions $u_A = u_{1,A} + u_{2,A}$, $w_A = w_{1,A} + w_{2,A}$. Then $u_A^p - v_A^p \le C(u_{1,A}^{p-1} + u_{2,A}^{p-1})(w_{1,A} + w_{2,A})$ for some $C = C(p) > 0$. Since $u_{1,A}^{p-1} \in L^{p'r}(\mathbf{R}^N)$ and $u_{2,A}^{p-1} \in L^{p's}(\mathbf{R}^N)$ we use Lemma 2.3, Hölder's inequality and Lemma 2.4 to write

$$||(|u|^{p-1}u - |v|^{p-1}v)\mathrm{I}_A||_{r,s} \le C(||u_{1,A}||_{pr}^{p-1} + ||u_{2,A}||_{ps}^{p-1})(||w_{1,A}||_{pr} + ||w_{2,A}||_{ps}) \le C||u||_{pr,ps}^{p-1}||w||_{pr,ps}. \tag{2.7}$$

If $B = \{x \in \mathbf{R}^N, v(x) < 0 \le u(x)\}$, $C = \{x \in \mathbf{R}^N, v(x) \le u(x) < 0\}$, using analogous computations we write

$$||(|u|^{p-1}u - |v|^{p-1}v)\mathrm{I}_B||_{r,s} \le C||u||_{pr,ps}^{p-1}||w||_{pr,ps}, \tag{2.8}$$

$$||(|u|^{p-1}u - |v|^{p-1}v)\mathrm{I}_C||_{r,s} \le C||u||_{pr,ps}^{p-1}||w||_{pr,ps}. \tag{2.9}$$

The lemma follows from (2.7), (2.8) and (2.9). $\square$

If $S(t)$ is the linear heat semigroup and $u = u_1 + u_2$ is an optimal decomposition of $u \in E_{r,s}$ then $S(t)u = S(t)u_1 + S(t)u_2 \in E_{r,s}$ and $||S(t)u||_{r,s} \le ||u_1||_r + ||u_2||_s = ||u||_{r,s}$. This shows that $S(t)$ is a continuous semigroup of contractions in $E_{r,s}$. It is also clear that given $T > 0$ and $\theta > 1$ there exists $C > 0$ such that

$$||S(t)u||_{\theta r,\theta s} \le C t^{-\frac{N(\theta-1)}{2\theta r}} ||u||_{r,s} \tag{2.10}$$

for all $t \le T$.

We will also make use of the following generalized Gronwall's inequality.

Lemma 2.6. *Let $T > 0$, $A \ge 0$, $\alpha, \gamma \ge 0$, $0 \le \beta < 1$ be such that $1 + \alpha > \beta + \gamma$. If $\varphi \in L^\infty(0,T)$ satisfies*

$$0 \le \varphi(t) \le A + t^\alpha \int_{t/2}^{t} (t-\tau)^{-\beta} \tau^{-\gamma} \varphi(\tau)\, d\tau \tag{2.11}$$

a.e. in $(0,T)$, then there exists $C(T,\alpha,\beta,\gamma)$ such that a.e. in $(0,T)$

$$\varphi(t) \le CA. \tag{2.12}$$

Proof. Set $\overline{\varphi}(t) = \operatorname{ess} \sup_{\tau \in (0,T)} \varphi(\tau)$. Since $\varphi(t) \le A + \overline{\varphi}(t) t^{1+\alpha-\beta-\gamma} \int_{1/2}^{1} (1-\tau)^{-\beta} \tau^{-\gamma}\, d\tau$, we can choose $\bar{t}$ such that

$$\overline{\varphi}(t) \le 2A, \tag{2.13}$$

for a.a. $t \le \bar{t}$. Suppose now $t > t_0$. For $1/2 < k < 1$ consider the splitting

$$t^\alpha \int_{t/2}^{t} (t-\tau)^{-\beta} \tau^{-\gamma} \varphi(\tau)\, d\tau = t^\alpha \Big(\int_{t/2}^{kt} + \int_{kt}^{t} \Big) (t-\tau)^{-\beta} \tau^{-\gamma} \varphi(\tau)\, d\tau. \tag{2.14}$$

Note that k can be chosen so that

$$t^\alpha \int_{kt}^{t} (t-\tau)^{-\beta} \tau^{-\gamma} \varphi(\tau)\, d\tau \le T^{1+\alpha-\beta-\gamma}\, \overline{\varphi}(t) \int_{k}^{1} (1-\tau)^{-\beta} \tau^{-\gamma}\, d\tau \le \overline{\varphi}(t)/2, \tag{2.15}$$

Also,

$$t^\alpha \int_{t/2}^{kt} (t-\tau)^{-\beta} \tau^{-\gamma} \varphi(\tau)\, d\tau \le T^{1+\alpha-\beta-\gamma} (1-k)^{-\beta} 2^\gamma \int_0^t \overline{\varphi}(\tau)\, d\tau. \tag{2.16}$$

It follows from (2.14), (2.15) and (2.16) that there exists $C = C(T,\alpha,\beta,\gamma)$ such that

$$\overline{\varphi}(t) \le 2A + C \int_0^t \overline{\varphi}(\tau)\, d\tau, \tag{2.17}$$

for a.a. $\bar{t} \le t \le T$. Then (2.12) follows from (2.13), (2.17) and the standard Gronwall's inequality. ☐

Remark 2.7. Lemma 2.6 is a variant of the following result of [2]. Assume also that $\gamma < 1$ and replace (2.11) by $\varphi(t) \le A + t^\alpha \int_0^t (t-s)^{-\beta} s^{-\gamma} \varphi(s)\, ds$,. Then (2.12) holds true.

We now discuss the well-posedness of (2.1) for $u_0 \in E_{r,s}$. A standard way of studying the nonlinear heat equation for unbounded initial data is to consider the integral formulation

$$u(t) = S(t)u_0 + \int_0^t S(t-\tau)|u|^{p-1}u(\tau)\,d\tau \tag{2.18}$$

and to eventually obtain a fixed point of the mapping $u \longrightarrow \Phi(u)$,

$$\Phi(u)(t) = S(t)u_0 + \int_0^t S(t-\tau)|u(\tau)|^{p-1}u(\tau)\,d\tau, \tag{2.19}$$

in a suitable metric space, see [28], [2], [25]. This idea will be carried through here to prove the following.

Theorem 2.8. *Let $1 \le r < s$, $p > 1$ be such that $\frac{p-1}{r} < \frac{2}{N}$. Given $u_0 \in E_{r,s}$ there exist $T > 0$ and a unique solution $u \in C((0,T]; E_{pr,ps}) \cap C([0,T]; E_{r,s})$ of (2.18). In addition, $u \in C((0,T]; L^q(\mathbf{R}^N))$ for all $s \le q \le \infty$.*

Moreover, the following continuous dependence on the initial data holds. Given $M > 0$ let $u_0, v_0 \in E_{r,s}$ satisfy $||u_0||_{r,s}, ||v_0||_{r,s} \le M$ and let u, v be their corresponding solutions, defined on some interval $[0,T]$. Then for $q \in [s,\infty]$ there exists $C = C(N,T,r,s,p,q,M) > 0$ such that for all $t \in (0,T]$

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{q})}||u(t) - v(t)||_q \le C||u_0 - v_0||_{r,s}. \tag{2.20}$$

Proof. We use the ideas developped in [2], [28], [25], pointing out the modifications we have introduced here. We divide the proof into various parts.

Existence

Let $\beta = \frac{N(p-1)}{2pr}$, $0 < T < 1$ and define $W = \{u \in C((0,T]; E_{pr,ps}),\ \sup_{t\in(0,T]} t^\beta ||u(t)||_{pr,ps} < +\infty\}$, which is a Banach space for the norm $||u||_W = \sup_{t\in(0,T]} t^\beta ||u(t)||_{pr,ps}$. Set $M = ||S(t)u_0||_W$ and let K be the closed ball of radius $M+1$ in W. Since $\beta p < 1$, using (2.6) (with $v = 0$) and (2.10) it follows that there exists $C > 0$ such that $||\Phi(u)||_W \le M + C(M+1)^p T^{1-\beta p}$ for all $u \in K$. A similar computation using that

$$\Phi(u)(t) - \Phi(u)(\tau) = \int_\tau^t S(t-\tau)|u(\tau)|^{p-1}u(\tau)\,d\tau + \int_0^\tau S(\tau-\tau)(S(t-\tau)-I)|u(\tau)|^{p-1}u(\tau)\,d\tau$$

if $0 < \tau < t \le T$ shows that $\Phi(u) \in C((0,T]; E_{pr,ps})$. Therefore, $\Phi(K) \subset K$ if T is small enough. To show that Φ is a strict contraction in K, consider $u, v \in K$. Then (2.6), (2.10) yield $||\Phi(u) - \Phi(v)||_W \le C(M+1)^{p-1}T^{1-p\beta}||u-v||_W \le \frac{1}{2}||u-v||_W$ if T is possibly smaller. Thus there exists a unique local solution $u \in K$ of (2.18). Clearly, $u \in C([0,T]; E_{r,s})$.

Uniqueness

The argument of [2] showing the uniqueness of solutions in the class $C([0,T]; L^r(\mathbf{R}^N)) \cap C((0,T]; L^{pr}(\mathbf{R}^N))$ for $r(p-1) > N/2$ (see also [25]) applies here with minor changes.

Regularity

To show that $u(t) \in L^q(\mathbf{R}^N))$ if $s \le q \le \infty$, we adapt to the present context the bootstrap procedure of [25]. Let $p \le \theta \le \tilde\theta \le \infty$ with $\dfrac{N}{2r}\left(\dfrac{p}{\theta} - \dfrac{1}{\tilde\theta}\right) < 1$ and suppose there exists $L(\theta) > 0$ such that

$$\sup_{(0,T]} t^{\frac{N(\theta-1)}{2\theta r}} ||u(t)||_{\theta r, \theta s} \le L(\theta) \tag{2.21}$$

(the existence part of the proof ensures that this is valid for $\theta = p$ and $L(p) = M + 1$). Since $u(t) = S(t)u(t/2) + \int_{t/2}^t S(t-\tau)|u|^{p-1}u(\tau)\, d\tau$ and $\displaystyle\int_{1/2}^1 (1-\tau)^{-\frac{N}{2r}\left(\frac{p}{\theta}-\frac{1}{\tilde\theta}\right)} \tau^{-\frac{Np(\theta-1)}{2\theta r}}\, d\tau < +\infty$, it follows from (2.6), (2.10) and (2.21) that

$$\begin{aligned} ||u(t)||_{\tilde\theta r, \tilde\theta s} &\le C(t^{-\frac{N}{2r}\left(\frac{1}{\theta}-\frac{1}{\tilde\theta}\right)} ||u(t/2)||_{\theta r, \theta s} + \int_{t/2}^t (t-\tau)^{-\frac{N}{2r}\left(\frac{p}{\theta}-\frac{1}{\tilde\theta}\right)} ||u(\tau)||^p_{\theta r, \theta s}\, d\tau) \\ &\le L(\tilde\theta)(t^{-\frac{N(\tilde\theta-1)}{2\tilde\theta r}} + t^{1-p\beta-\frac{N(\tilde\theta-1)}{2\tilde\theta r}}). \end{aligned}$$

Then

$$t^{\frac{N(\tilde\theta-1)}{2\tilde\theta r}} ||u(t)||_{\tilde\theta r, \tilde\theta s} \le L(\tilde\theta). \tag{2.22}$$

Therefore, we see from (2.21) and (2.22) that one can bootstrap starting from $\theta = p$. It is easy to see that $\theta = \infty$ is reached in a finite number of steps, so we conclude that (2.21) holds for all $p \le \theta \le \infty$. Also, if $1 \le \theta < p$, once again (2.6), (2.10) yield

$$\begin{aligned} t^{\frac{N(\theta-1)}{2\theta r}} ||u(t)||_{\theta r, \theta s} &\le C(||u_0||_{r,s} + M^p t^{\frac{N(\theta-1)}{2\theta r}} \int_0^t (t-\tau)^{-\frac{N(\theta-1)}{2\theta r}} \tau^{-\beta p}\, d\tau) \\ &= C(||u_0||_{r,s} + M^p t^{1-\beta p} \int_0^1 (1-\tau)^{-\frac{N(\theta-1)}{2\theta r}} \tau^{-\beta p}\, d\tau) \le L(\theta). \end{aligned}$$

So, in fact,

$$\sup_{(0,T]} t^{\frac{N\theta}{2r(1-\theta)}} ||u(t)||_{\theta r, \theta s} \le L(\theta) \tag{2.23}$$

for all $1 \le \theta \le +\infty$. Setting $\theta = q/r$ and $\theta = q/s$ in (2.23) and using Lemma 2.2, we see that $u(t) \in L^q(\mathbf{R}^N)$ for all $q \ge s$ and that there exists $L'(q) > 0$ verifying

$$\sup_{(0,T]} t^{\frac{N}{2}\left(\frac{1}{r}-\frac{1}{q}\right)} ||u(t)||_q \le L'(q). \tag{2.24}$$

Finally, writting $u(t+\delta) - u(t) = (S(\delta) - I)u(t) + \displaystyle\int_0^\delta S(\tau)|u|^{p-1}u(t+\tau)\, d\tau$ and using (2.24), it follows easily that $u \in C((0,T]; L^q(\mathbf{R}^N)$ for all $q \ge s$.

Continuous dependence

Let $u_0, v_0 \in E_{r,s}$ with $\max\{||u_0||_{r,s}, ||v_0||_{r,s}\} \le M$ and let $u(t)$, $v(t)$ be their corresponding solutions. Given $\theta \in [1, +\infty]$, define $\alpha = \dfrac{N}{2r}\left(1 - \dfrac{1}{\theta}\right)$ and set $w(t) = u(t) - v(t)$, $w_0 = u_0 - v_0$. To show (2.20), we first prove that

$$||w(t)||_{\theta r, \theta s} \le C t^{-\alpha} ||w_0||_{r,s}. \tag{2.25}$$

We have that

$$w(t) = S(t-\tilde{t})w(\tilde{t}) + \int_{\tilde{t}}^{t} S(t-\tau)(|u(\tau)|^{p-1}u(\tau) - |v(\tau)|^{p-1}v(\tau))\,d\tau, \tag{2.26}$$

for $0 \leq \tilde{t} < t$. Consider $\varphi(t) = t^{\alpha}||w(t)||_{\theta r,\theta s}$.

Suppose first that $\theta \in [1,p]$. Using (2.26) (for $\tilde{t} = 0$), (2.10), (2.6) and (2.23) it follows that

$$\varphi(t) \leq C(||w_0||_{r,s} + t^{\alpha}\int_0^t (t-\tau)^{-\alpha}\tau^{-p\beta}\varphi(\tau)\,d\tau).$$

Then (2.25) holds, see Remark 2.7.

Take next $p < \theta \leq \infty$. We use (2.26) (with $\tau = t/2$), (2.25) (with $\theta = 1$), (2.10) (with $\theta = 1/p$ and θr, θs replacing r, s), (2.6), (2.23) to get that

$$\varphi(t) \leq C(||w_0||_{r,s} + t^{\alpha}\int_{t/2}^t (t-s)^{-\frac{N(p-1)}{2\theta r}} s^{-p\alpha}\varphi(s)\,ds).$$

Since $\dfrac{N(p-1)}{2r} < 1$, Lemma 2.6 applies and (2.25) is obtained for $q > p$. Using (2.25) for $\theta = q/s$ and for $\theta = q/r$, (2.20) follows from Lemma 2.2. □

The critical case $\dfrac{p-1}{r} = \dfrac{2}{N}$, $r > 1$ can also be handled with analogous techniques [2], leading to the following result.

Theorem 2.9. *Suppose* $\dfrac{p-1}{r} = \dfrac{2}{N}$, $1 < r < s$. *Given* $u_0 \in E_{r,s}$ *there exists* $T > 0$ *and a unique solution* $u \in C((0,T);E_{pr,ps}) \cap C([0,T);E_{r,s})$ *of (2.18). In addition,* $u \in C((0,T);L^q(\mathbf{R}^N))$ *for all* $s \leq q \leq \infty$.

Moreover, the following continuous dependence on the initial data holds. Given $M > 0$ *let* $u_0, v_0 \in E_{r,s}$ *satisfy* $||u_0||_{r,s}, ||v_0||_{r,s} \leq M$ *and let* u, v *be their corresponding solutions, defined on some interval* $[0,T]$. *Then for* $q \in [s,\infty]$ *there exists* $C = C(N,T,r,s,p,q,M) > 0$ *such that for all* $t \in (0,T]$

$$t^{\frac{N}{2}(\frac{1}{r}-\frac{1}{q})}||u(t)-v(t)||_q \leq C||u_0 - v_0||_{r,s}. \tag{2.27}$$

In the rest of this section we treat the case of finite measures as initial data. Initial conditions in measure spaces have been considered in related situations, see [4] for the discussion of the "good sign case", $f(u) = -|u|^{p-1}u$ and [1] for the study of the heat equation with nonlinear boundary conditions. The nonlinear heat equation (2.1) was treated in [3]. We present and prove the result below for the sake of completeness. We denote $\mathcal{M}(\mathbf{R}^N)$ the space of finite measures of $\mathbf{R}^N$ and $||.||_{\mathcal{M}}$ its usual norm. Theorem 2.10 shows that finite measures behave as elements of $L^1(\mathbf{R}^N)$.

Theorem 2.10. *Suppose* $N(p-1) < 2$. *Given* $u_0 = \mu_0 \in \mathcal{M}(\mathbf{R}^N)$ *there exists* $T > 0$ *and a unique solution* $u \in C((0,T];L^p(\mathbf{R}^N) \cap C((0,T];L^1(\mathbf{R}^N)$ *of (2.18) such that* $u(t) \longrightarrow \mu_0$ *as* $t \longrightarrow 0$ *in the weak-* topology of* $\mathcal{M}(\mathbf{R}^N)$.

Moreover, the following continuous dependence on the initial data holds. Given $M > 0$ let μ_0, $\nu_0 \in \mathcal{M}(\mathbf{R}^N)$ be such that $||\mu_0||_{\mathcal{M}}$, $||\nu_0||_{\mathcal{M}} \le M$ and let u, v be their corresponding solutions, defined on some interval $[0,T]$. Then for $q \in [1,\infty]$ there exists $C = C(N,M,p,q) > 0$ such that for all $t \in (0,T]$

$$t^{\frac{N}{2}(1-\frac{1}{q})}||u(t) - v(t)||_q \le C||\mu_0 - \nu_0||_{\mathcal{M}}. \tag{2.28}$$

Proof. Let $||\mu_0||_{\mathcal{M}} \le M$ and $\beta = \dfrac{N}{2p'}$, where p' is the conjugate of p. For $T > 0$ consider the Banach space $W = C((0,T]; L^p(\mathbf{R}^N))$, endowed with the norm $||u||_W = \sup\limits_{t\in(0,T]} t^\beta ||u(t)||_p$. Let $K(M,T)$ be the closed ball of radius $M+1$ in W and define $\Phi(u)(t) = S(t)\mu_0 + \int_0^t S(t-\tau)|u(\tau)|^{p-1}u(\tau)\,d\tau$. Using that $t^\beta||S(t)\mu_0||_p \le M$, it is not difficult to see that for some $C > 0$, $||\Phi(u)||_W \le M + C(M+1)^pT^{1-p\beta}$, $||\Phi(u)(t)||_1 \le M + (M+1)^pt^{1-p\beta}$, $||\Phi(u) - \Phi(v)||_W \le C(M+1)^{p-1}T^{1-p\beta}||u-v||_W$, with $\Phi(u) \in C((0,T]; L^p(\mathbf{R}^N))$. Therefore, for T small enough Φ is a strict contraction in K verifying $\sup\limits_{t\in(0,T]} ||u(t)||_1 \le (M+1)$. This gives the existence of a solution $u \in C((0,T]; L^p(\mathbf{R}^N))$ of (2.18). We have that $S(t)\mu_0 \in C((0,T); L^1(\mathbf{R}^N))$ and $|| \int_{\tilde{t}}^t S(t-\tau)|u(\tau)|^{p-1}u(\tau)\,d\tau||_1 \le \int_{\tilde{t}}^t ||u(\tau)||_p^p\,d\tau \le (M+1)^p(1-p\beta)^{-1}(t^{1-p\beta} - \tilde{t}^{1-p\beta})$. Hence, $u \in C((0,T]; L^1(\mathbf{R}^N))$. Since $p\beta < 1$ and $u(t) - S(t)\mu_0 \longrightarrow 0$ as $t \longrightarrow 0$ in $L^1(\mathbf{R}^N)$ we see that $u(t) \longrightarrow \mu_0$ weak-* in $\mathcal{M}(\mathbf{R}^N)$.

To show the uniqueness, consider $v \in C((0,T]; L^p(\mathbf{R}^N)) \cap C((0,T]; L^1(\mathbf{R}^N))$ a solution of (2.18) such that $v(t) \longrightarrow \mu_0$ weak-* in $\mathcal{M}(\mathbf{R}^N)$. Choose M such that $||\mu_0||_{\mathcal{M}} \le M$, $\sup_{0<t\le T} ||v(t)||_1 \le M+1$. Given $a \in (0,T/2]$, set $v_a(t) = v(t+a)$ and let u_a be the solution constructed above having $u_a(0) = v(a)$ as initial value. Note that we can take $u_a \in K(M+1,T_1)$ for some $T_1 > 0$ independent of $a \in (0,T/2]$. On the other hand, using that $v_a(t) = S(t)v(a) + \int_0^t S(t-\tau)|v_a(\tau)|^{p-1}v_\tau(\tau)\,d\tau$ and that $v_a(t) \in C([0,T/2]; L^p(\mathbf{R}^N)) \cap C([0,T/2]; L^1(\mathbf{R}^N))$, we see that $v_a \in K(M+1,T_2)$ for some $T_2 > 0$ (which depends on τ). Defining $T_3 = \min\{T_1,T_2\}$ it follows from the uniqueness of the fixed point of Φ in $K(M+1,T_3)$ that $v_a(t) = u_a(t)$ for $0 \le t \le T_3$ and for all $0 < a \le T/2$. But since u_a and v_a are regular solutions, they must coincide in $0 \le t \le T/2$. Hence, for $0 < t, a \le T/2$,

$$t^\beta ||v(t+a)||_p \le M' + 1.$$

Letting $a \longrightarrow 0$, we conclude that v is the solution obtained by the fixed point argument.

We now prove (2.28). Consider $\mu_0, \nu_0 \in \mathcal{M}(\mathbf{R}^N)$ such that $||\mu_0||_{\mathcal{M}}$, $||\nu_0||_{\mathcal{M}} \le M$ and let u, v be their corresponding solutions verifying

$$\sup_{t\in(0,T]} t^\beta \max\{||u(t)||_p, ||v(t)||_p\} \le M+1 \tag{2.29}$$

for some $T > 0$. We have that $u(t) - v(t) = S(t)(\mu_0 - \nu_0) + \displaystyle\int_0^t S(t-\tau)(|u|^{p-1}u - |v|^{p-1}v)(\tau)\,d\tau$, so that (2.29) yields

$$||u(\tilde{t}) - v(\tilde{t})||_1 \le ||\mu_0 - \nu_0||_{\mathcal{M}} + 2(M+1)^p \int_0^{\tilde{t}} \tau^{-p\beta} d\tau = ||\mu_0 - \nu_0||_{\mathcal{M}} + 2(M+1)^p(1-p\beta)^{-1}\tilde{t}^{1-p\beta}.$$

Thus

$$||u(\tilde{t}) - v(\tilde{t})||_1 \leq 2||\mu_0 - \nu_0||_{\mathcal{M}},$$

for $\tilde{t}$ sufficiently small. Now $u(\tilde{t}), v(\tilde{t}) \in L^1(\mathbf{R}^N)$ and $||u(\tilde{t})||_1, ||v(\tilde{t})||_1|| \leq M'$, for some $M' > 0$ and for all $\tilde{t} \in (0, T)$. Now the results of [2] concerning initial data on $L^1(\mathbf{R}^N)$ ensures that for all $q \geq 1$ there exists $C = C(q)$ verifying

$$t^{\frac{n}{2}(1-\frac{1}{q})}||u(t+\tilde{t}) - v(t+\tilde{t})||_q \leq C||u(\tilde{t}) - v(\tilde{t})||_1 \leq 2C||\mu_0 - \nu_0||_{\mathcal{M}},$$

for $t \in (0, T - \tilde{t}]$, and $\tilde{t}$ small. Then (2.28) follows from the L^q-continuity of the solutions for $t > 0$ □

3. Proof of the main results

In the proof of Theorem 1.3, Theorem 1.4 and Theorem 1.5 we use the following results about the solution behaviour near blowup. Below, we suppose that $u_0 \in C_0(\mathbf{R}^N)$ and that its corresponding solution $u(t)$ blows up at $T = T(u_0)$. We denote $B = B(u_0)$ the set of blowup points.

Fact 1 - In [21] it is shown that if $u(t)$ is positive, radial and radially decreasing then $B = \{0\}$, that is, there is single point blowup.

Fact 2 - When $p < p_s$ then the blow up is of type I, which means that

$$\sup_{t\in[0,T)} (T-t)^{p-1}||u(t)||_\infty < \infty \tag{3.1}$$

This was proven in [10] for positive solutions and extended to sign-changing solutions in [11], [12]. It is not clear, however, if p_s is a sharp critical value for type I blowup. Indeed, let $p^* = 1 + \dfrac{4}{N-4-2(N-1)^{1/2}}$ if $N \geq 11$ and $p^* = +\infty$ if $N < 11$. When $p > p^*$ the existence of solutions which are not of type I was shown in [15]. Most recently, (3.1) was proved for $p < p^*$ in the case of radial solutions and under some technical restrictions, see Theorem 2 of [19]).

Fact 3 - Let x_0 be a blowup point of u and consider the similarity transformation $w(y,s) = (T-t)^{\frac{1}{p-1}} u(x,t)$, where $y = \dfrac{(x-x_0)}{\sqrt{T-t}}$, $s = -\log(T-t)$. Note that $s \longrightarrow +\infty$ when $t \longrightarrow T$. Let $y = (y_1, y_2, \ldots, y_N) \in \mathbf{R}^N$ and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$, where α_i is a nonnegative integer for each $1 \leq i \leq N$. Define $\mathbf{H}_\alpha(y) = \Pi_{i=1}^N H_{\alpha_i}(y_i/2)$, where $H_n(y) = (-1)^n \dfrac{d^n}{d\,y^n} e^{-y^2}$ is the standard Hermite polynomial of order n. In [26] (see also [7] for an analogous result) the following classification of singularities of positive solutions was established. Either there exists $C_p > 0$ such that (after applying an orthogonal transformation in the space variables)

$$w(y,s) = (p-1)^{-\frac{1}{p-1}} - C_p(k - \frac{1}{2}\sum_{i=1}^k y_i^2)s^{-1} + o(s^{-1}), \tag{3.2}$$

or for some $m \geq 2$

$$w(y,s) = (p-1)^{-\frac{1}{p-1}} - e^{-(m-1)s} \sum_{|\alpha|=2m} c_\alpha \mathbf{H}_\alpha(y) + o(e^{-(m-1)s}), \tag{3.3}$$

where $C_\alpha = C_\alpha(N,p) \geq 0$ is such that the homogeneous multilinear form $B(x) = \sum_{|\alpha|=2m} C_\alpha x^\alpha$ is nonzero and nonnegative. Convergence in (1.15) and in (1.4) takes place in $C^k_{\rm loc}(\mathbf{R}^N)$ for any $k \geq 0$. These correspond to the nondegenerate and degenerate behaviours, respectively.

Fact 4 - Consider u_0 positive, radial and radially decreasing. Then (3.2) holds with $k = N$, see [5].

Fact 5 - There exists a blowup profile $u^* \in C^\infty(\mathbf{R}^N \setminus B)$ such that $u(t) \longrightarrow u^*$ uniformy on compact sets of $\mathbf{R}^N \setminus B$, see [20]. (Continuity of the blowup profile with respect to the initial data is discussed in Proposition 2.3 of [20].)

Fact 6 - In [6] it is shown that continuity of the blowup time for type I solutions holds in the following sense. Let $\tilde{u}_0 \in C(\mathbf{R}^N)$ and assume that its corresponding solution $\tilde{u}(t)$ blows up and is of type I. Then there exists a $L^\infty(\mathbf{R}^N)$-neighborhood $\mathcal{O}$ of $\tilde{u}_0$ such that $u(t)$ blows up and is of type I if $u_0 \in \mathcal{O}$. In addition, the application

$$u_0 \longrightarrow T(u_0) \text{ is continuous} \tag{3.5}$$

in $\mathcal{O}$ (the result is in general false, see [9]). Stability of the blowup set B and of the blowup profile u^* was proven under the supplementary assumptions of single point blowup, of uniform boundedness at infinity and of full nondegeneracy. More precisely, assume further that

- $\tilde{u}(t)$ blows up at a single point $\tilde{x}^*$,

- for some $M, R > 0$ we have $|\tilde{u}(x,t)| \leq M$ if $t < T$ and $||x|| \geq R$,

- $\tilde{u}$ satisfies (3.2) with $k = N$.

Then there exists a $L^\infty(\mathbf{R}^N)$-neighborhood $\mathcal{O}_1$ of $\tilde{u}_0$ such that, if $u_0 \in \mathcal{O}_1$, the corresponding solution $u(t)$ blows up at a single point x^* and its blowup profile u^* verifies

$$\lim_{x \longrightarrow x^*} \left(\frac{|x-x^*|^2}{\log|x-x^*|} \right)^{\frac{1}{p-1}} u^*(x) = \left(\frac{8p}{(p-1)^2} \right)^{\frac{1}{p-1}}. \tag{3.6}$$

We proceed now to the proof of our main results.

Proof of Theorem 1.3. Let u_λ satisfy (1.1) with $u_\lambda(0) = \lambda\,\varphi$. Define

$$v_\mu(x,t) = \mu^{\frac{2}{p-1}}\, u_\lambda(\mu x, \mu^2 t), \tag{3.7}$$

where $\mu = \lambda^{-(\frac{2}{p-1}-\sigma)^{-1}}$. Then v_μ is also a solution of (1.1) verifying $v_\mu(0) = \mu^\sigma \varphi(\mu x)$ and blowing up at

$$T(v_\mu) = \mu^{-2} T_\lambda = \lambda^{(\frac{1}{p-1}-\frac{\sigma}{2})^{-1}} T_\lambda .$$

Let $r, s > 1$ be such that $r\sigma < N < s\sigma$. Given $0 < \varepsilon < 1$ use (1.4) to choose $M > 0$, $\delta > 0$ such that $||x|^\sigma \varphi(x) - 1| \leq \varepsilon$ if $|x| > M$ and that

$$\int_{|x|<\delta} |x|^{-r\sigma} \, dx \leq \varepsilon^r / 2^{r+1}. \tag{3.8}$$

It follows that, for $\mu > M/\delta$,

$$\int_{M/\mu<|x|<\delta} v_\mu^r(0,x) \, dx \leq \int_{|x|<\delta} (2|x|^{-\sigma})^r \, dx \leq \varepsilon^r/2. \tag{3.9}$$

Next, take $\overline{\mu} \geq M/\delta$ such that

$$\int_{|x|<M/\mu} v_\mu^r(0,x) \, dx = \mu^{-(N-r\sigma)} \int_{|y|<M} \varphi^r(x) \, dx \leq \varepsilon^r/4 \tag{3.10}$$

if $\mu > \overline{\mu}$. Using (3.8), (3.9) and (3.10) we obtain that for $\mu > \overline{\mu}$

$$||(v_\mu(0,x) - |x|^{-\sigma}) \mathrm{I}_{|x|<\delta}||_r \leq \varepsilon. \tag{3.11}$$

Using again (1.4) and taking $\overline{\mu}$ eventually larger it follows from dominated convergence that

$$||(v_\mu(0,x) - |x|^{-\sigma}) \mathrm{I}_{|x|>\delta}||_s \leq \varepsilon. \tag{3.12}$$

if $\mu > \overline{\mu}$. Now (3.11) and (3.12) imply that $v_\mu(0) \longrightarrow |x|^{-\sigma}$ in $E_{r,s}$ as $\mu \longrightarrow +\infty$ (that is, as $\lambda \longrightarrow 0$). It follows from Theorem 2.8 that $v_\mu(\tau) \longrightarrow U(\tau)$ in $L^\infty(\mathbf{R}^N)$ for some $\tau > 0$, where $U(t)$ is the solution of (1.3).

We want to apply Fact 6 above and we have to verify that we can do so. First, defining $v_{\mu,\tau}(t) = v_\mu(t+\tau)$ and $U_\tau(t) = U(t+\tau)$ we place ourselves in the context of regular solutions. Next, note that since $p < p_s$ (3.1) is verified, see Fact 2. Also, U defined by (1.7) is radial and radially decreasing (this is shown in [21] for regular initial data and holds here due to the continuity results of Theorem 2.8). This has the following consequences: (i) U blows up at a single point, see Fact 1; (ii) using Fact 5, we see that blow up at $+\infty$ is precluded; (iii) by Fact 4 (3.2) holds with $k = N$; (iv) by (2.20) and the smoothing effect of the heat operator, there exists $t > 0$ such that $v_{\mu,\tau}(t) \longrightarrow U_\tau(t)$ uniformly in $\mathbf{R}^N$ as $\mu \longrightarrow \infty$.

These observations allow us to apply the results of Fact 6. It follows then that $v_{\mu,\tau}$ has a single blowup point z_μ for μ large (and so does v_μ). Hence, $x_\lambda = \mu z_\mu$ is the unique blowup point of u_λ for λ small, see (3.7). Call v_μ^* the blowup profile of $v_{\mu,\tau}$. Since $v_\mu^*(x) = \mu^{\frac{2}{p-1}} u_\lambda(\mu x)$, (1.5) follows directly from (3.6) applied to $v_{\mu,\tau}$. Finally, (3.5) yields

$$T(v_{\mu,\tau}) = T(v_\mu) - \tau = \lambda^{(\frac{1}{p-1}-\frac{\sigma}{2})^{-1}} T_\lambda - \tau \longrightarrow T(U_\tau) = T_U - \tau.$$

This shows (1.6). ∎

We present a preliminary lemma before proving Theorem 1.4.

Lemma 3.1. *Given $\varphi \in L^1(\mathbf{R}^N)$, define $\psi_\mu(x) = \mu^N \varphi(\mu x)$, $\mu \geq 1$. Then*

$$\sup_{t\geq 0}\sup_{\mu\geq 1} t^{\frac{N}{2}} ||S(t)\psi_\mu||_\infty \leq ||\varphi||_1. \tag{3.13}$$

Moreover, given $0 < \tau < T$ and $\varepsilon > 0$ there exists $M > 0$ such that

$$\sup_{t\in[0,T]}\sup_{\mu\geq 1}\int_{|x|>M} |S(t)\psi_\mu|(x)\,dx \leq \varepsilon, \tag{3.14}$$

$$\sup_{t\in[\tau,T]}\sup_{\mu\geq 1}\sup_{|x|>M} |S(t)\psi_\mu(x)|\,dx \leq \varepsilon. \tag{3.15}$$

Proof. If $\varphi > 0$ then $||S(t)\psi_\mu||_1 = ||\psi_\mu||_1 = ||\varphi||_1$. For general $\varphi \in L^1(\mathbf{R}^N)$,

$$||S(t)\psi_\mu||_1 \leq ||S(t)\psi_\mu^+||_1 + ||S(t)\psi_\mu^-||_1 = ||\varphi^+||_1 + ||\varphi^-||_1 = ||\varphi||_1.$$

Now (3.13) follows from usual parabolic regularity effect. To show (3.14) we may suppose that $\varphi > 0$. Then for any $M > 0$,

$$\begin{aligned}\int_{|x|>M} S(t)\psi_\mu(x)\,dx &= \mu^N(4\pi t)^{-N/2}\int_{|x|>M}\int_{y\in\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}}\varphi(\mu y)\,dy\,dx\\ &= (4\pi)^{-N/2}\int_{z\in\mathbf{R}^N}\int_{|w\sqrt{t}+z/\mu|>M} e^{-\frac{|w|^2}{4}}\varphi(z)\,dw\,dz.\end{aligned} \tag{3.16}$$

Given $\varepsilon > 0$ let $K > 0$ be such that $\int_{|z|>K}\varphi(z)\,dz \leq \varepsilon/2$. Thus

$$(4\pi)^{-N/2}\int_{|z|>K}\int_{|w\sqrt{t}+z/\mu|>M} e^{-\frac{|w|^2}{4}}\varphi(z)\,dw\,dz \leq (4\pi)^{-N/2}\int_{w\in\mathbf{R}^N} e^{-\frac{|w|^2}{4}}\,dw\int_{|z|>K}\varphi(z)\,dz \leq \varepsilon/2. \tag{3.17}$$

If $|z| < K$, $|w\sqrt{t} + z/\mu| > M$, $\mu \geq 1$ and $t \leq T$ then $|w| > (M-K)/\sqrt{T}$. Therefore, we can choose $M > K$ large enough so that

$$(4\pi)^{-N/2}\int_{|z|<K}\int_{|w\sqrt{t}+z/\mu|>M} e^{-\frac{|w|^2}{4}}\varphi(z)\,dw\,dz \leq (4\pi)^{-N/2}\int_{z\in\mathbf{R}^N}\varphi(z)\,dz\int_{|w|>\frac{M-K}{\sqrt{T}}} e^{-\frac{|w|^2}{4}}\,dw \leq \varepsilon/2. \tag{3.18}$$

. Then (3.14) follows from (3.16), (3.17) and (3.18). We next prove (3.15). Given $0 < \tau < T$, $\varepsilon > 0$, let $K > 0$ be such that

$$(4\pi\tau)^{-N/2}\Big(\int_{|y|>K}|\varphi(y)|\,dy + e^{-\frac{K^2}{4T}}||\varphi||_1\Big) \leq \varepsilon. \tag{3.19}$$

If $|y| < K$, $\mu > 1$ and $|x| > 2K$ then $|\mu x - y| \geq \mu K$. Using this and (3.19), we see that

$$|S(t)\psi_\mu(x)| = (4\pi t)^{-N/2}\Big|\int_{|y|>K} + \int_{|y|<K} e^{-\frac{|\mu x-y|^2}{4t\mu^2}}\varphi(y)\,dy\Big| \leq \varepsilon.$$

□

Proof of Theorem 1.4. Let $\int_{\mathbf{R}^N} \varphi = 1$ and let V be the solution defined by (1.7), blowing up at T_V. Call u_λ the solution of (1.1) verifying $u_\lambda(0) = \lambda\varphi$ and define $v_\mu = \mu^{\frac{2}{p-1}} u_\lambda(\mu x, \mu^2 t)$, for $\mu = \lambda^{-(\frac{2}{p-1}-N)^{-1}}$. Then v_μ is also a solution of (1.1), having $\psi_\mu = \mu^N \varphi(\mu x)$ as initial datum and blowing up at

$$T(v_\mu) = \mu^{-2} T_\lambda = \lambda^{(\frac{1}{p-1}-\frac{N}{2})^{-1}} T_\lambda .$$

Clearly, $\psi_\mu \longrightarrow \delta_0$ weak-* in $\mathcal{M}(\mathbf{R}^N)$ as $\mu \longrightarrow +\infty$. We claim that for $t < T_V$ sufficiently small it holds that $t < T(v_\mu)$ if μ is large enough, with $||v_\mu(t) - V(t)||_\infty \longrightarrow 0$ as $\mu \longrightarrow \infty$. Indeed, since $p < p_f$, by (3.13) we see that

$$\overline{v}_\mu(t) = \left(1 - (p-1)\int_0^t ||S(s)\psi_\mu||_\infty^{p-1}\, ds\right)^{\frac{1}{1-p}} S(t)\psi_\mu$$

is well defined in some interval $[0, \tilde{T}]$ which is independent of $\mu > 1$. Clearly, $||\overline{v}_\mu(t) - \psi_\mu||_\infty \longrightarrow 0$ as $t \longrightarrow 0$. A straightforward computation shows that $\overline{v}_\mu$ is a supersolution of (1.1). Since $-\overline{v}_\mu$ is a corresponding subsolution, $T(v_\mu) \geq \tilde{T}$ for all $\mu \geq 1$. Let B_R be the closed ball of radius R in $\mathbf{R}^N$ and $\tau < \tilde{T}$. Applying again the smoothing effect of the heat operator we obtain that

$$\sup_{t\in[0,\tilde{T}]} t^{-\frac{N}{2}(1-\frac{1}{p})} ||v_\mu(t)||_p \leq C. \tag{3.20}$$

and that $\{v_\mu\}_{\mu\geq 1}$ is bounded in $C^1([\tau, \tilde{T}] \times B_R)$. Using Arzelà-Ascoli, a standard diagonal procedure and (3.14), we obtain a subsequence μ_k such that $v_{\mu_k} \longrightarrow v \in C_{\rm loc}((0, \tilde{T}], L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N))$ for some v satisfying (3.20). Note that

$$v_{\mu_k}(t+\tau) = S(t)v_{\mu_k}(\tau) + \int_\tau^{t+\tau} S(t+\tau-s)|v_{\mu_k}(s)|^{p-1} v_{\mu_k}(s)\, ds \tag{3.21}$$

for all $0 \leq \tau$, $t > 0$ with $t + \tau \leq \tilde{T}$. We then let $k \longrightarrow \infty$ to obtain that

$$v(t+\tau) = S(t)v(\tau) + \int_\tau^{t+\tau} S(t+\tau-s)|v(s)|^{p-1} v(s)\, ds \tag{3.22}$$

for all $t, \tau > 0$ such that $t + \tau \leq \tilde{T}$. We next show that we can take $\tau = 0$ in (3.22). Indeed, since $p < p_f$, we observe that given $\varepsilon > 0$ it follows from (3.20) that there exists τ_0 independent of k such that $|| \int_0^\tau S(\tau - s)|v_{\mu_k}(s)|^{p-1} v_{\mu_k}(s)\, ds||_1 \leq \varepsilon$ if $\tau < \tau_0$. Taking $\tau = 0$ in (3.21) (and renaming $t = \tau$) yields $||v_{\mu_k}(\tau) - S(\tau)v_{\mu_k}(0)||_1 \leq \varepsilon$ if $\tau \leq \tau_0$. Letting $k \longrightarrow \infty$ we get that $||v(\tau) - S(\tau)\delta_0||_1 \leq \varepsilon$. Therefore, $||S(t)v(\tau) - S(t)\delta_0||_1 \longrightarrow 0$ as $\tau \longrightarrow 0$. Using this and (3.20) (for v) in (3.22) we get

$$v(t) = S(t)\delta_0 + \int_0^t S(t-s)|v(s)|^{p-1} v(s)\, ds.$$

This shows that v solves (1.1) and that $v(t) \rightharpoonup \delta_0$ weak-* in $\mathcal{M}(\mathbf{R}^N)$ as $t \longrightarrow 0$. By uniqueness, $v = V$ and so, in fact, $v_\mu \longrightarrow V$ in $C_{\rm loc}((0, \tilde{T}], L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N))$ as $\mu \longrightarrow \infty$.

We now argue as in the proof of Theorem 1.1 to obtain that for λ small enough u_λ blows up at a finite time T_λ, at a single point x_λ and to show that (1.8), (1.9) hold. $\square$

In the proof of Theorem 1.5 we use the following result.

Lemma 3.2. *Let $\varphi \in C_0(\mathbf{R}^N)$ be such that $|x|^N|\varphi(x)| \le 1$ if $|x| \ge 2$. Define $\psi_\mu(x) = \mu^N(\log\mu)^{-1}\varphi(\mu x)$, $\mu \ge e$. Then given $T > 0$ there exists $C > 0$ such that*

(i) $\sup_{t\ge0}\sup_{|x|\ge2}\sup_{\mu\ge e}|x|^N|S(t)\psi_\mu(x)| \le C.$

(ii) $\sup_{t\in[0,T]}\sup_{\mu\ge e} t^{N/2}||S(t)\psi_\mu||_\infty \le C.$

Proof. We first prove (i). It suffices to show the result for $\varphi_1(x) = |x|^{-N}\mathrm{I}_{\{|x|>1\}}$ and for $\varphi_2(x) = \mathrm{I}_{\{|x|<1\}}$. Consider $\psi_\mu(x) = \mu^N\varphi_1(\mu x)$. Then

$$|x|^N S(t)\psi_\mu(x) = (4\pi t)^{-\frac{N}{2}}(\log\mu)^{-1}\int_{\mu|y|>1} e^{-\frac{|x-y|^2}{4t}}|x|^N|y|^{-N}\,dy. \tag{3.23}$$

We split the above domain of integration in two parts, $R_1 = \{\mu^{-1} < |y| < \frac{|x|}{2}\}$ and $R_2 = \{|y| > \frac{|x|}{2}\}$. Note that if $|y| \le |x|/2$ then $t^{-\frac{N}{2}}e^{-\frac{|x-y|^2}{4t}}|x|^N \le \sup_{z\in\mathbf{R}^N} e^{-\frac{|z|^2}{16}}|z|^N \le Ce^{-\frac{|z|^2}{32}}$. Using this, we have that

$$(4\pi t)^{-\frac{N}{2}}(\log\mu)^{-1}\int_{R_1} e^{-\frac{|x-y|^2}{4t}}|x|^N|y|^{-N}\,dy \le Ce^{-\frac{|x|^2}{32}}(\log\mu)^{-1}\int_{\mu^{-1}<|y|<|x|/2}|y|^{-N}\,dy \le$$

$$Ce^{-\frac{|x|^2}{32}}(\log\mu)^{-1}(\log|x|/2 + \log\mu) \le C, \tag{3.24}$$

if $\mu \ge e$. Also,

$$(4\pi t)^{-\frac{N}{2}}(\log\mu)^{-1}\int_{R_2} e^{-\frac{|x-y|^2}{4t}}|x|^N|y|^{-N}\,dy \le 2^N(4\pi t)^{-\frac{N}{2}}\int_{\mathbf{R}^N} e^{-\frac{|x-y|^2}{4t}}\,dy = 2^N. \tag{3.25}$$

Then (3.23), (3.24) and (3.25) yield (i) for $\psi_\mu(x) = \mu^N\varphi_1(\mu x)$.

Suppose now $\psi_\mu(x) = \mu^N(\log\mu)^{-1}\varphi_2(\mu x)$ and let $\overline{x} = x/\sqrt{t}$, $\rho = \mu\sqrt{t}$. Then

$$|x|^N S(t)\psi_\mu(x) = (4\pi t)^{-\frac{N}{2}}(\mu|x|)^N(\log\mu)^{-1}\int_{\mu|y|<1} e^{-\frac{|x-y|^2}{4t}}\,dy = (4\pi)^{-\frac{N}{2}}(\rho|\overline{x}|)^N(\log\mu)^{-1}\int_{\rho|z|<1} e^{-\frac{|\overline{x}-z|^2}{4}}\,dz.$$

If $\rho|\overline{x}| \le 2$ then

$$|x|^N S(t)\psi_\mu(x) = (4\pi)^{-\frac{N}{2}}(\rho|\overline{x}|)^N(\log\mu)^{-1}\int_{\rho|z|<1} e^{-\frac{|\overline{x}-z|^2}{4}}\,dz \le 2^N(\log\mu)^{-1}. \tag{3.26}$$

If $\rho|\overline{x}| > 2$ then $\rho|z| < 1$ implies that $|\overline{x} - z| > \dfrac{|\overline{x}|}{2}$. Therefore, for $\mu > e$,

$$|x|^N S(t)\psi_\mu(x) \le (4\pi)^{-\frac{N}{2}}e^{-\frac{|\overline{x}|^2}{16}}|\overline{x}|^N(\log\mu)^{-1} \le C. \tag{3.27}$$

Then (i) for $\psi_\mu(x)=\mu^N\varphi_2(\mu x)$ follows from (3.26) and (3.27).

To prove (ii), we may suppose that φ is positive, radial and radially decreasing. In this case, the maximum of $S(t)\psi_\mu$ is reached at $x=0$. For $t\le T$ and $\mu\ge e$,

$$\mu^N\int_{\mathbf{R}^N} e^{-\frac{|y|^2}{4t}}\varphi(\mu y)\,dy\le \mu^N\int_{\mu|y|<1}\varphi(\mu y)\,dy+\int_{1<\mu|y|} e^{-\frac{|y|^2}{4T}}|y|^{-N}\,dy\le C\log\mu.$$

Hence,

$$t^{N/2}S(t)\psi_\mu(0)=(4\pi)^{-N/2}\mu^N(\log\mu)^{-1}\int_{\mathbf{R}^N} e^{-\frac{|y|^2}{4t}}\varphi(\mu y)\,dy\le C.$$

□

Proof of Theorem 1.5. Let u_λ be the solution of (1.1) such that $u_\lambda(0)=\lambda\,\varphi$. For $\mu>\overline{\mu}=e^{(\frac{2}{p-1}-N)^{-1}}$ set $g(\mu)=S_N\,\mu^{\frac{2}{(p-1)}-N}\log\mu$ and $h=g^{-1}$, S_N being the measure of the unit sphere of $\mathbf{R}^N$. Set $v_\mu=\mu^{\frac{2}{p-1}}u_\lambda(\mu x,\mu^2 t)$, where $\mu=h(\lambda^{-1})$. Then v_μ is also a solution of (1.1) having $\psi_\mu=S_N^{-1}\mu^N(\log\mu)^{-1}\varphi(\mu x)$ as initial value and blowing up at

$$T(v_\mu)=\mu^{-2}T_\lambda=(h(\lambda^{-1}))^{-2}T_\lambda\,. \tag{3.28}$$

We recall the assumption

$$\lim_{|x|\longrightarrow\infty}|x|^N\varphi(x)=1. \tag{3.29}$$

Write $\psi_\mu=\psi_\mu^1+\psi_\mu^2$ where $\psi_\mu^1=\psi_\mu \mathrm{I}_{\{|x|<1\}}\in L^1(\mathbf{R}^N)$ and $\psi_\mu^2=\psi_\mu \mathrm{I}_{\{|x|>1\}}\in L^p(\mathbf{R}^N)$. It follows from (3.29) that

$$\lim_{\mu\longrightarrow\infty}||\psi_\mu^2||_p=0. \tag{3.30}$$

To prove that

$$\psi_\mu^1\rightharpoonup\delta_0 \text{ weak-* in } \mathcal{M}(\mathbf{R}^N), \tag{3.31}$$

take $\xi\in C_b(\mathbf{R}^N)$ and $0<\varepsilon<1$. Let $0<\delta<1$ be such that

$$|\xi(x)-\xi(0)|\le\varepsilon \tag{3.32}$$

if $|x|\le\delta$. Using (3.29), choose $R>1$ such that

$$||x|^N\varphi(x)-1|\le\varepsilon \tag{3.33}$$

if $|x|>R$. For $\mu>R/\delta$ write

$$\begin{aligned}\int_{\mathbf{R}^N}\psi_\mu^1(x)\xi(x)\,dx-\xi(0)=&\int_{|x|<R/\mu}\psi_\mu(x)\xi(x)\,dx+\int_{R/\mu<|x|<\delta}\psi_\mu(x)(\xi(x)-\xi(0))\,dx\\&+\xi(0)\int_{R/\mu<|x|<\delta}\psi_\mu(x)-(S_N\log\mu)^{-1}|x|^{-N}\,dx\\&-\xi(0)(1-(S_N\log\mu)^{-1}\int_{R/\mu<|x|<\delta}|x|^{-N}\,dx)+\int_{\delta<|x|<1}\psi_\mu(x)\xi(x)\,dx.\end{aligned} \tag{3.34}$$

Let $C(R) = \int_{|y|<R} |\varphi(y)|\,dy$. Then

$$|\int_{|x|<R/\mu} \psi_\mu(x)\xi(x)\,dx| = \mu^N(\log\mu)^{-1}|\int_{|x|<R/\mu} \varphi(\mu x)\xi(x)\,dx| \le C(R)||\xi||_\infty(\log\mu)^{-1}. \tag{3.35}$$

By (3.32) and (3.33) we have that

$$|\int_{R/\mu<|x|<\delta} \psi_\mu(x)(\xi(x)-\xi(0))\,dx| \le 2\varepsilon(S_N\log\mu)^{-1}\int_{R<|y|<\delta\mu} |y|^{-N}\,dy < 2\varepsilon, \tag{3.36}$$

$$|\int_{R/\mu<|x|<\delta} \psi_\mu(x) - (S_N\log\mu)^{-1}|x|^{-N}\,dx| \le \varepsilon(S_N\log\mu)^{-1}\int_{R/\mu<|x|<\delta} |x|^{-N}\,dx \le \varepsilon, \tag{3.37}$$

$$1-(S_N\log\mu)^{-1}\int_{R/\mu<|x|<\delta} |x|^{-N}\,dx = \log(R/\delta)(\log\mu)^{-1} \tag{3.38}$$

and that

$$|\int_{\delta<|x|<1} \psi_\mu(x)\xi(x)\,dx| \le 2||\xi||_\infty(S_N\log\mu)^{-1}\int_{\delta<|x|<1} |x|^{-N}\,dx = 2||\xi||_\infty \log 1/\delta(\log\mu)^{-1}. \tag{3.39}$$

Using (3.35)-(3.39) in (3.34) we can choose $\overline{\mu}$ such that $|\int_{\mathbf{R}^N} \psi^1_\mu(x)\xi(x)\,dx - \xi(0)| \le 4\varepsilon$ if $\mu > \overline{\mu}$. This shows (3.31) . By (3.30) and (3.31), $\{v_\mu\}$ is bounded in $E_{1,p} = L^1(\mathbf{R}^N) + L^p(\mathbf{R}^N)$.

We now argue as in the proof of Theorem 1.4. Since $p < p_f$, it follows from Lemma 3.2 (ii) that

$$\overline{v}_\mu(t) = \left(1-(p-1)\int_0^t ||S(s)\psi_\mu||_\infty^{p-1}\,ds\right)^{\frac{1}{1-p}} S(t)\psi_\mu.$$

is a supersolution of (1.1) well defined in some interval $[0,\tilde{T}]$ which is independent of μ and that $||\overline{v}_\mu(t) - \psi_\mu||_\infty \longrightarrow 0$ as $t \longrightarrow 0$. Since $-\overline{v}_\mu$ is a corresponding subsolution, v_μ exists in $[0,\tilde{T}]$ for all $\mu \ge e$.

By Lemma 3.2 (i) $v_\mu(t)$ decays in x uniformly with respect to $t \in [0,\tilde{T}]$ and $\mu \ge e$. This allows us to proceed as in the proof of Theorem 1.4 to show that $||v_\mu(t) - U(t)||_\infty \longrightarrow 0$ as $\mu \longrightarrow \infty$ for all $t \le \tilde{T}$, where $U(t)$ solves (1.3). It follows then from the arguments of Theorem 1.3 that for λ small enough u_λ blows up at a finite time T_λ, at a single point x_λ and that (1.10), (1.11) hold. □

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