2 Renormalization

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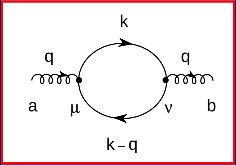
ULTRAVIOLET DIVERGENCES



$$i\,\Pi^{\mu\nu}_{ab}(q) \;=\; -g_s^2\,\delta_{ab}\,T_F \int \frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}[\gamma^\mu \not{k}\gamma^\nu(\not{k}-\not{q})]}{k^2\,(k-q)^2}$$

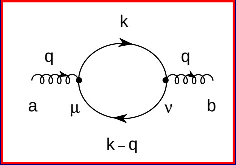
$$=\; i\,\delta_{ab}\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\;\Pi(q^2)$$

Divergent Integral ➡ $D = 4 + 2\epsilon$

$$\int \frac{d^Dk}{(2\pi)^D}\,\frac{k^\alpha\,(k-q)^\beta}{k^2\,(k-q)^2} \;=\; \frac{-i\,\Gamma(-\epsilon)}{6(4\pi)^2}\left(\frac{-q^2}{4\pi}\right)^{\epsilon}\left(1-\frac{5}{3}\epsilon\right)\left\{\frac{q^2 g^{\alpha\beta}}{2\,(1+\epsilon)} + q^\alpha q^\beta\right\}$$

$$\Gamma(-\epsilon) \;=\; -\frac{1}{\epsilon} - \gamma_E + O(\epsilon) \qquad ; \qquad \gamma_E = 0.577215\ldots$$

$$\mu^{2\epsilon}\left(\frac{-q^2}{4\pi\mu^2}\right)^{\epsilon}\Gamma(-\epsilon) \;=\; -\mu^{2\epsilon}\left\{\frac{1}{\epsilon} + \gamma_E - \ln 4\pi + \ln\left(\frac{-q^2}{\mu^2}\right) + O(\epsilon)\right\}$$



$$i\,\Pi^{\mu\nu}_{ab}(q) \;=\; -g_s^2\,\delta_{ab}\,T_F \int \frac{d^4k}{(2\pi)^4}\,\frac{\mathrm{Tr}[\gamma^\mu \not{k} \gamma^\nu (\not{k}-\not{q})]}{k^2\,(k-q)^2}$$

$$=\; i\,\delta_{ab}\,\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\;\Pi(q^2)$$

$$\Pi(q^2) \;=\; -\frac{4}{3}\,T_F\,\left(\frac{g_s\,\mu^\epsilon}{4\pi}\right)^2\;\left\{\frac{1}{\epsilon}+\gamma_E-\ln 4\pi+\ln\left(\frac{-q^2}{\mu^2}\right)-\frac{5}{3}+O(\epsilon)\right\}$$

$$\Pi(q^2) \;=\; \Pi(q_0^2)\;-\;\frac{4}{3}\,T_F\,\left(\frac{g_s}{4\pi}\right)^2\,\ln\left(\frac{q^2}{q_0^2}\right)$$

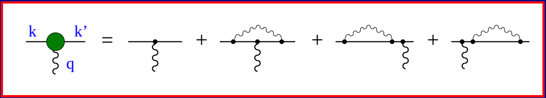
QFT predicts the q^2 dependence

Schemes: $\Pi(q^2) \equiv \Delta\Pi_\epsilon(\mu^2) + \Pi_R(q^2/\mu^2)$

$$\Delta\Pi_\epsilon(\mu^2) = \begin{cases} -\frac{T_F}{3\pi}\,\frac{g_s^2}{4\pi}\,\mu^{2\epsilon}\left[\frac{1}{\hat\epsilon}-\frac{5}{3}\right] & (\mu) \\ -\frac{T_F}{3\pi}\,\frac{g_s^2}{4\pi}\,\mu^{2\epsilon}\,\frac{1}{\epsilon} & (\mathrm{MS}) \\ -\frac{T_F}{3\pi}\,\frac{g_s^2}{4\pi}\,\mu^{2\epsilon}\,\frac{1}{\hat\epsilon} & (\overline{\mathrm{MS}}) \end{cases} \quad ; \quad \frac{1}{\hat\epsilon} \equiv \frac{1}{\epsilon} + \gamma_E - \ln 4\pi$$

$$\Pi_R(q^2/\mu^2) = \begin{cases} -\frac{T_F}{3\pi}\,\frac{g_s^2}{4\pi}\,\ln\left(\frac{-q^2}{\mu^2}\right) & (\mu) \\ -\frac{T_F}{3\pi}\,\frac{g_s^2}{4\pi}\left[\ln\left(\frac{-q^2}{\mu^2}\right)+\gamma_E-\ln(4\pi)-\frac{5}{3}\right] & (\mathrm{MS}) \\ -\frac{T_F}{3\pi}\,\frac{g_s^2}{4\pi}\left[\ln\left(\frac{-q^2}{\mu^2}\right)-\frac{5}{3}\right] & (\overline{\mathrm{MS}}) \end{cases}$$

SYMMETRY: CVC



$$\mathcal{L} = e\,Q\,\left(\bar{\psi}\,\gamma^\mu \psi\right) A_\mu = J^\mu A_\mu$$

$$\partial_\mu J^\mu = 0$$

$$\langle\, \psi(k') \,|\, J^\mu \,|\, \psi(k) \,\rangle = \mathrm{e}^{iq\cdot x}\; e\,Q\;\; \bar{u}(k') \left[\gamma^\mu\, F_1(q^2) + \frac{i}{2m}\,\sigma^{\mu\nu} q_\nu\, F_2(q^2) \right] u(k)$$

Lowest Order: $F_1(q^2) = 1 \quad ; \quad F_2(q^2) = 0$

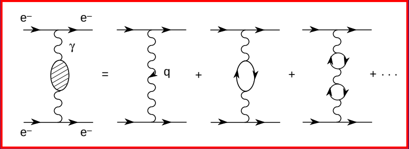
$$e\,Q\,\Delta_{\vec{k}\vec{k}'} \;=\; \langle \psi(k')|\,\mathcal{Q}\,|\psi(k)\rangle \;=\; \langle \psi(k')| \int d^3x\, J^0(x)\, |\psi(k)\rangle$$

$$=\; (2\pi)^3\,\delta^{(3)}(\vec{q})\;\; e\,Q\,F_1(0)\;\; u^\dagger(k)\,u(k) \;\;=\;\; e\,Q\,F_1(0)\,\Delta_{\vec{k}\vec{k}'}$$

$$\Delta_{\vec{k}\vec{k}'} \;\equiv\; (2\pi)^3\, 2k^0\;\delta^{(3)}(\vec{k} - \vec{k}')$$

$\Longrightarrow$ $\boxed{F_1(0) = 1}$ **to all orders in α**

RENORMALIZATION



$$T(q^2) \sim -J^\mu J_\mu \, \frac{\alpha}{q^2} \left\{ 1 - \Pi(q^2) + \cdots \right\}$$

$$\alpha_0 \left\{ 1 - \Delta\Pi_\epsilon(\mu^2) - \Pi_R(q^2/\mu^2) \right\} \;\equiv\; \alpha_R(\mu^2) \left\{ 1 - \Pi_R(q^2/\mu^2) \right\}$$

$$= \;\alpha_R(\mu^2) \left\{ 1 + \frac{\alpha_R(\mu^2)}{3\pi} \left[\ln\left(-q^2/\mu^2\right) + C \right] + \cdots \right\}$$

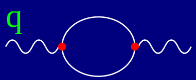
$$= \;\alpha_R(Q^2) \left\{ 1 + \frac{\alpha_R(Q^2)}{3\pi} \, C + \cdots \right\}$$

$$\alpha_R(\mu^2) = \alpha_0 \left\{ 1 + \frac{\alpha_0}{3\pi} \, \mu^{2\epsilon} \left[\frac{1}{\epsilon} + C' \right] + \cdots \right\}$$

QED UV DIVERGENCES

Superficial Degree of Divergence:

$$\mathcal{D} = 4 - B_E - \frac{3}{2}\, F_E$$

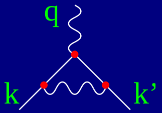


$$i\,\Pi^{\mu\nu}(q) \;=\; i\,\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\,\Pi(q^2)$$

$$\Pi(q^2) \;=\; -\frac{4}{3}\,\frac{\alpha_0\,\mu^{2\epsilon}}{4\pi}\,\left[\frac{1}{\hat\epsilon} + \cdots\right]$$

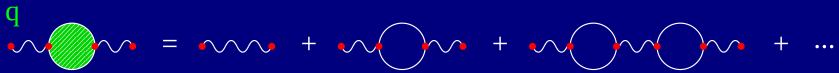
$$-i\,\Sigma(\not p) \;=\; -i\left[\Sigma_1(p^2) + (\not p - m)\,\Sigma_2(p^2)\right]$$

$$\Sigma_1(q^2) = m_0\,\frac{\alpha_0\,\mu^{2\epsilon}}{4\pi}\,\left[\frac{-3}{\hat\epsilon} + \cdots\right] \qquad ; \qquad \Sigma_2(q^2) = \xi_0\,\frac{\alpha_0\,\mu^{2\epsilon}}{4\pi}\,\left[\frac{1}{\hat\epsilon} + \cdots\right]$$



$$i\,e\,Q\,\Gamma^\mu \qquad ; \qquad \Gamma^\mu = \frac{\alpha_0\,\mu^{2\epsilon}}{4\pi}\,\left\{\gamma^\mu\,\left[-\xi_0\,\frac{1}{\hat\epsilon} + \cdots\right] + \cdots\right\}$$

PHOTON SELF-ENERGY: Dyson Summation



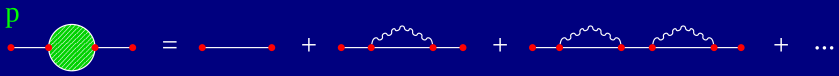
$$i\, D^{\mu\nu}(q) \;=\; i\, D^{\mu\nu}_{(0)}(q) \;+\; i\, D^{\mu\lambda}_{(0)}(q)\; i\,\Pi_{\lambda\rho}(q)\; i\, D^{\rho\nu}_{(0)}(q) \;+\; \cdots$$

$$D^{\mu\nu}(q) \;=\; \frac{1}{q^2}\left\{\left(-g^{\mu\nu}+\frac{q^\mu q^\nu}{q^2}\right)\frac{1}{1+\Pi(q^2)} \;-\; \xi_0\,\frac{q^\mu q^\nu}{q^2}\right\}$$

$$\equiv\; Z_3\; D^{\mu\nu}_R(q,\mu)$$

$$\frac{1}{1+\Pi(q^2)} \;=\; \frac{Z_3}{1+\Pi_R(q^2/\mu^2)} \quad ; \quad \xi_0 \;=\; Z_3\,\xi \quad ; \quad Z_3 \;=\; 1-\Delta\Pi_\epsilon(\mu^2)$$

FERMION SELF-ENERGY: Dyson Summation



$$i\,S(\not p) \;=\; i\,S^{(0)}(\not p) \;+\; i\,S^{(0)}(\not p)\;[-i\,\Sigma(\not p)]\;i\,S^{(0)}(\not p) \;+\; \cdots$$

$$S(\not p) \;=\; \frac{1}{\not p - m_0 - \Sigma(\not p)} \;\equiv\; Z_2\;\; S_R(\not p,\mu)$$

$$S_R(\not p,\mu)^{-1} \;=\; (\not p - m)\left[1-\Sigma_2^R(p^2/\mu^2)\right] - \Sigma_1^R(p^2/\mu^2)$$

$$m \;=\; m_0 \;+\; \frac{\Delta\Sigma_1^\epsilon(\mu^2)}{1-\Delta\Sigma_2^\epsilon(\mu^2)} \;\equiv\; Z_2\, Z_4^{-1}\;\; m_0$$

$$Z_2 = 1+\Delta\Sigma_2^\epsilon(\mu^2) \qquad ; \qquad Z_4 = 1+\Delta\Sigma_2^\epsilon(\mu^2) - \frac{\Delta\Sigma_1^\epsilon(\mu^2)}{m_0}$$

RENORMALIZATION

$$\mathcal{L}_{QED} = -\frac{1}{4} F^{(0)}_{\mu\nu} F^{\mu\nu}_{(0)} - \frac{1}{2\,\xi_0}\left[\partial_\mu A^\mu_{(0)}\right]^2 + i\,\bar{\psi}^{(0)}\gamma^\mu\partial_\mu\psi^{(0)} - m_0\,\bar{\psi}^{(0)}\psi^{(0)}$$

$$+\, e_0\, Q\, A^{(0)}_\mu \left[\bar{\psi}^{(0)}\gamma^\mu\psi^{(0)}\right]$$

$$= -Z_3\,\frac{1}{4} F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\,\xi}\left[\partial_\mu A^\mu\right]^2 + Z_2\, i\,\bar{\psi}\,\gamma^\mu\partial_\mu\psi - Z_4\, m\,\bar{\psi}\,\psi$$

$$+\, Z_1\, e\, Q\, A_\mu \left[\bar{\psi}\,\gamma^\mu\psi\right]$$

$$A^{(0)}_\mu \equiv Z_3^{1/2}\, A_\mu \quad ; \quad \psi^{(0)} \equiv Z_2^{1/2}\,\psi \quad ; \quad \xi_0 \equiv Z_3\,\xi$$

$$m_0 \equiv Z_4\, Z_2^{-1}\, m \quad ; \quad e_0 \equiv Z_1\, Z_2^{-1}\, Z_3^{-1/2}\, e$$

Gauge Invariance (CVC): $Z_1 = Z_2$

$\overline{\mathrm{MS}}$ SCHEME

$$Z_1 = Z_2 = 1 + \frac{\alpha_0\,\mu^{2\epsilon}}{4\pi}\;\xi_0\;\frac{1}{\hat{\epsilon}}$$

$$Z_3 = 1 + \frac{\alpha_0\,\mu^{2\epsilon}}{3\pi}\;\frac{1}{\hat{\epsilon}}$$

$$Z_4 = 1 + \frac{\alpha_0\,\mu^{2\epsilon}}{4\pi}\;(3+\xi_0)\;\frac{1}{\hat{\epsilon}}$$

$$\frac{1}{\hat{\epsilon}} \equiv \frac{2}{D-4} + \gamma_E - \ln 4\pi$$