The SM Construction

- Choice of the **gauge group**.
- Choice of **fermion representations**.
- Choice of the **pattern** of **symmetry breaking**.
- **Fermion masses** and mixings

Choice of the gauge group

Let us consider only one generation of fermions (ν, e, u, d). In the IVB model lepton charged current and electromagnetic interactions are

$$\mathcal{L} = \frac{g}{2\sqrt{2}} \left(J^\mu W_\mu^{(+)} + \text{h.c.} \right) + e J_\mu^{\text{em}} A^\mu \,,$$

$$\begin{aligned} J_\mu &= 2 \left(\bar{\nu}_L \gamma_\mu e_L + \bar{u}_L \gamma_\mu d_L \right) \\ J_\mu^{\text{em}} &= -\bar{e}\gamma_\mu e + \frac{2}{3}\bar{u}\gamma_\mu u - \frac{1}{3}\bar{d}\gamma_\mu d \end{aligned}$$

Which involve three bosons $W^{(\pm)}$ and A, coupled to three currents $J_\mu, J_\mu^\dagger, J_\mu^{\text{em}}$. The generators of the group should be the charges associated with those currents.

Define the week and electric charges as

$$T_+ = \frac{1}{2}\int d^3x J_0(x)\,,\ T_- = T_+^\dagger,\ Q = \int d^3x J_0^{\rm em}(x)$$

One can easily show that

$$[T_+, T_-] = 2T_3\,, \quad [T_3, T_\pm] = \pm T_\pm$$

$SU(2)$ **algebra** in terms of the ladder operators, however

$$T_3 = \frac{1}{2}\int d^3x \left(\nu_L^\dagger \nu_L - e_L^\dagger e_L + u_L^\dagger u_L - d_L^\dagger d_L\right)$$

cannot be expressed in terms of only the charge Q:

$$J_\mu^{\rm em} = J_\mu^3 + \frac{1}{2} J_\mu^Y \quad \rightarrow \quad Q = T_3 + \frac{Y}{2}$$

with

$$J_\mu^Y \equiv -\left(\bar{\nu}_L\gamma_\mu\nu_L + \bar{e}_L\gamma_\mu e_L\right) + \frac{1}{3}\left(\bar{u}_L\gamma_\mu u_L + \bar{d}_L\gamma_\mu d_L\right)$$
$$-2\bar{e}_R\gamma_\mu e_R + \frac{4}{3}\bar{u}_R\gamma_\mu u_R - \frac{2}{3}\bar{d}_R\gamma_\mu d_R$$

Q cannot be the third generator of $SU(2)$. The reasons for that are obvious:

i) Q is pure vector, while **the $SU(2)$ charges are left-handed**.

ii) If Q is going to be a **generator of $SU(2)$ it should be traceless**, with only electrons and neutrinos that is not possible.

Additional independent generator T_3 required to close the algebra of $SU(2) \Rightarrow$ additional gauge boson W_3 .

Contribution of this extra gauge boson, with its appropriate non-Abelian couplings, essential to cure the bad high energy behavior of $\nu\,\bar{\nu} \to W^+ W^-$.
Other choices to close the algebra need to introduce extra leptons, which have not been observed, and do not predict week neutral currents, which do have been observed.
One can also check that

$$[Y, T_\pm] = 0\,, \quad [Y, T_3] = 0$$

therefore we can take the gauge group as the direct product of $SU(2)$ and and Abelian group with generator Y, $SU(2)_L \otimes U(1)_Y$ and the electromagnetic charge will be just a combination of Y and T_3.

Given the gauge group and its gauge field content

$$\begin{array}{lcl} SU(2)_L & \longrightarrow & W^1_\mu\,,\;\; W^2_\mu\,,\;\; W^3_\mu \\ U(1)_Y & \longrightarrow & B_\mu \end{array}$$

one obtains the interaction Lagrangian by using the covariant derivative

$$\partial_\mu \;\longrightarrow\; D_\mu = \partial_\mu - i g \vec{T}\,\vec{W}_\mu - i g' \frac{Y}{2} B_\mu$$

while the strength tensors are

$$\begin{array}{lcl} \vec{W}_{\mu\nu} & \equiv & \partial_\mu \vec{W}_\nu - \partial_\nu \vec{W}_\mu + g\,\vec{W}_\mu \times \vec{W}_\nu \\ B_{\mu\nu} & \equiv & \partial_\mu B_\nu - \partial_\nu B_\mu \end{array}$$

then the Lagrangian for pure gauge fields is,

$$\mathcal{L}_{\rm gauge} = -\frac{1}{4}\vec{W}_{\mu\nu}\vec{W}^{\mu\nu} - \frac{1}{4}B_{\mu\nu}B^{\mu\nu} \ .$$

it contains a non-Abelian $W_3\,W^+W^-$**couplings** with the exact coefficient needed to solve the problems of the IVB model.

Choice of the fermion representations

In the one-family approximation we have seen that the $SU(2)$ is generated by the charges

$$
\begin{aligned}
T_+ &= \int d^3x \left(\nu_L^\dagger e_L + u_L^\dagger d_L\right) \\
T_- &= (T_+)^\dagger \\
T_3 &= \frac{1}{2}\int d^3x \left(\nu_L^\dagger \nu_L - e_L^\dagger e_L + u_L^\dagger u_L - d_L^\dagger d_L\right)
\end{aligned}
$$

while

$$
\begin{aligned}
Y \equiv \int d^3x \Big[& -\left(\nu_L^\dagger \nu_L + e_L^\dagger e_L\right) \\
& + \frac{1}{3}\left(u_L^\dagger u_L + d_L^\dagger d_L\right) \\
& - 2 e_R^\dagger e_R + \frac{4}{3} u_R^\dagger u_R - \frac{2}{3} d_R^\dagger d_R \Big]
\end{aligned}
$$

Y commutes with all $\vec{T}$, then all members of the same $SU(2)$ multiplet should have the same hypercharge:

- ν_L and e_L should be organized into a doublet with $Y = -1$
- left-handed quarks into a doublet of $Y = 1/3$
- right-handed fermions are singlets of $SU(2)$.

$$L_L \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \sim (1, 2, -1),$$

$$e_R \sim (1, 1, -2), \quad \nu_R \sim (1, 1, 0),$$

$$Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim (3, 2, \tfrac{1}{3}),$$

$$d_R \sim (3, 1, -\tfrac{2}{3}), \quad u_R \sim (3, 1, \frac{4}{3})$$

- **Right-handed neutrinos,** are completely neutral with respect to the gauge group and **are not necessary**. The simplest choice is to remove them from the spectrum (other choices will be discussed later on).
- Given the gauge group and the fermion representations we can immediately build the interacting Lagrangian by changing ordinary derivatives by covariant derivatives.
- This leads to the **correct electromagnetic** and **charged current** interactions.
- In addition it **predicts** the structure of the **neutral current** interactions.

- Y assignment: it has been just a consequence of the charge assignment of fermion charges $Y = 2(Q - T_3)$ which was done on **purely phenomenological grounds**.
- Since $U(1)_Y$ **is Abelian** the theory does not say anything about charge assignments, however our theory is chiral and, **chiral gauge theories** could have **pathologies** if charges are not assigned correctly. In the following section we will show that with our hypercharge assignments all pathologies cancel. In fact the **cancellation of those pathologies** can be taken as a **hint to select the physical charges of fermions**.

Anomaly Cancellation

Universitat de València
Dept. de Física Teòrica
IFIC (UV-CSIC)

Anomalies could spoil the gauge invariance of a theory with chiral fermions. To preserve the gauge invariance, **anomalies should cancel**.

The contribution to the anomaly of triangle diagrams with three gauge bosons G_a, G_b, G_c (associated to generators T^a, T^b, T^c), is proportional to

$$\mathcal{A} = \mathrm{Tr}\left(\left\{T^a, T^b\right\} T^c\right)_L - \mathrm{Tr}\left(\left\{T^a, T^b\right\} T^c\right)_R$$

In our case the generators are $T^a = \tau^a/2,\ Y$.

Contributions of three W's or one W and two B's cancel because $\mathrm{Tr}(\tau^a) = 0$.

The only possible contributions come from two W's and one B or three B's.

$$SU(2)^2U(1) \quad : \quad \mathrm{Tr}\left(\{\tau^a,\tau^b\}Y\right) = 2\delta^{ab}\sum_{\text{doublets}} Y$$

$$\propto \quad -1 + N_C\frac{1}{3}$$

$$U(1)^3 \quad : \quad \mathrm{Tr}\left(Y^3\right) \propto \sum_{\text{doublets}} Y^3 - \sum_{\text{singlets}} Y^3$$

$$= \quad 2(-1)^3 + 2N_C\left(\frac{1}{3}\right)^3$$

$$- \left((-2)^3 + N_C\left(\frac{4}{3}\right)^3 + N_C\left(\frac{-2}{3}\right)^3\right)$$

$$= \quad -6\left(-1 + N_C\frac{1}{3}\right)$$

- Anomalies cancel within a family if $N_C = 3$. $\Rightarrow$ existence of complete families.
- Anomalies involving gluons (or gravitons) and weak gauge bosons also cancel with these assignments.
- It is interesting to note that cancellation of all anomalies fix uniquely hypercharges, and charges, of all particles in agreement with observed values.
- More complete unification which could explain representation assignments and anomaly cancellation.

The Lagrangian before SSB

Universitat de València
Dept. de Física Teòrica
IFIC (UV-CSIC)

Each family contains 5 multiplets (6 if ν_R exist)

$$\psi_i = (Q_L, d_R, u_R, L_L, e_R)$$

then we can write the free Lagrangian for fermions as

$$\mathcal{L}_0 = \sum_i i\,\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i$$

which has a $SU(2)_L \otimes U(1)_Y$ **symmetry**.
Note that **mass terms are forbidden** by the global $SU(2)_L \otimes U(1)_Y$ invariance: Masses involve left-handed fields and right-handed fields, since **left-handed fields are doublets** and **right-handed fields are singlets**: **mass terms are doublets**.

To find the interacting Lagrangian **promote** the **ordinary derivatives** to **covariant derivatives**

$$\partial_\mu \psi_i \rightarrow D_\mu \psi_i \equiv \left(\partial_\mu - ig\vec{T}\vec{W}_\mu - ig' \frac{Y_i}{2} B_\mu \right) \psi_i$$

with $\vec{T} = \vec{\tau}/2$ acting on doublets and $\vec{T} = 0$ acting on singlets. Then

$$\mathcal{L}_\psi = \sum_i i\, \bar{\psi}_i \gamma^\mu D_\mu \psi_i$$

which **correctly generates** **charged currents** and **neutral currents**.

Neutral currents appear in terms of W_3 and B which at this point are unphysical because they are massless.

Physical particles, with definite mass, are the **photon A and the Z**, which will be a **mixture of the W_3 and B**. This can **better seen after SSB.**

Spontaneous symmetry breaking

Universitat de València
Dept. de Física Teòrica
IFIC (UV-CSIC)

To give masses to the $W^\pm$ and the Z via the Higgs mechanism we need to introduce a set of scalars in such a way that the symmetry is broken according to the following pattern

$$SU(2)_L \otimes U(1)_Y \stackrel{\mathrm{SSB}}{\longrightarrow} U(1)_Q$$

this requires at least **three real scalars** (to be absorbed by the $W^\pm$, and the Z) and **one** of them **must be neutral** in order to be able to develop a charge conserving VEV. In addition we would like the SSB mechanism to be able to generate also fermion masses. We have seen that **fermion mass terms are doublets**, therefore a **complex doublet** will satisfy all these requirements

$$\Phi \equiv \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} , \quad Y(\Phi) = 1$$

We add the Lagrangian

$$\mathcal{L}_\Phi = (D_\mu \Phi)^\dagger \; D^\mu \Phi - V(\Phi)$$

with

$$V(\Phi) = \mu^2 \; \Phi^\dagger \Phi + \lambda \; (\Phi^\dagger \Phi)^2$$

For $\mu^2 < 0$ there is SSB. To preserve the **exact** $U(1)_Q$ symmetry we choose the VEV of the doublet scalar as

$$\langle \Phi \rangle \equiv \langle 0 | \, \Phi \, | 0 \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$$

where

$$v = \sqrt{-\frac{\mu^2}{\lambda}}$$

In this case the corresponding gauge boson, **the photon**, **will remain massless**. In order to reach the physical spectrum we parametrize the doublet scalar as

$$\Phi = \frac{(v+H)}{\sqrt{2}} \exp\left(i \frac{\vec{\tau}}{2} \vec{\theta}/v \right) \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

which is written in terms of four real fields, the **three Goldstone**, $\vec{\theta}$, and the **Higgs field** H.

Gauge invariance allows us to choose $\vec{\theta}$ at will. The **choice** $\vec{\theta} = 0$, **the unitary gauge**, removes all unphysical degrees of freedom from the spectrum.

Substitution in $\mathcal{L}_\Phi$ leads to

$$\mathcal{L}_\Phi = \left| \left(\partial_\mu - i g \frac{\vec{\tau}}{2} \vec{W}_\mu - i \frac{g'}{2} B_\mu \right) \frac{(v+H)}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right|^2 - \mu^2 \frac{(v+H)^2}{2} - \lambda \frac{(v+H)^4}{4} \; .$$

We find a mass term for the Higgs boson, $m_H = \sqrt{-2\mu^2} = \sqrt{2\lambda}\, v$, and mass terms for the gauge bosons

$$\mathcal{L}_M = \frac{v^2}{8} g^2 \left(W^1_\mu W^{1\mu} + W^2_\mu W^{2\mu} \right) + \frac{v^2}{8} g^2 W^3_\mu W^{3\mu} - 2 g g' W^3_\mu B^\mu + g'^2 B_\mu B^\mu$$

the W_μ^1 and W_μ^2 are degenerate and can be combined into fields of definite charge

$$W_\mu^\pm = \frac{1}{\sqrt{2}}\left(W_\mu^1 \mp i\, W^2\right)$$

the W_μ^3 and the B_μ mass terms can be rewritten in matrix form as

$$\frac{v^2}{8}\left(W_\mu^3, B_\mu\right)\begin{pmatrix} g^2 & -gg' \\ -gg' & g'^2 \end{pmatrix}\begin{pmatrix} W_\mu^3 \\ B_\mu \end{pmatrix}$$

which can be diagonalized by an orthogonal transformation leading to two eigenstates, **one massless** that we will identify with the **photon,** A_μ and the other **massive, with mass** m_Z, which will be identified with the Z**-gauge boson**.

$$Z_\mu = \cos\theta_W W^3_\mu - \sin\theta_W B_\mu$$
$$A_\mu = \sin\theta_W W^3_\mu + \cos\theta_W B_\mu$$

with $\tan\theta_W = g'/g$.
When rewriting $\mathcal{L}_\psi$ in terms of the mass eigenstates we will see that this is the correct identification.
Then the mass terms are written as

$$\mathcal{L}_M = m_W^2 W_\mu^+ W^{-\,\mu} + \frac{1}{2} m_Z^2 Z_\mu Z^\mu \ ,$$

with

$$m_W^2 = \frac{v^2}{4} g^2 \,, \qquad m_Z^2 = \frac{v^2}{4}(g^2 + g'^2) = \frac{v^2}{4} \frac{g^2}{\cos^2\theta_W}$$

The precise relationship between masses and the mixing $\cos\theta_W$ (which also gives the **ratio of neutral to charged current couplings**)

$$\rho = \frac{m_W^2}{m_Z^2 \cos^2\theta_W} = 1$$

is a consequence of an accidental symmetry (the **custodial symmetry**) of the Higgs potential for doublets. This symmetry does not exist is for other multiplets of scalars. The custodial symmetry is also broken by other pieces of the Lagrangian (B-interactions and Yukawa couplings). This will show up at the loop level in corrections to the ρ parameter.

Fermion masses and mixings

Universitat de València
Dept. de Física Teòrica
IFIC (UV-CSIC)

For N_g families of fermions (and no ν_R's) we can write the Yukawa interactions

$$\mathcal{L}_Y = -\bar{L}_L Y_e \Phi e_R - \bar{Q}_L Y_d \Phi d_R - \bar{Q}_L Y_u \tilde{\Phi} u_R + \text{h.c.}$$

where, L_L, Q_L, e_R, d_R, u_R are all vectors in family space and $\tilde{\Phi} = i\tau_2 \Phi^*$ is an iso-doublet with $Y(\tilde{\Phi}) = -1$. The Yukawa couplings, Y_e, Y_d, Y_u, are **general $N_g \times N_g$ matrices**.
The gauge-kinetic part of the Lagrangian is invariant under the following transformations in family space:

$$L_L \to U_L L_L\,, \quad e_R \to U_e e_R\,,$$
$$Q_L \to U_Q Q_L\,, \quad d_R \to U_d d_R\,, \quad u_R \to U_u u_R$$

where all the U's are unitary matrices.

The Yukawa Lagrangian,however, is not invariant,

$$Y_e \to U_L^\dagger Y_e U_e \,, \quad Y_d \to U_Q^\dagger Y_d U_d \,, \quad Y_u \to U_Q^\dagger Y_u U_u$$

Yukawa couplings related by these transformations are **physically equivalent**.

Choose U_L, and U_e to make Y_e **diagonal** and **positive definite**.

Choose U_Q, and U_u to make Y_u **diagonal** and **positive definite**.

Choose U_d to make Y_d **hermitian** (we cannot longer use U_Q to make it diagonal).

After SSB we obtain

$$\mathcal{L}_Y = -(1+\frac{H}{v})\left(e_L M_e e_R + \bar{d}_L M_d d_R + \bar{u}_L M_u u_R + \text{h.c.}\right)$$

where (M_e and M_u **diagonal** and M_d hermitian)

$$M_e = Y_e v/\sqrt{2}\,, \quad M_d = Y_d v/\sqrt{2}\,, \quad M_u = Y_u v/\sqrt{2}$$

The full Lagrangian is invariant under phase transformations

$$L_L \longrightarrow K_L L_L\,, \quad e_R \longrightarrow K_L e_R$$

with K_L a diagonal matrix of phases: **Conservation of family lepton numbers**.
To diagonalize M_d we use a unitary matrix V

$$M_d = V D_d V^\dagger$$

and then perform the following transformation on the d-quark fields

$$d_R \longrightarrow V d_R\,, \quad d_L \longrightarrow V d_L$$

Does not leave gauge interactions invariant, since it only transform one component of the quark doublet.
Neutral current interactions have the form $\bar{d}_R \gamma^\mu d_R$ and $\bar{d}_L \gamma^\mu d_L$ then V cancels in neutral current interactions: **No flavor changing neutral currents** (GIM mechanism).
However, it does show up in charged currents. The replacement $d_L \to V d_L$ immediately leads to the following charged current interactions:

$$\mathcal{L}_{\rm CC} = \frac{g}{\sqrt{2}} \left\{ W_\mu^+ \left[\bar{u}_L \gamma^\mu V d_L + \bar{\nu}_L \gamma^\mu e_L \right] + \textbf{h.c.} \right\}$$

V **cannot be removed** completely **from the Lagrangian**.
V Is a physical quantity (the **CKM** mixing matrix)