δN formalism for curvature perturbations from inflation

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1. Introduction

2. Linear perturbation theory

metric perturbation & time slicing,
δN formalism

3. Nonlinear extension on superhorizon scales

gradient expansion, conservation law,
local Friedmann equation

4. Nonlinear ΔN formula

ΔN for slowroll inflation

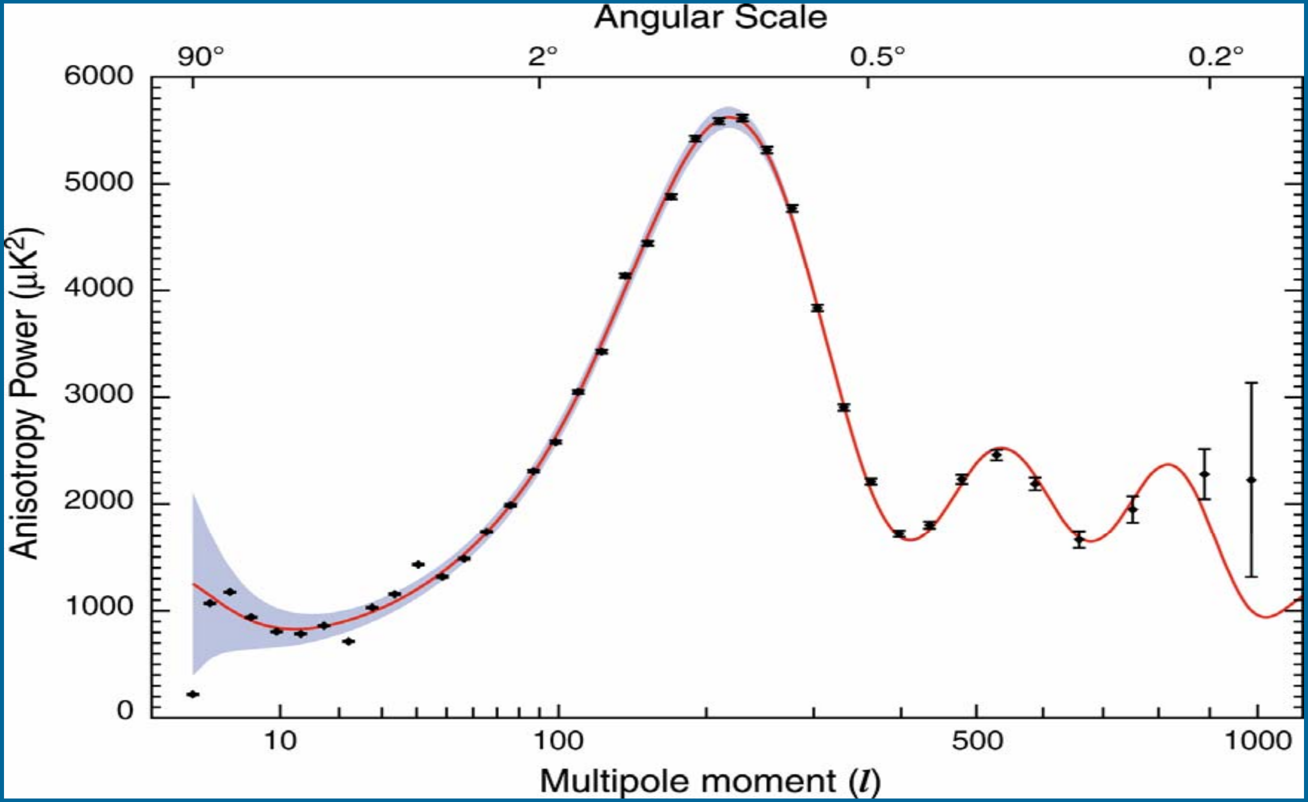
diagrammatic method for ΔN
IR divergence issue

} partial answer to points raised by David Lyth in the discussion session.

5. Summary

1. Introduction

- Standard (single-field, slowroll) inflation predicts scale-invariant Gaussian curvature perturbations.



- CMB (WMAP) is consistent with the prediction.
- Linear perturbation theory seems to be valid.

- So, why bother doing more research on inflation?

  Because observational data does not exclude other models.

  Tensor perturbations have not been detected yet.

  T/S ~ 0.2 - 0.3? or smaller?

- In fact, inflation may not be so simple.

  multi-field, non-slowroll, extra-dim's, string theory...

- PLANCK, CMBpol, ... may detect non-Gaussianity

$$\Psi = \Psi_{gauss} + f_{NL}\Psi^2_{gauss} + \cdots \; ; \qquad |f_{NL}| \gtrsim 5?$$

- Nonlinear backreaction on superhorizon scales?

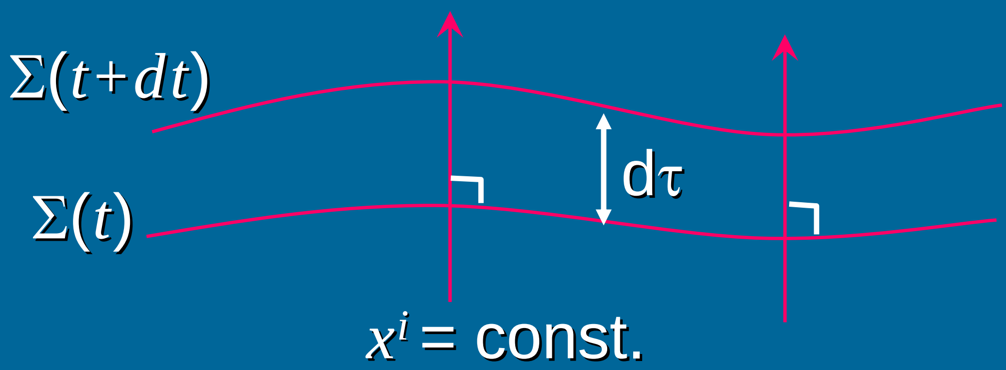
Re-consider the dynamics on super-horizon scales

2. Linear perturbation theory

Bardeen '80, Mukhanov '81, Kodama & MS '84, ….

- metric on a spatially flat background (g_{0j}=0 for simplicity)

$$ds^2 = -\left(1+2A\right)dt^2 + a^2(t)\left[(1+2\mathcal{R})\delta_{ij} + H_{ij}\right]dx^i dx^j$$



$$\left(H_{ij}\right)_{\text{scalar}} = \partial_i \partial_j E$$

$$\left(H_{ij}\right)_{\text{tensor}} = \text{transverse-traceless}$$

- propertime along x^i = const.: $d\tau = (1+A)dt$

- curvature perturbation on $\Sigma(t)$: $\mathcal{R} \Leftrightarrow {}^{(3)}R = -\dfrac{4}{a^2}\,{}^{(3)}\Delta\,\mathcal{R}$

- expansion (Hubble parameter): $\tilde{H} = H\left(1-A\right) + \partial_t\left[\mathcal{R} + \dfrac{1}{3}\,{}^{(3)}\Delta E\right]$

Choice of time-slicing

matter-based slices

- comoving slicing $T^{\mu}{}_{i} = 0 \quad \left(\phi=\phi(t) \text{ for a scalar field}\right)$
- uniform density slicing $-T^{0}{}_{0} \equiv \rho = \rho(t)$

geometrical slices

- uniform Hubble slicing

$$\tilde{H} = H(t) \Leftrightarrow -H A + \partial_t \left[\mathcal{R} + \frac{1}{3} \overset{(3)}{\Delta} E \right] = 0$$

- flat slicing

$$\overset{(3)}{R} = -\frac{4}{a^2} \overset{(3)}{\Delta} \mathcal{R} = 0 \Leftrightarrow \mathcal{R} = 0$$

- Newton (shear-free) slicing

$$\partial_t \left[H_{ij} \right]^{\text{scalar}}_{\text{traceless}} \equiv \left[\partial_i \partial_j - \frac{1}{3} \delta_{ij} \overset{(3)}{\Delta} \right] \partial_t E = 0 \Leftrightarrow \partial_t E = 0 \Leftrightarrow E = 0$$

comoving = uniform ρ = uniform H on superhorizon scales

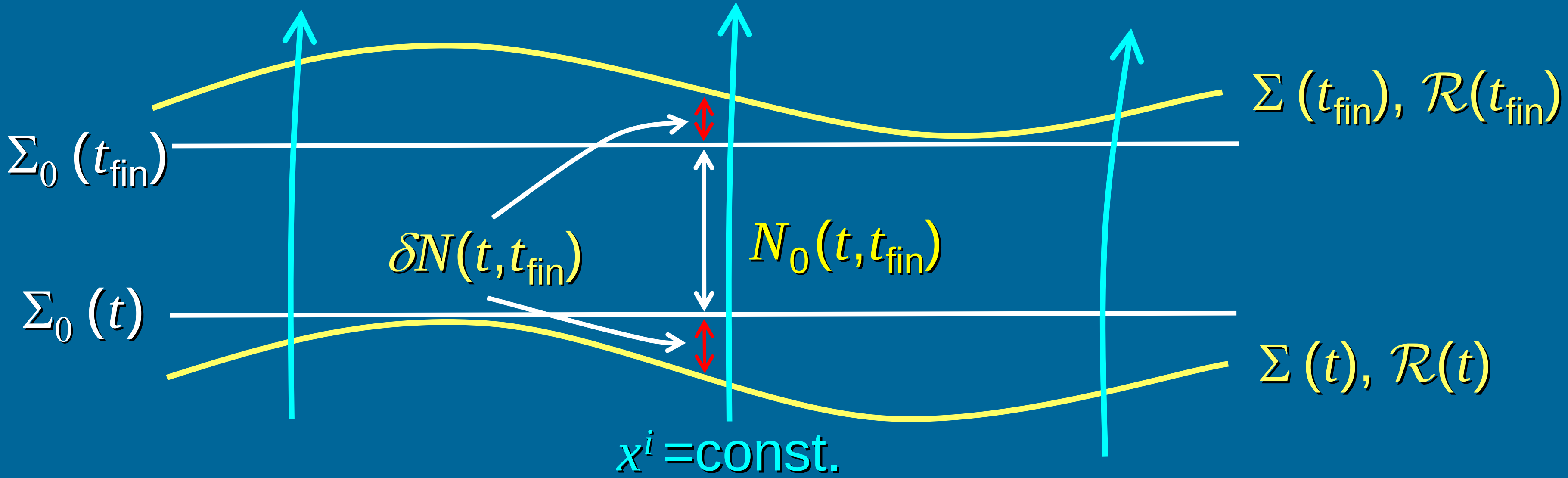
• δN formalism in linear theory

MS & Stewart '96

e-folding number perturbation between $\Sigma(t)$ and $\Sigma(t_{\rm fin})$:

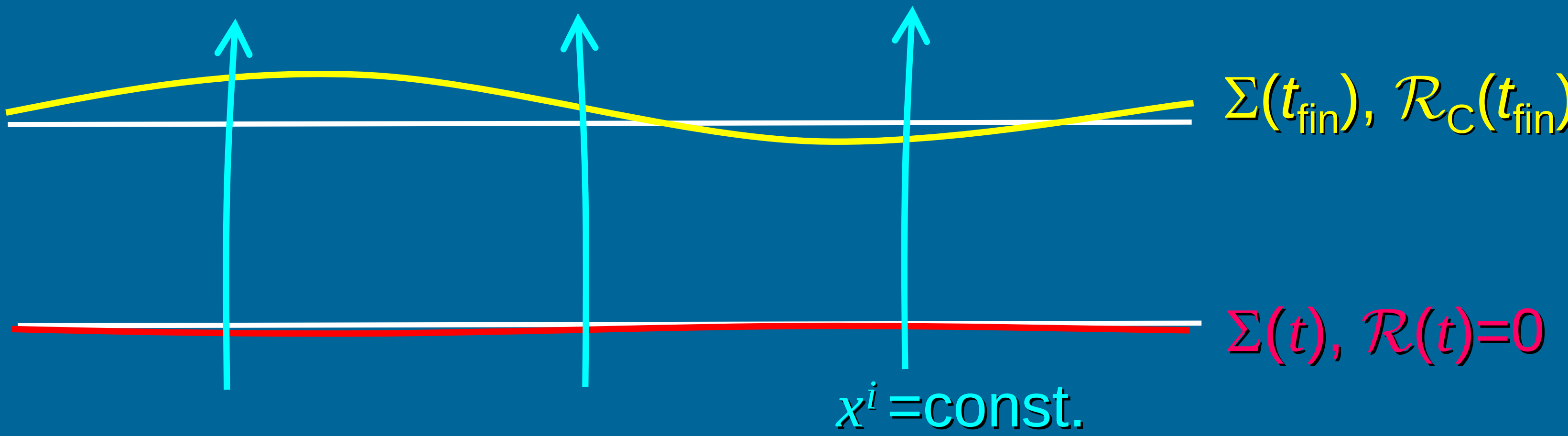
$$\delta N(t;t_{\rm fin}) \equiv \int_t^{t_{\rm fin}} \tilde{H}\, d\tau - \left(\int_t^{t_{\rm fin}} H\, d\tau\right)_{\rm background}$$

$$= \int_t^{t_{\rm fin}} \partial_t \left[\mathcal{R} + \frac{1}{3} \overset{(3)}{\Delta} E\right] dt = \mathcal{R}(t_{\rm fin}) - \mathcal{R}(t) + O(\varepsilon^2)$$



δN=0 if both $\Sigma(t)$ and $\Sigma(t_{\rm fin})$ are chosen to be 'flat' ($\mathcal{R}$=0).

Choose $\Sigma(t)$ = flat ($\mathcal{R}$=0) and $\Sigma(t_{\rm fin})$ = comoving:



$\Rightarrow$ $\delta N(t;t_{\rm fin}) = \mathcal{R}_{\rm C}(t_{\rm fin})$

curvature perturbation on comoving slice
(suffix 'C' for comoving)

The gauge-invariant variable 'ζ' used in the literature
is related to $\mathcal{R}_{\rm C}$ as $\zeta = -\mathcal{R}_{\rm C}$ or $\zeta = \mathcal{R}_{\rm C}$

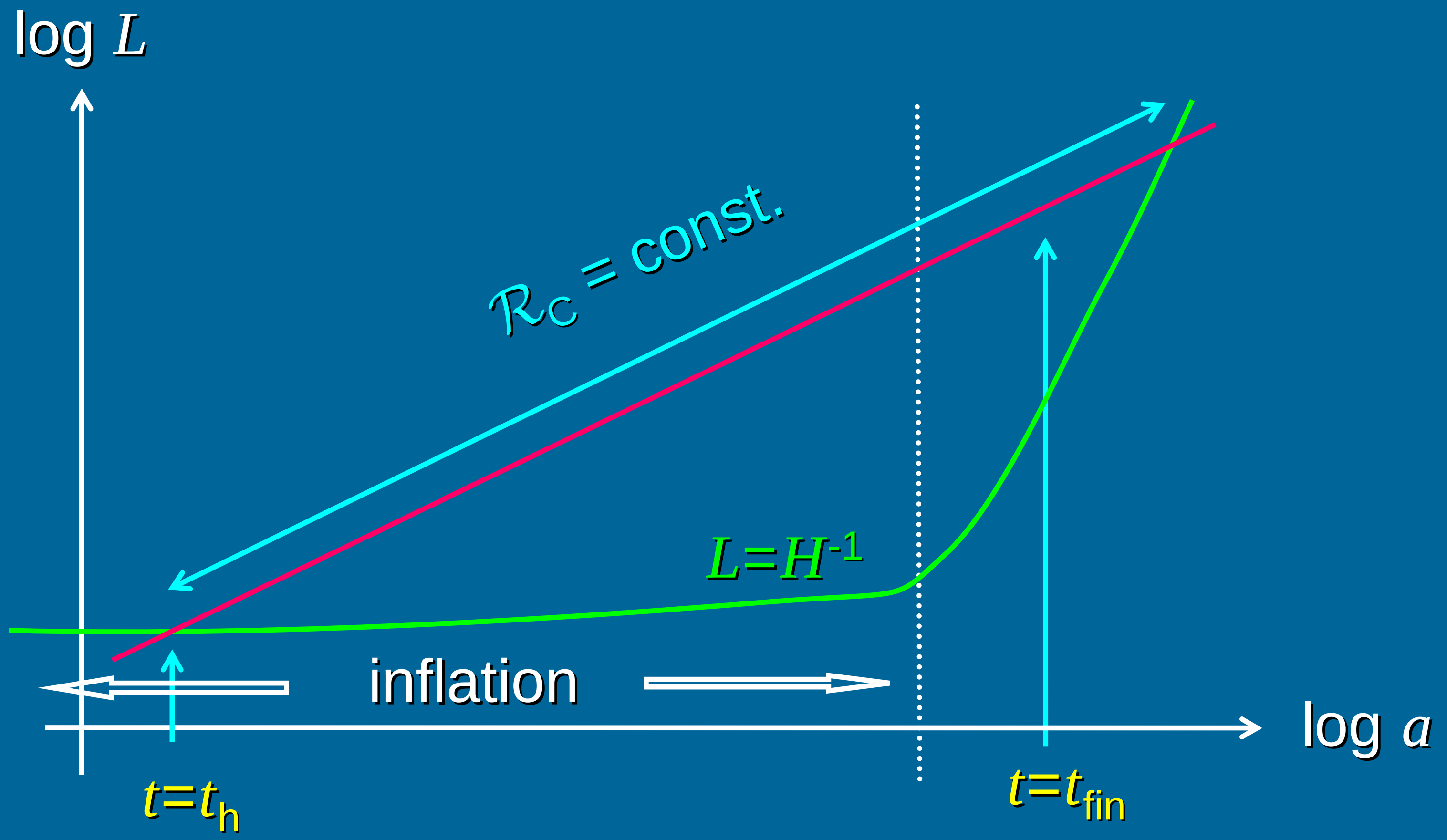
By definition, $\delta N(t;\, t_{\rm fin})$ is t-independent

Example: slow-roll inflation

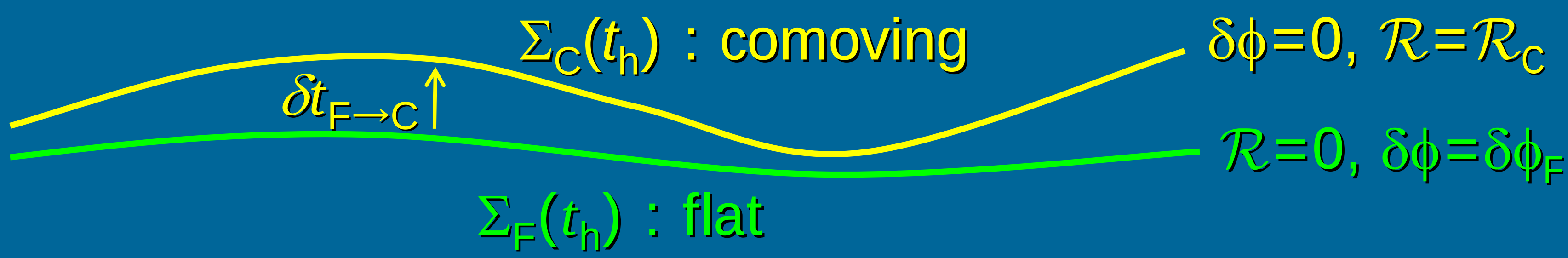
- single-field inflation, no extra degree of freedom

$\mathcal{R}_C$ becomes constant soon after horizon-crossing ($t=t_h$):

$$\delta N\left(t_h;t_{fin}\right)=\mathcal{R}_C\left(t_{fin}\right)=\mathcal{R}_C\left(t_h\right)$$



Also $\delta N = H(t_h)\,\delta t_{F\to C}$, where $\delta t_{F\to C}$ is the time difference between the comoving and flat slices at $t=t_h$.



$$\phi_F\left(t_h+\delta t_{F\to C},\,x^i\right)=\phi_C\left(t_h\right)\ \Rightarrow\ \delta\phi_F+\dot{\phi}\left(t_h\right)\delta t_{F\to C}=0$$

$$\Rightarrow\ \mathcal{R}_C\left(t_{\rm fin}\right)=\delta N\left(t_h;t_{\rm fin}\right)=-\frac{H}{d\phi/dt}\delta\phi_F\left(t_h\right)\ \Leftarrow\ dN=-Hdt$$

$$=\frac{dN}{d\phi}\delta\phi_F\left(t_h\right)\quad\cdots\ \delta N\ \text{formula}$$

Starobinsky '85

Only the knowledge of the background evolution is necessary to calculate $\mathcal{R}_C(t_{\rm fin})$.

- δN for a multi-component scalar:
  (for slowroll inflation)

$$\mathcal{R}_{\mathrm{C}}(t_{\mathrm{fin}}) = \delta N = \sum_a \frac{\partial N}{\partial \phi^a} \delta\phi_{\mathrm{F}}^a (t_{\mathrm{h}})$$

MS & Stewart '96

N.B. $\mathcal{R}_{\mathrm{C}}$ $(=\zeta)$ is no longer constant in time:

$$\mathcal{R}_{\mathrm{C}}(t) = -H \frac{\dot{\phi}\cdot\delta\phi_{\mathrm{F}}}{\|\dot{\phi}\|^2}$$

… time varying even on superhorizon scales

$$\left\langle \left|\mathcal{R}_{\mathrm{C}}\right|^2 (t_{\mathrm{fin}}) \right\rangle = \|\nabla N\|^2 \|\delta\phi_{\mathrm{F}}\|^2 = \|\nabla N\|^2 \frac{H^2(t_{\mathrm{h}})}{(2\pi)^2}$$

$$\nabla_a N \equiv \frac{\partial N}{\partial \phi^a}$$

Further extension to non-slowroll case is possible, if general slow-roll condition is satisfied at horizon-crossing.

Lee, MS, Stewart, Tanaka & Yokoyama '05

$$\frac{\dot{\phi}^2}{2H^2} = O(\xi), \frac{\ddot{\phi}}{H\dot{\phi}} = O(\xi), \frac{\dddot{\phi}}{H^2\dot{\phi}} = O(\xi), \ldots, \quad \xi \ll 1$$

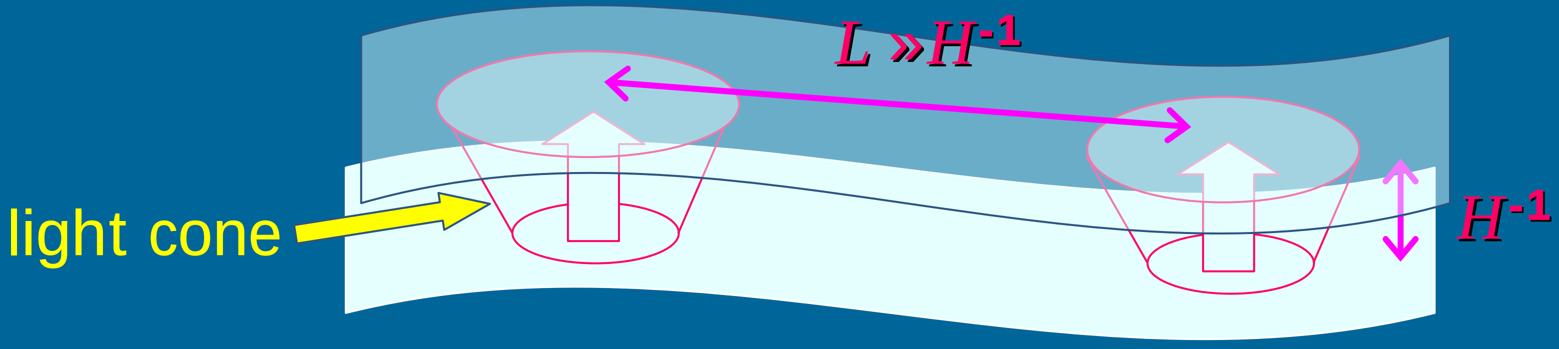
3. Nonlinear extension

- On superhorizon scales, gradient expansion is valid:

$$\left|\frac{\partial}{\partial x^i}Q\right| \ll \left|\frac{\partial}{\partial t}Q\right| \sim HQ; \;\; H \sim \sqrt{G\rho}$$

Belinski et al. '70, Tomita '72, Salopek & Bond '90, ...

This is a consequence of causality:



- At lowest order, no signal propagates in spatial directions.

Field equations reduce to ODE's

metric on superhorizon scales

- gradient expansion:

$$\partial_i \to \varepsilon \partial_i \,, \qquad \varepsilon = \text{expansion parameter}$$

- metric:

$$ds^2 = -\mathcal{N}^2 dt^2 + e^{2\alpha} \tilde{\gamma}_{ij} \left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right)$$

$$\det \tilde{\gamma}_{ij} = 1, \qquad \beta^{\mathrm{i}} = O(\varepsilon)$$

the only non-trivial assumption

contains GW (~ tensor) modes

$$\alpha\left(t, x^i\right) = \ln a(t) + \psi\left(t, x^i\right); \; \psi \sim \text{curvature perturbation}$$

e.g., choose $\psi(t_*, 0) = 0$

fiducial `background'

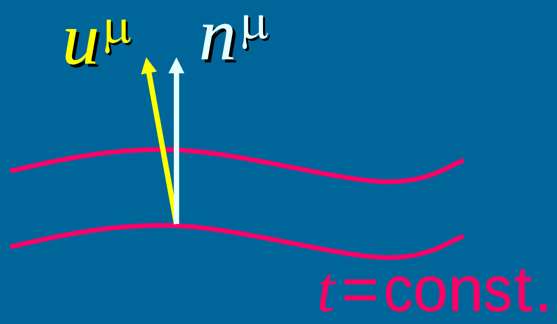
- Energy momentum tensor:

$$T^{\mu\nu} = \rho u^\mu u^\nu + p\left(g^{\mu\nu} + u^\mu u^\nu\right); \quad u_\mu \nabla_\nu T^{\mu\nu} = 0$$

$$\Rightarrow \frac{d}{d\tau}\rho + \nabla_\mu u^\mu (\rho + p) = 0\ ; \ \nabla_\mu u^\mu = 3\frac{\partial_t \alpha}{\mathcal{N}} + O\left(\varepsilon^2\right)$$

assumption: $v^i \equiv \frac{u^i}{u^0} = O\left(\varepsilon\right)$ ⟹ $u^\mu - n^\mu = O(\varepsilon)$

(absence of vorticity mode)



- Local Hubble parameter:

$$\tilde{H} \equiv \frac{1}{3}\nabla_\mu n^\mu = \frac{1}{3}\nabla_\mu u^\mu + O\left(\varepsilon^2\right)$$

$$n_\mu dx^\mu = -\mathcal{N}dt \quad \cdots \text{normal to } t = \text{const.}$$

At leading order, local Hubble parameter on any slicing is equivalent to expansion rate of matter flow.

So, hereafter, we redefine $\tilde{H}$ to be $\tilde{H} \equiv \frac{1}{3}\nabla_\mu u^\mu$

Local Friedmann equation

$$\tilde{H}^2(t,x^i) = \frac{8\pi G}{3}\rho(t,x^i) + O(\varepsilon^2)$$

x^i : comoving (Lagrangean) coordinates.

$$\frac{d}{d\tau}\rho + 3\tilde{H}\left(\rho + p\right) = 0$$

$d\tau = \mathcal{N}\,dt$: proper time along matter flow

- exactly the same as the background equations.

  "separate universe"

- uniform ρ slice = uniform Hubble slice = comoving slice

  as in the case of linear theory

- no modifications/backreaction due to super-Hubble perturbations.

cf. Hirata & Seljak '05
Noh & Hwang '05

4. Nonlinear ΔN formula

- energy conservation:

  (applicable to each independent matter component)

$$\frac{\partial_t \rho}{3(\rho + p)} + O(\varepsilon^2) = -\partial_t \alpha = -\left(\frac{\dot{a}}{a} + \partial_t \psi\right) = -\tilde{H}\mathcal{N} + O(\varepsilon^2)$$

- e-folding number:

$$N(t_2, t_1; x^i) \equiv \int_{t_1}^{t_2} \tilde{H}\mathcal{N} dt = -\frac{1}{3}\int_{t_1}^{t_2} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt$$

  where x^i=const. is a comoving worldline.

This definition applies to any choice of time-slicing.

$$\Rightarrow \quad \psi(t_2, x^i) - \psi(t_1, x^i) = \Delta N(t_2, t_1; x^i)$$

where

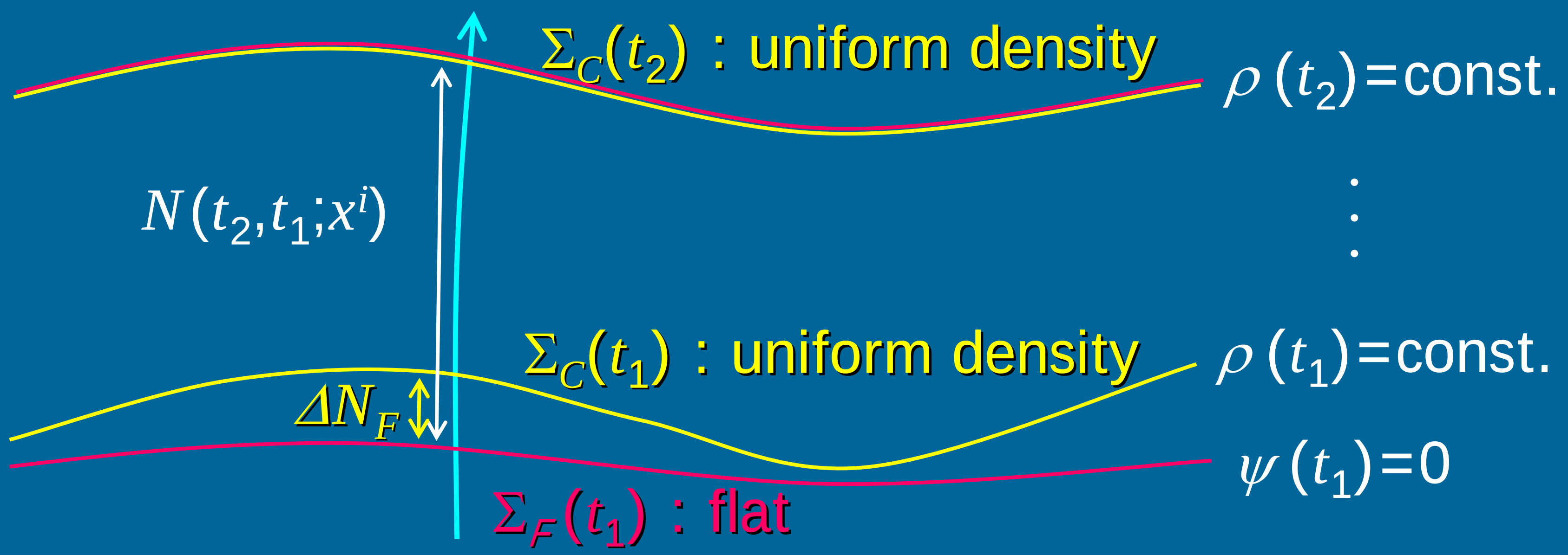
$$\Delta N(t_2, t_1; x^i) \equiv N(t_2, t_1; x^i) - \ln\left(\frac{a(t_2)}{a(t_1)}\right)$$

• ΔN - formula

Lyth & Wands '03, Malik, Lyth & MS '04,
Lyth & Rodriguez '05, Langlois & Vernizzi '05

Let us take slicing such that $\Sigma(t)$ is flat at $t = t_1$ [$\Sigma_F(t_1)$]
and uniform density/uniform H/comoving at $t = t_2$ [$\Sigma_C(t_1)$] :

('flat' slice: $\Sigma(t)$ on which $\psi = 0 \leftrightarrow e^{\alpha} = a(t)$)



$$N(t_2, t_1; x^i) = N_0(t_2, t_1) + \Delta N_F$$

$$N_0(t_2, t_1) = \ln\left(\frac{a(t_2)}{a(t_1)}\right)$$ between $\Sigma_C(t_1)$ and $\Sigma_C(t_2)$

Then

$$\Delta N_F = \psi\left(t_2, x^i\right) - \psi\left(t_1, x^i\right) = \psi_{\mathrm{C}}\left(t_2, x^i\right)$$

suffix C for comoving/uniform ρ/uniform H

where ΔN_F is equal to e-folding number from $\Sigma_F(t_1)$ to $\Sigma_C(t_1)$:

$$\Delta N_F = -\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_2)} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt + \frac{1}{3}\int_{\Sigma_C(t_1)}^{\Sigma_C(t_2)} \frac{\partial_t \rho}{\rho + P} dt$$

$$= -\frac{1}{3}\int_{\Sigma_F(t_1)}^{\Sigma_C(t_1)} \left.\frac{\partial_t \rho}{\rho + P}\right|_{x^i} dt$$

For slow-roll inflation in linear theory, this reduces to

$$\psi_C(t_2) \equiv \mathcal{R}_{\mathrm{C}}(t_2) = \delta N\left(t_1; t_2\right) = H\left(t_1\right)\delta t_{F\to C} = \left[\sum_a \frac{\partial N}{\partial \phi^a}\delta\phi^a_F\right](t_1)$$

• ΔN for 'slowroll' inflation

MS & Tanaka '98, Lyth & Rodriguez '05

- In slowroll inflation, all decaying mode solutions of the (multi-component) inflaton field ϕ die out.
- If ϕ is slow rolling when the scale of our interest leaves the horizon, N is only a function of ϕ (indep't of $d\phi/dt$, apart from trivial dep. on time t_{fin} from which N is measured), no matter how complicated the subsequent evolution would be.
- Nonlinear ΔN for multi-component inflation :

$$\Delta N = N\left(\phi^A + \delta\phi^A\right) - N\left(\phi^A\right)$$

$$= \sum_n \frac{1}{n!} \frac{\partial^n N}{\partial\phi^{A_1}\partial\phi^{A_2}\cdots\partial\phi^{A_n}} \delta\phi^{A_1}\delta\phi^{A_2}\cdots\delta\phi^{A_n}$$

where $\delta\phi = \delta\phi_F$ (on flat slice) at horizon-crossing.

($\delta\phi_F$ may contain non-gaussianity from subhorizon interactions)

cf. Maldacena '03, Weinberg '05, ...

- Caveat   (may not be so important, but ...)

In linear theory, δN can be evaluated for each Fourier mode at horizon-crossing $k=aH$ ($t=t_k$) during inflation.

This is no longer possible for nonlinear ΔN because it is formulated in real space.

We must evaluate ΔN at $t=t_*$ ($>t_k$) when all the relevant scales are outside Hubble horizon.

Linear (Gaussian) random field $\delta\phi_F$ must be evolved to $t=t_*$:

$$\delta\phi_F^A(t_*,x)=\int\frac{d^3k}{(2\pi)^{3/2}}\left(\hat{a}_k\varphi_k^A(t_*)e^{ik\cdot x}+h.c.\right)$$

For slowroll inflation,

$$\varphi_k^A(t)=\frac{\partial\phi^A}{\partial N}(t)\,\frac{\varphi_k(t_k)}{\dfrac{\partial\phi^A}{\partial N}(t_k)},\qquad \frac{4\pi k^3}{(2\pi)^3}\left|\varphi_k(t_k)\right|^2=\left(\frac{H}{2\pi}\right)^2_{t_k}$$

- simple solvable (hybrid-type) model

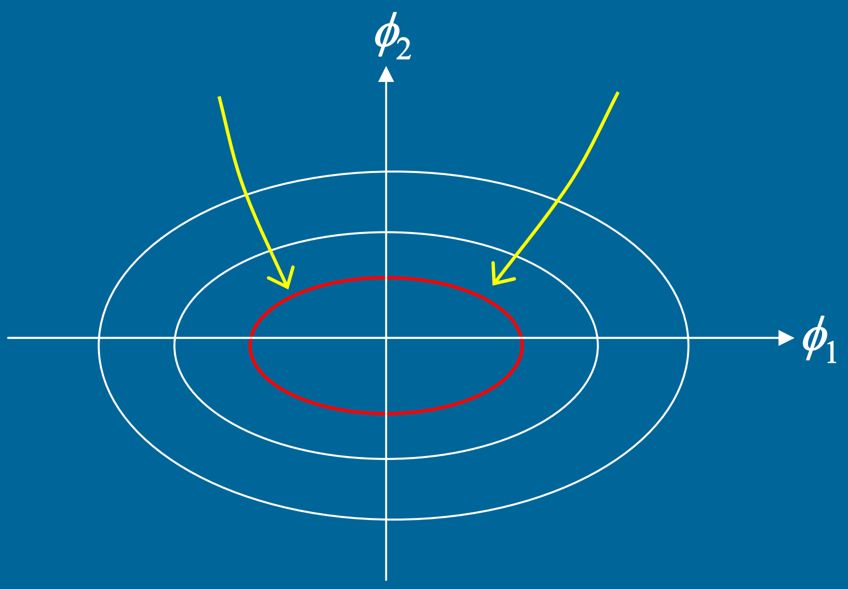
$$V(\phi) = V_0 \exp\left[\frac{1}{2}\sum_{A=1}^{M} m_A^2 \phi_A^2\right], \quad 3H^2 = V(\phi) \qquad 8\pi G = 1$$

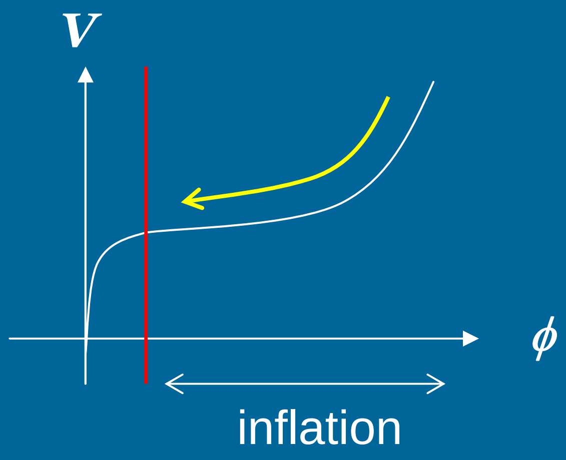
$$\frac{\partial \phi_A}{\partial N} = -\frac{\partial_A V}{3H^2} = -\frac{\partial_A V}{V} = -m_A^2 \phi_A \;\Rightarrow\; \phi_A = C_A(\boldsymbol{\lambda}) \exp\left[-m_A^2 N\right]$$

$\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{M-1})$ parametrizes different trajectories

Inflation assumed to end at $\sum_A m_A^4 \phi_A^2 = D^2$ set $N \to N_{\rm end} - N$

For M=2: $m_1^2 \phi_1 = D \cos\lambda \exp\left[m_1^2 N\right], \quad m_2^2 \phi_2 = D \sin\lambda \exp\left[m_2^2 N\right]$





$$\Rightarrow \quad \sum_A m_A^4 \phi_A^2 \exp\left[-2 m_A^2 N\right] = D^2 \;\rightarrow\; N = N(\phi_1, \phi_2, \cdots, \phi_M)$$

Power spectrum (linear δN):

$$\frac{4\pi k^2}{(2\pi)^3} P(k) \equiv \left\langle \mathcal{R}_c^2(t_{\mathrm{fin}}) \right\rangle_k = \left\langle \delta N^2 \right\rangle_k = \left(\frac{H}{2\pi}\right)_{t_k}^2 \sum_A \left(\frac{\partial N}{\partial \phi_A}\right)_{t_k}^2$$

For 2-component field,

$$\frac{\partial N}{\partial \phi_1} = \frac{m_1^2 \cos\lambda \exp[-m_1^2 N]}{D\left(m_1^2 \cos^2\lambda + m_2^2 \sin^2\lambda\right)}, \quad \frac{\partial N}{\partial \phi_2} = \frac{m_2^2 \sin\lambda \exp[-m_2^2 N]}{D\left(m_1^2 \cos^2\lambda + m_2^2 \sin^2\lambda\right)}$$

In general,

$$\frac{\partial N}{\partial \phi_A} = \frac{n_A(\lambda)\, m_A^2 \exp[-m_A^2 N]}{D \sum_B n_B^2(\lambda)\, m_B^2}\;; \qquad \sum_B n_B^2(\lambda) = 1.$$

Nonlinear ΔN can be easily evaluated to an arbitrary order:

$$\Delta N = \sum_n \frac{1}{n!} \frac{\partial^n N}{\partial \phi_{A_1} \partial \phi_{A_2} \cdots \partial \phi_{A_n}} \delta\phi_{A_1} \delta\phi_{A_2} \cdots \delta\phi_{A_n}$$

Diagrammatic method for nonlinear ΔN

(still primitive; need to be elaborated)

$$\zeta = \Delta N = \sum_n \frac{N_{A_1 A_2 \cdots A_n}}{n!} \delta\phi^{A_1} \delta\phi^{A_2} \cdots \delta\phi^{A_n} \; ; \qquad N_{A_1 A_2 \cdots A_n} \equiv \frac{D^n N}{\partial\phi^{A_1} \partial\phi^{A_2} \cdots \partial\phi^{A_n}}$$

'basic' 2-pt function: $\left\langle \delta\phi^A(x) \delta\phi^B(y) \right\rangle = h^{AB}(\phi) G_0(x-y)$

field space metric (h^{AB})

$\delta\phi$ is assumed to be Gaussian

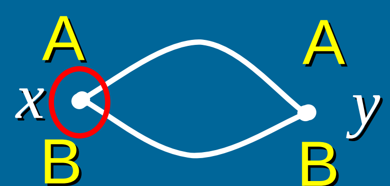
for non-Gaussian $\delta\phi$, there will be basic n-pt functions

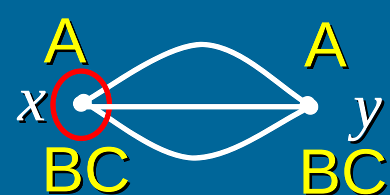
- connected n-pt function of ζ:

2-pt function

$$\left\langle \zeta(x) \zeta(y) \right\rangle_c = N_A N^A G_0(x-y) + \frac{1}{2!} N_{AB} N^{AB} G_0(x-y)^2 + \frac{1}{3!} N_{ABC} N^{ABC} G_0(x-y)^3 + \cdots$$

⟺ 

$+ \frac{1}{2!}$ 

$+ \frac{1}{3!}$ 

$+ \cdots$

3-pt function

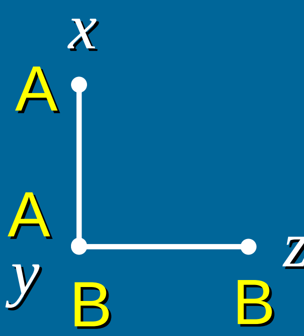
$$\left\langle \zeta(x)\zeta(y)\zeta(z)\right\rangle_c = N^A N_{AB} N^B G_0(x-y)G_0(y-z) + \text{perm.}$$

$$+ N^{AB} N_{BC} N^{CA} G_0(x-y)G_0(y-z)G_0(z-x)$$

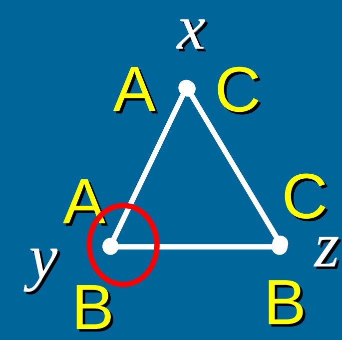
$$+ \frac{1}{2!} N^A N_{ABC} N^{BC} G_0(x-y)^2 G_0(y-z) + \text{perm.}$$

$$+ \cdots$$

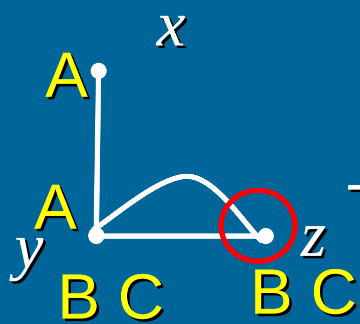
⇔



+ perm. + 2! x $\frac{1}{2!}$



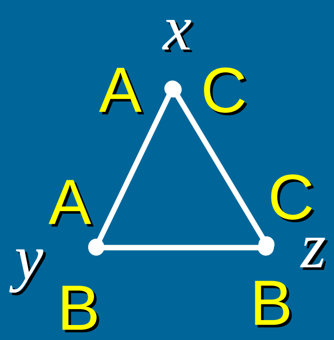
+ $\frac{1}{2!}$



+ perm. + ···

IR divergence problem

Loop diagrams like



$\Leftrightarrow$ $N^{AB} N_{BC} N^{CA} G_0(x-y) G_0(y-z) G_0(z-x)$

in the m-pt function give rise to IR divergence in the (m-1)-spectrum if $P(k) \sim k^{n-4}$ with $n \leq 1$.

Boubekeur & Lyth '05

eg, the above diagram gives

$$P(k_1, k_2, k_3) \sim \delta^3(k_1 + k_2 + k_3) \int d^3p \, P(p) P(|k_1 + p|) P(|k_2 - p|)$$

cutoff-dependent!

Is this IR cutoff physically observable?

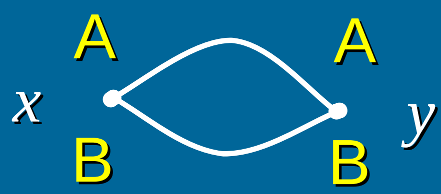
(real space 3-pt fcn is perfectly regular if $G_0(x)$ is regular.)

- Possible prescription for CMB (need justification)

Sachs-Wolfe: $$\frac{\delta T}{T}(\vec{n}) = \sum_{l,m} a_{lm} Y_{lm}(\Omega_{\vec{n}}) \sim \frac{1}{3}\Psi(\vec{x} = \vec{n}\, r_{\rm lss})$$

conformal distance to LSS

consider a loop diagram of 2-pt function:



$$\Leftrightarrow \quad N^{AB} N_{AB} G_0(x-y)^2$$

$$\Rightarrow \quad C_{l,\rm loop} = \sum_m \left\langle a_{lm}^2 \right\rangle_{\rm loop}$$

$$\sim \sum_{l_1 l_2} \begin{pmatrix} l_1 & l_2 & l \\ 0 & 0 & 0 \end{pmatrix}^2 \int k_1^2 dk_1 P(k_1) j_{l_1}^2(k_1 r_{\rm lss}) \int k_2^2 dk_2 P(k_2) j_{l_2}^2(k_2 r_{\rm lss})$$

Wigner 3j-symbol

The divergence disappears by excluding l_1=0 & l_2=0.

8. Summary

- Superhorizon scale perturbations can never affect local (horizon-size) dynamics, hence never cause backreaction.

  nonlinearity on superhorizon scales are always local.

  However, nonlocal nonlinearity (non-Gaussianity) may appear due to quantum interactions on subhorizon scales.

  cf. Weinberg '06

- There exists a nonlinear generalization of δN formula which is useful in evaluating non-Gaussianity from inflation.

  elaboration of diagrammatic method may need to be done.

  prescription to remove IR divergence from loop diagrams is given. ··· need to be justified.