Non-Hermitian operators in QM

&

$\mathcal{PT}$- symmetry

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Based on :

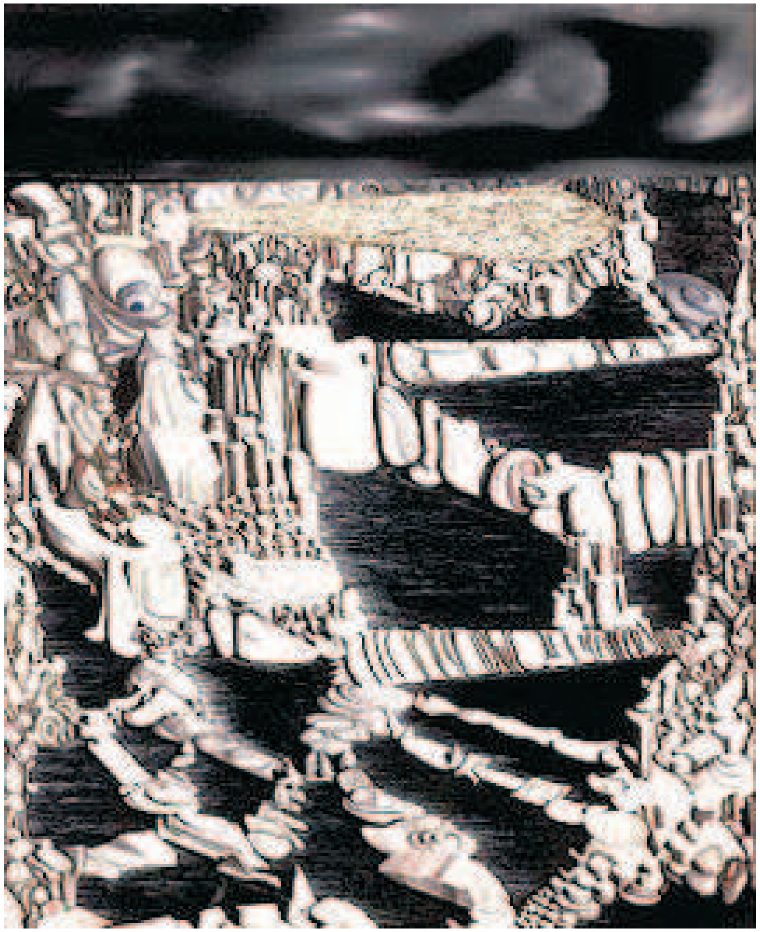
- D.K., H. Bíla, M. Znojil, *J. Phys. A 39 (2006), 10143*; [math-ph/0604055]
- D.K., *submitted*; arXiv:0707.1781 [math-ph]
- D. Borisov, D.K., *submitted*; arXiv:0707.3039 [math-ph]

¿ QM with non-Hermitian operators ?

$$\mathbb{C}$$

$$\mathbb{R}$$

$$H^* = H$$



Imaginary Numbers by Yves Tanguy, 1954

(Museo Thyssen-Bornemisza, Madrid)

$$\mathbb{I}$$

$$H^{\mathcal{PT}} = H$$

Insignificant non-Hermiticity

Example 1. evolution operator $U(t) = \exp(-itH)$:

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Theorem (spectral theorem).
Let $H = H^*$ have discrete spectrum, $H\psi_j = E_j\psi_j$.
Then

$$f(H) = \sum_j f(E_j)\,\psi_j \langle \psi_j, \cdot \rangle$$

for any complex-valued continuous function f.

$\Longrightarrow$ important consequences: minimax principle

Technical non-Hermiticity

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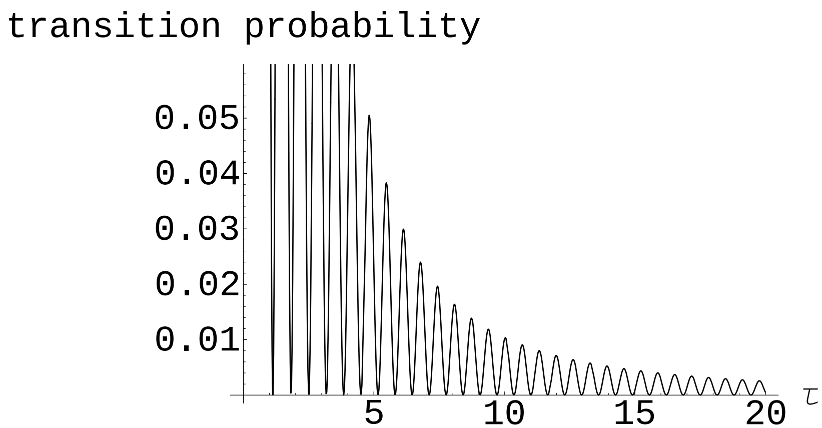
Example 1. adiabatic transition probability for $H(t) := \vec{\gamma}(t/\tau) \cdot \vec{\sigma}$, $\tau \to \infty$



[Berry 1990], [Joye, Kunz, Pfister 1991], [Jakšić, Segert 1993], . . .

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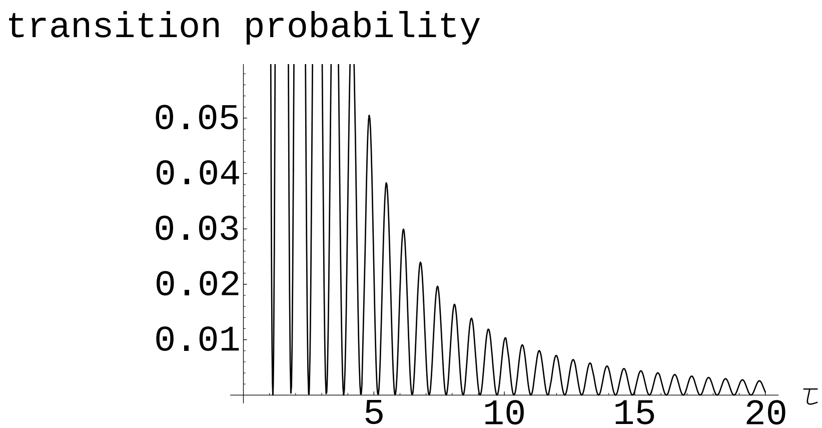
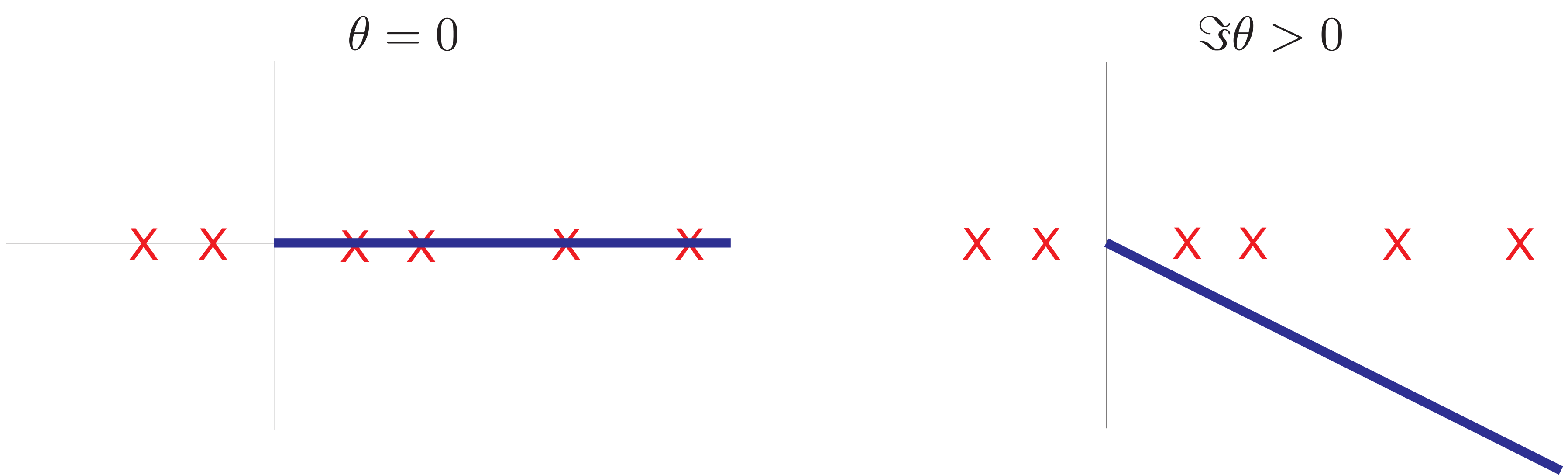


[Berry 1990], [Joye, Kunz, Pfister 1991], [Jakšić, Segert 1993], . . .

Example 2. complex scaling $H_\theta := S_\theta(-\Delta + V)S_\theta^{-1}$, $\quad (S_\theta\psi)(x) := e^{\theta/2}\,\psi(e^\theta x)$

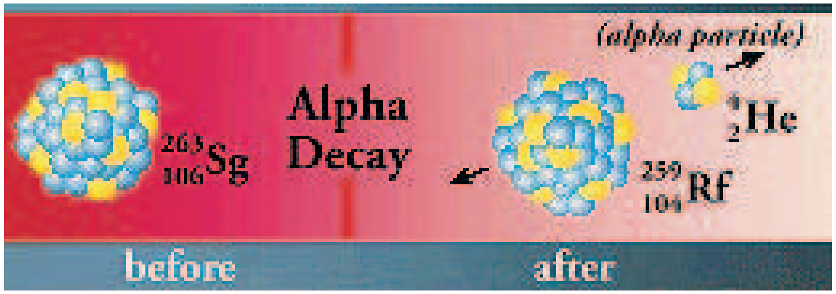


[Aguilar/Balslev, Combes 1971], [Simon 1972], [Van Winter 1974], . . .

Approximate non-Hermiticity

open systems

Example 1. radioactive decay



Example 2. dissipative Schrödinger operators in semiconductor physics

Baro, Behrndt, Kaiser, Neidhardt, Rehberg 2002–. . .

¿ Fundamental non-Hermiticity ?

without violating the “physical axioms” of QM

Non-Hermitian Hamiltonians with real spectra

$$-\Delta + V \quad \text{in} \quad L^2(\mathbb{R})$$

$$V(x) = x^2 + ix^3$$

[Caliceti, Graffi, Maioli 1980]

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$$V(x) = \begin{cases} i\,\mathrm{sgn}(x) & \text{if} \quad x \in (-1,1) \\ \infty & \text{elsewhere} \end{cases}$$

[Znojil 2001]

$$V(x) = \begin{cases} -i\,\delta(x+\frac{1}{2}) + i\,\delta(x-\frac{1}{2}) & \text{if} \quad x \in (-1,1) \\ \infty & \text{elsewhere} \end{cases}$$

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¿ What is behind the reality of the spectrum ?

What is $\mathcal{PT}$-symmetry ?

$$[H, \mathcal{PT}] = 0$$

$$\begin{aligned} (\mathcal{P}\psi)(x) &= \psi(-x) \\ (\mathcal{T}\psi)(x) &= \overline{\psi(x)} \end{aligned}$$

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unbroken $\mathcal{PT}$-symmetry $:\Leftrightarrow$ H and $\mathcal{PT}$ have the same eigenstates $\Leftrightarrow$ $\sigma(H) \subset \mathbb{R}$

Here we assume that $H = -\Delta + V$ has purely discrete spectrum.

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$H^* \phi_n = E_n \phi_n$

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$\Longrightarrow$ H is Hermitian in $\big(X, \langle\cdot, \Theta\,\cdot\rangle\big)$, *i.e.* $\Theta^{1/2} H \Theta^{-1/2}$ is Hermitian in $\big(X, \langle\cdot,\cdot\rangle\big)$

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Albeverio-Fei-Kurasov, Bender-Boettcher, Caliceti-Graffi-Sjöstrand, Jones-Mateo, Langer-Tretter, Mostafazadeh, Scholtz-Geyer-Hahne, Znojil, . . .

Attempts to calculate the metric

1. Perturbative

For instance, [Bender 2004] for $\frac{1}{2}p^2 + \frac{1}{2}x^2 + \varepsilon i x^3$: $\Theta^{-1} = \exp(\varepsilon Q_1 + \varepsilon^3 Q_3 + \dots)$

$$\begin{aligned}
Q_1 &= -\tfrac{4}{3}\mu^{-4}p^3 - 2\mu^{-2}S_{1,2},\\
Q_3 &= \tfrac{128}{15}\mu^{-10}p^5 + \tfrac{40}{3}\mu^{-8}S_{3,2} + 8\mu^{-6}S_{1,4} - 12\mu^{-8}p\,,\\
Q_5 &= -\tfrac{320}{3}\mu^{-16}p^7 - \tfrac{544}{3}\mu^{-14}S_{5,2} - \tfrac{512}{3}\mu^{-12}S_{3,4}\\
&\quad - 64\mu^{-10}S_{1,6} + \tfrac{24\,736}{45}\mu^{-14}p^3 + \tfrac{6\,368}{15}\mu^{-12}S_{1,2}\,,\\
Q_7 &= \tfrac{553\,984}{315}\mu^{-22}p^9 + \tfrac{97\,792}{35}\mu^{-20}S_{7,2} + \tfrac{377\,344}{105}\mu^{-18}S_{5,4}\\
&\quad + \tfrac{721\,024}{315}\mu^{-16}S_{3,6} + \tfrac{1\,792}{3}\mu^{-14}S_{1,8} - \tfrac{2\,209\,024}{105}\mu^{-20}p^5\\
&\quad - \tfrac{2\,875\,648}{105}\mu^{-18}S_{3,2} - \tfrac{390\,336}{35}\mu^{-16}S_{1,4} + \tfrac{46\,976}{5}\mu^{-18}p\,.
\end{aligned}$$

$S_{0,0} = 1$, $S_{0,3} = x^3$, $S_{1,1} = \frac{1}{2}(xp + px)$, $S_{1,2} = \frac{1}{3}\left(x^2p + xpx + px^2\right)$, and so on.

Other models: [Mostafazadeh, Batal 2004], [Scholtz, Geyer 2006], ...

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2. Formal
   - Is Θ well defined ?
   - Is Θ bounded ?
   - Does $\Theta D(H) \subseteq D(H^*)$ hold ? (*NB* $H^*\Theta = \Theta H$)

What is $\mathcal{PT}$-symmetry ?

a mathematical approach

Special case of J-self-adjointness [Edmunds, Evans 1987]

$$H^* = JHJ$$

where J is a conjugation operator: $\begin{cases} (J\phi, J\psi) = (\psi, \phi) \\ J^2\psi = \psi \end{cases}$ $\forall \phi, \psi \in \mathcal{H}$

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Fact. Let H be densely defined, closed, J-self-adjoint operator in a Hilbert space. Then the residual spectrum of H is empty.

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Fact. Let H be densely defined, closed, J-self-adjoint operator in a Hilbert space. Then the residual spectrum of H is empty.

N.B. $\begin{cases} \sigma_{\mathsf{p}}(H) = \{\lambda \mid H - \lambda \text{ is not injective}\} \\ \sigma_{\mathsf{c}}(H) = \{\lambda \mid H - \lambda \text{ is not surjective} \quad \& \quad \mathfrak{R}(H-\lambda) \text{ is dense}\} \\ \sigma_{\mathsf{r}}(H) = \{\lambda \mid H - \lambda \text{ is injective} \quad \& \quad \mathfrak{R}(H-\lambda) \text{ is not dense}\} \end{cases}$

Proof. $\lambda \in \sigma_{\mathsf{r}}(H) \Leftrightarrow \overline{\lambda} \in \sigma_{\mathsf{p}}(H^*) \Leftrightarrow \lambda \in \sigma_{\mathsf{p}}(H)$ *q.e.d.*

The simplest $\mathcal{PT}$-symmetric model

[D.K., Bíla, Znojil 2006]

$$\mathcal{H} := L^2(0,\pi), \quad H_\alpha \psi := -\psi'', \quad D(H_\alpha) := \left\{ \psi \in W^{2,2}(0,\pi) \,\middle|\, \begin{array}{l} \psi'(0) + i\alpha\psi(0) = 0 \\ \psi'(\pi) + i\alpha\psi(\pi) = 0 \end{array} \right\}$$

$\alpha \in \mathbb{R}$

$$\frac{d\psi}{dn} - i\alpha\psi = 0 \qquad\qquad \frac{d\psi}{dn} + i\alpha\psi = 0$$

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$-\Delta$

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Theorem 1. H_α is an m-sectorial operator with compact resolvent satisfying

$$\boxed{H_\alpha^* = H_{-\alpha} = \mathcal{T} H_\alpha \mathcal{T}} \qquad (\mathcal{T}\text{-self-adjointness})$$

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Theorem 2. $\boxed{\sigma(H_\alpha) = \{\alpha^2\} \cup \{n^2\}_{n=1}^{\infty}}$

Corollary. The spectrum of H_α is $\begin{cases} \text{always real,} \\ \text{simple if } \quad \alpha \notin \mathbb{Z} \setminus \{0\}. \end{cases}$

Biorthonormal eigenbasis

$\alpha \notin \mathbb{Z} \setminus \{0\}$

$H_\alpha \psi_n^\alpha = E_n^\alpha \psi_n^\alpha$

$H_\alpha^* \phi_n^\alpha = E_n^\alpha \phi_n^\alpha$

$$\psi_n^\alpha(x) = \begin{cases} A_0^\alpha \, \exp(-i\alpha x) \\ A_n^\alpha \left(\cos(nx) - i\frac{\alpha}{n} \, \sin(nx)\right) \end{cases}$$

$$\phi_n^\alpha(x) = \begin{cases} B_0^\alpha \, \exp(i\alpha x) \\ B_n^\alpha \left(\cos(nx) + i\frac{\alpha}{n} \, \sin(nx)\right) \end{cases}$$

Special normalisation:

$$A_n^\alpha = \begin{cases} \sqrt{\frac{1}{\pi}} \frac{i2\pi\alpha}{1 - \exp(-i2\pi\alpha)} \\ \sqrt{\frac{2}{\pi}} \frac{n^2}{n^2 - \alpha^2} \end{cases}$$

$$B_n^\alpha = \begin{cases} \sqrt{\frac{1}{\pi}} \\ \sqrt{\frac{2}{\pi}} \end{cases}$$

Theorem 3. $\langle \phi_n^\alpha, \psi_m^\alpha \rangle = \delta_{nm}$ and $\psi = \sum_{n=0}^{\infty} \psi_n^\alpha \langle \phi_n^\alpha, \psi \rangle = \sum_{n=0}^{\infty} \phi_n^\alpha \langle \psi_n^\alpha, \psi \rangle$

Corollary. $f(H) = \sum_j f(E_j) \, \psi_j \langle \phi_j, \cdot \rangle$ for $f(E) = E^N$, *etc.*

Calculation of the metric operator

[D.K. 2007]

$$\Theta \equiv \sum_{n=0}^{\infty} \phi_n^\alpha \langle \phi_n^\alpha, \cdot \rangle = \phi_0^\alpha \langle \phi_0^\alpha, \cdot \rangle + \sum_{n=1}^{\infty} \left(\chi_n^N + i\frac{\alpha}{n}\,\chi_n^D \right) \left\langle \left(\chi_n^N + i\frac{\alpha}{n}\,\chi_n^D \right), \cdot \right\rangle =: \Theta_1 + \Theta_2$$

$$\Theta_2 = \sum_{n=1}^{\infty} \left\{ \chi_n^N \langle \chi_n^N, \cdot \rangle + \frac{\alpha^2}{n^2}\,\chi_n^D \langle \chi_n^D, \cdot \rangle + \frac{\alpha}{n^2}\,p\,\chi_n^D \langle \chi_n^D, \cdot \rangle + \frac{\alpha}{n^2}\,p^* \chi_n^N \langle \chi_n^N, \cdot \rangle \right\}$$

$$p\psi := -i\psi', \qquad D(p) := W_0^{1,2}(0,\pi)$$

$$= I - \chi_0^N \langle \chi_0^N, \cdot \rangle + \alpha^2 (-\Delta_D)^{-1} + \alpha p\,(-\Delta_D)^{-1} + \alpha p^* (-\Delta_N^\perp)^{-1}$$

$$-\Delta_N^\perp := (I - P_0^N)(-\Delta_N)(I - P_0^N)$$

$$P_0^N := \chi_0^N \langle \chi_0^N, \cdot \rangle$$

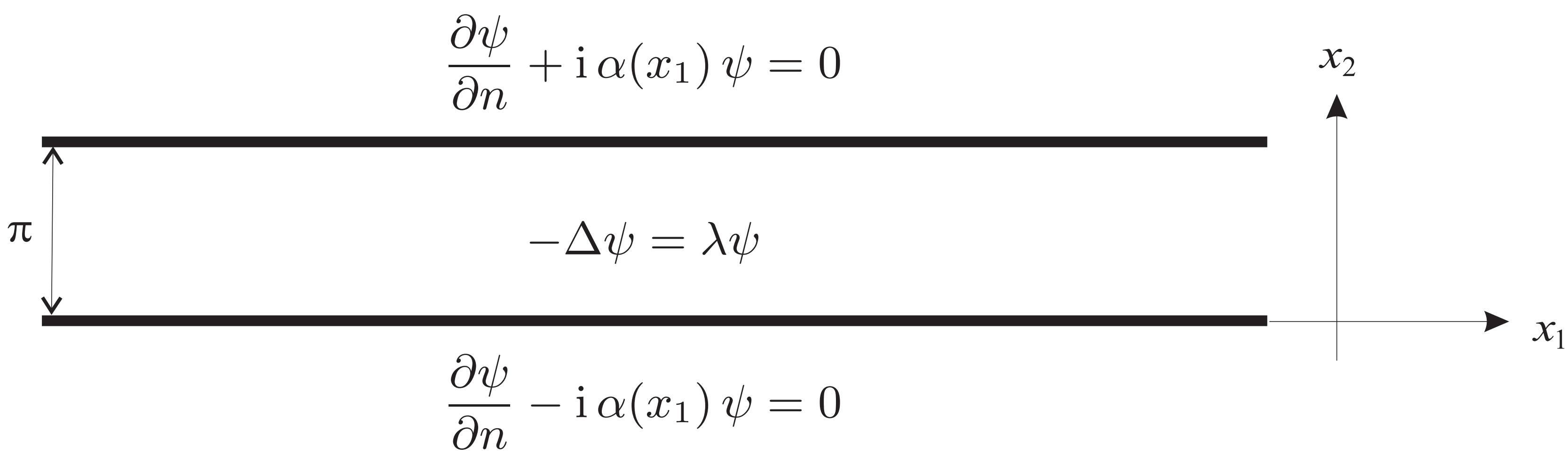
Theorem 4. Θ is bounded, symmetric, non-negative and satisfies

$$\forall \psi \in D(H_\alpha), \qquad H_\alpha^* \Theta\,\psi = \Theta\,H_\alpha\,\psi$$

Moreover, Θ is positive if $\alpha \notin \mathbb{Z} \setminus \{0\}$.

$\mathcal{PT}$-symmetric waveguide

[Borisov, D.K. 2007]



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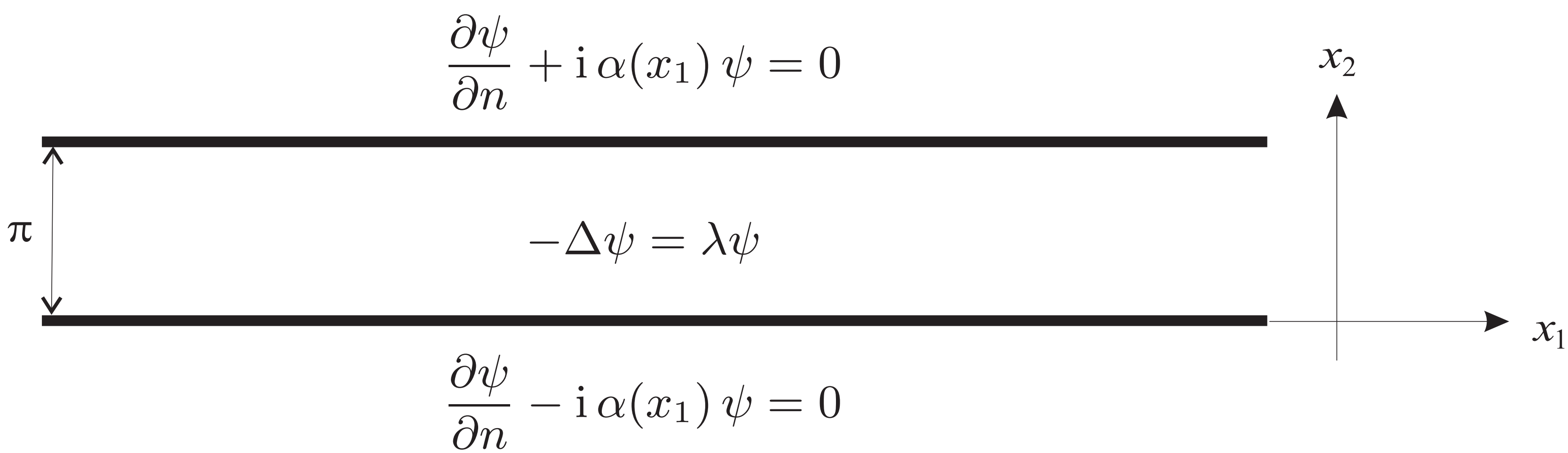

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Theorem. Let $\alpha \in W^{1,\infty}(\mathbb{R})$. Then H_α is an m-sectorial operator satisfying

$$H_\alpha^* = H_{-\alpha} = \mathcal{T} H_\alpha \mathcal{T} \qquad (\mathcal{T}\text{-self-adjointness})$$

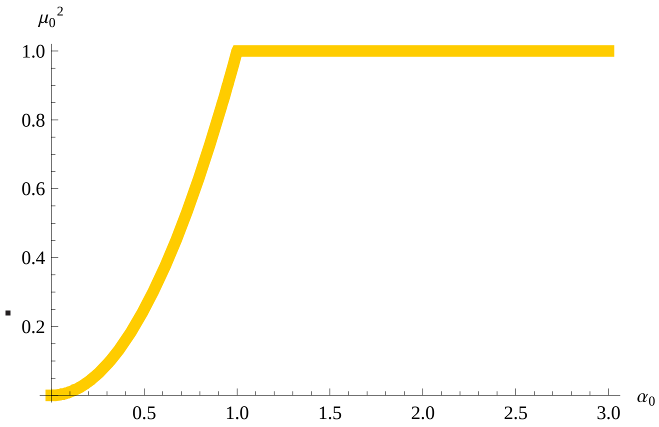
Corollary. $\sigma_{\mathsf{r}}(H_\alpha) = \varnothing$

Spectral analysis

Stability of the continuous spectrum

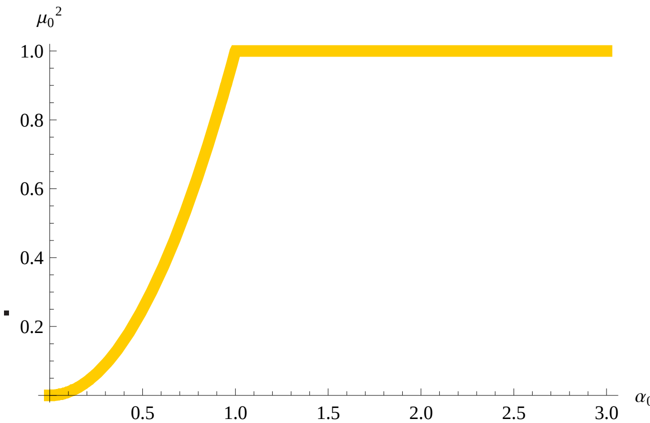
Theorem. Let $\alpha - \alpha_0 \in C_0(\mathbb{R}) \cap W^{1,\infty}(\mathbb{R})$.

Then $\boxed{\sigma_c(H_\alpha) = [\mu_0^2, \infty)}$ where $\mu_0 := \min\{|\alpha_0|, 1\}$.



Spectral analysis

Stability of the continuous spectrum



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Weakly-coupled bound states

$\alpha(x_1) = \alpha_0 + \varepsilon\, \beta(x_1)$

$\varepsilon \longrightarrow 0+$ $\quad \beta \in C_0^2(\mathbb{R})$

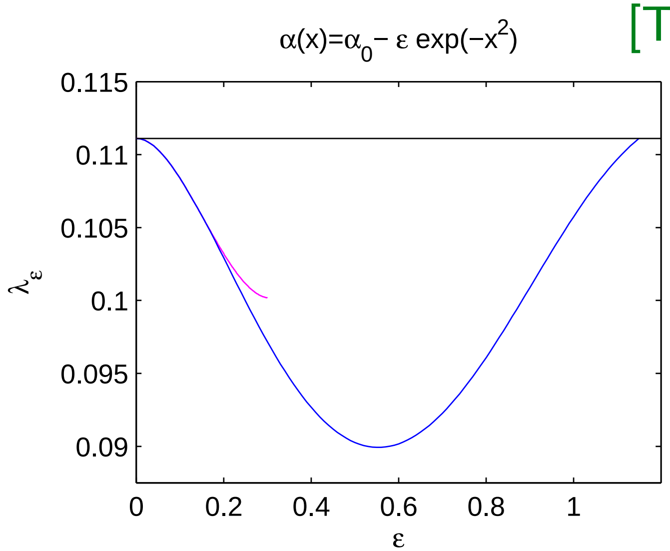
Theorem. Let $|\alpha_0| < 1$.

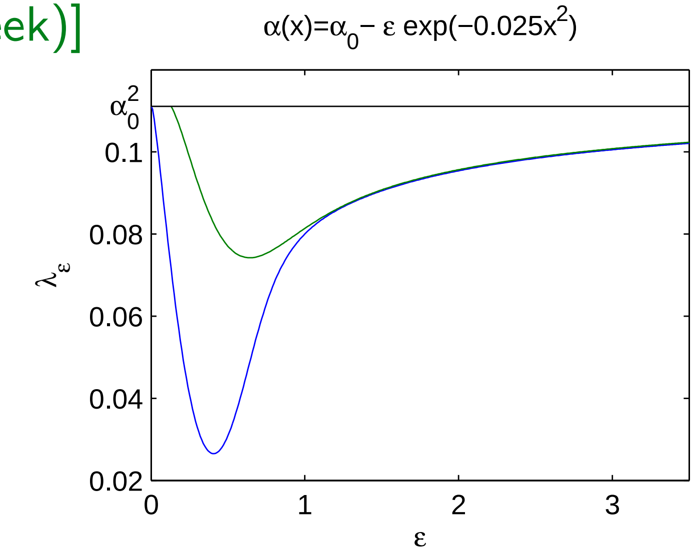
1. $\alpha_0 \langle\beta\rangle < 0 \quad \Longrightarrow \quad \exists!\ \lambda_\varepsilon = \mu_0^2 - \varepsilon^2 \alpha_0^2 \langle\beta\rangle^2 + 2\varepsilon^3 \alpha_0 \langle\beta\rangle \tau + \mathcal{O}(\varepsilon^4) \in \mathbb{R}$
2. $\alpha_0 \langle\beta\rangle > 0 \quad \Longrightarrow$ no
3. $\alpha_0 = 0 \quad \Longrightarrow$ no

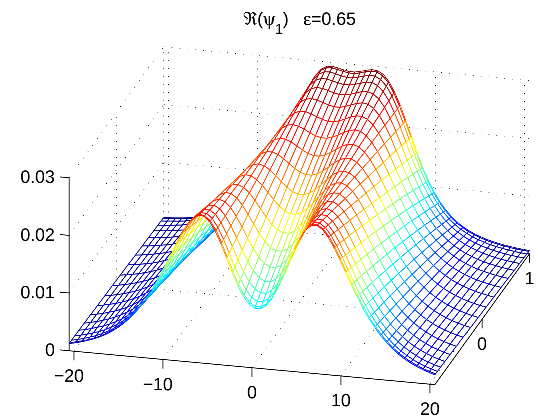
$$\langle\beta\rangle := \int_{\mathbb{R}} \beta(x_1)\, \mathrm{d}x_1$$

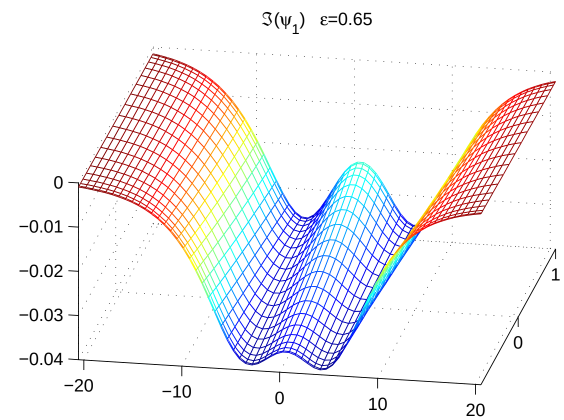
Numerical results

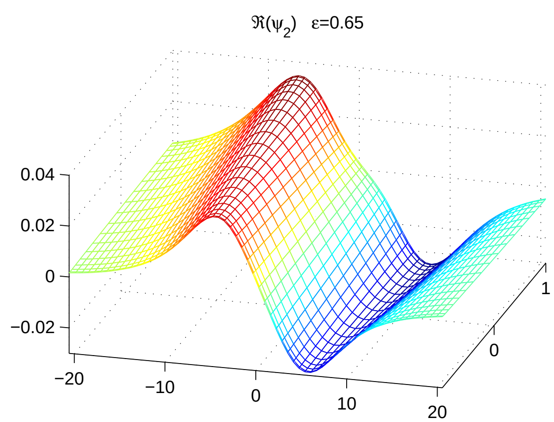
[Tater 2007 (last week)]

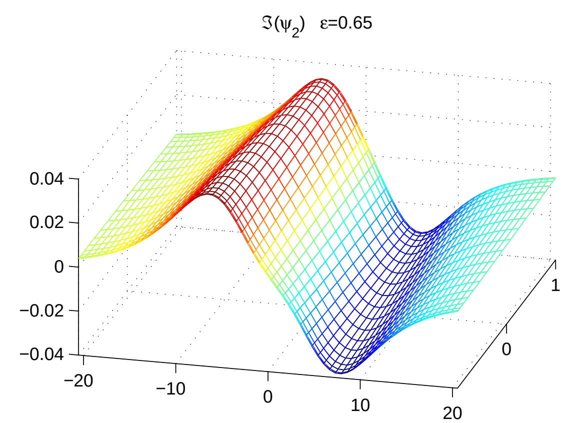












Conclusions

Ad $\mathcal{PT}$-symmetry :

$\longrightarrow$ no extension of QM

$\longrightarrow$ rather an alternative (pseudo-Hermitian) representation

$\longrightarrow$ overlooked for over 70 years

¡ rigorous formulation is still missing !

¿ phenomenological relevance ?

Conclusions

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Ad $\mathcal{PT}$-symmetric waveguide :

→ rigorous treatment

¿ reality of the total spectrum ?

¿ non-perturbative proof of the existence of the point spectrum ?

¿ calculation of the metric operator ?

¿ physical motivation ?