The Standard Model of Electroweak Interactions

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Introduction to weak interactions and gauge theories

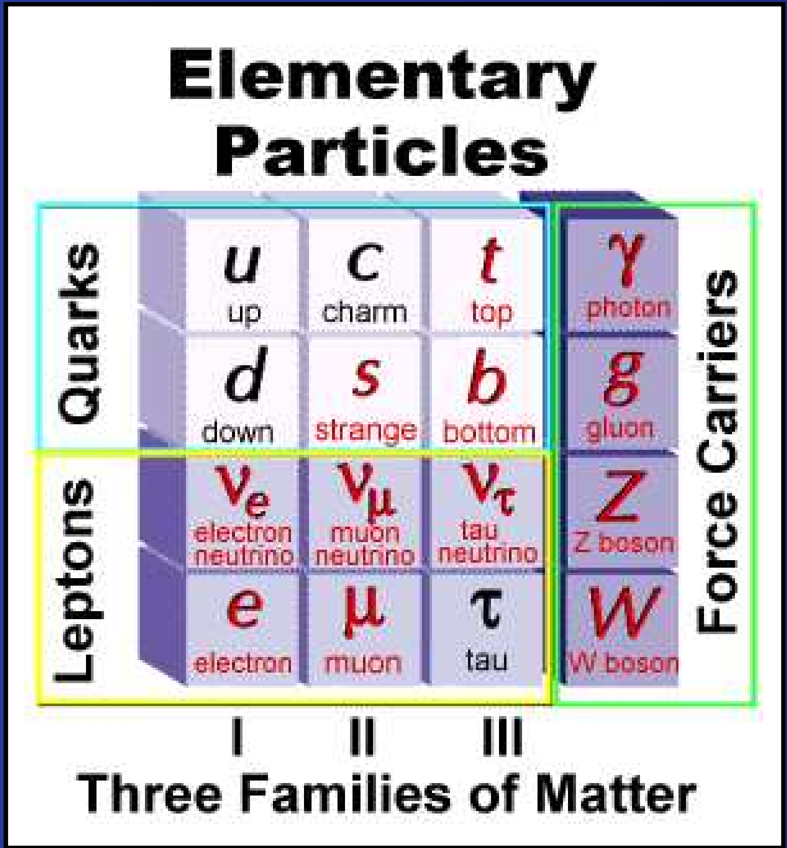
Introduction

The existence of the atomic nucleus and spontaneous radioactivity require two additional short-range forces:

- **Strong Interactions**: Keep nucleus bound.
- **Week interactions**: Allow beta decay of nuclei.

Since then a long way has lead to the Standard Model (SM) which describes accurately the interactions of all known particles.
The Standard Model (SM) is a **gauge theory** (exchange of spin–1 fields) based on the group $SU(3)_C \otimes SU(2)_L \otimes U(1)_Y$, which describes **strong** (**8 massless gluons**), **electromagnetic** (**1 massless photon**) and **weak** (3 massive bosons, $W^{\pm}$ **and** Z) interactions.

In the SM fermion matter, leptons an quarks, are organized in three families (or generations) which feel the same interactions and only differ in their masses:



The fermions of each family $\begin{bmatrix} \nu_e & u \\ e & d \end{bmatrix}$, are embedded in different chiral representations of the gauge group:

$\psi_L = P_L \psi$ and $\psi_R = P_R \psi_R$

$P_L \equiv \frac{1}{2}(1 - \gamma_5)$,

$P_R \equiv \frac{1}{2}(1 + \gamma_5)$

$$\begin{bmatrix} \nu_e & u \\ e & d \end{bmatrix} \equiv \begin{cases} L_L \equiv \begin{pmatrix} \nu_{eL} \\ e_L \end{pmatrix} \sim (1, 2, -1) \\ e_R \sim (1, 1, -2)\,, \ \nu_R \sim (1, 1, 0) \\ Q_L \equiv \begin{pmatrix} u_L \\ d_L \end{pmatrix} \sim (3, 2, \frac{1}{3}) \\ d_R \sim (3, 1, -\frac{2}{3})\,, \ u_R \sim (3, 1, \frac{4}{3}) \end{cases}$$

Left-handed fields are $SU(2)_L$ **doublets**.

Right-handed partners transform as **singlets**.

W, and Z are massive $\Longrightarrow$ **spont. symmetry breaking** (SSB)

$$SU(3)_C \otimes SU(2)_L \otimes U(1)_Y \xrightarrow{\text{SSB}} SU(3)_C \otimes U(1)_{QED}$$

Need for a physical scalar particle, the **Higgs** boson.

$\mu^- \to e^- \bar{\nu}_e \nu_\mu$ decay

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If one tries to describe $\mu^- \to e^- \bar{\nu}_e \nu_\mu$ (and the related $\nu_\mu e^- \to \mu^- \nu_e$ process) with the most general four-fermion interaction, one finds experimentally that the decay amplitude only involves left-handed fermions, with an effective interaction of the $V - A$ type:

$$\mathcal{L}_{\text{eff}} = -\frac{G_F}{\sqrt{2}} \left[\bar{e}\gamma^\alpha(1-\gamma_5)\nu_e\right] \left[\bar{\nu}_\mu \gamma_\alpha (1-\gamma_5)\mu\right]$$

Where the so-called Fermi coupling constant G_F is fixed by the total decay width. One obtains

$$G_F = (1.16639 \pm 0.00002) \times 10^{-5}\,\text{GeV}^{-2} \approx \frac{1}{(293\,\text{GeV})^2}$$

Beta decay

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- Weak transitions $n \to p e^- \bar{\nu}_e$ and $p \to n e^+ \nu_e$ (in nuclei) can be described by the effective interaction

$$\mathcal{L}_{\rm eff} = -\frac{G}{\sqrt{2}} \left[\bar{p}\gamma^\alpha(1 - g_A\gamma_5)n\right] \left[\bar{e}\gamma_\alpha(1 - \gamma_5)\nu_e\right]$$

where $G \approx 0.975\, G_F$, $g_A = 1.2573 \pm 0.0028$.

- The **strength** approximately the **same as for μ decay**
- **only left-handed leptons** are involved.

Universal interaction at the quark-lepton level:

$$\mathcal{L}_{\rm eff} = -\frac{G}{\sqrt{2}} \left[\bar{u}\gamma^\alpha(1 - \gamma_5)d\right] \left[\bar{e}\gamma_\alpha(1 - \gamma_5)\nu_e\right]$$

g_A understood as a **QCD** correction.

$\Delta S = 1$ transitions and ν flavors

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$\Delta S = 1$ decays [$K \to (\pi) l^- \bar{\nu}_l$, $\Lambda \to p e^- \bar{\nu}_e$, $\ldots$] show:

- The weak interaction is always of the $V - A$ type.
- The strength is the same in all decays but smaller than in $\Delta S = 0$ processes: $G \;\approx\; 0.22\, G_F$
- All decays satisfy the $\Delta S = \Delta Q$ rule (i.e. decays such as $\Sigma^+ \to n e^+ \nu_e$ or $\bar{K}^0 \to \pi^- l^+ \nu_l$ never occur)

Neutrino flavors

$\bar{\nu}_\mu$ can produce μ^+ but never e^+

$$\bar{\nu}_\mu X \to \mu^+ X' \,, \qquad\qquad \bar{\nu}_\mu X \not\to e^+ X' \,.$$

$\bar{\nu}_e$ produces e^+ but never $\mu^+ \Longrightarrow$ the neutrino partners of the electron and the muon are two different particles: $\nu_e \neq \nu_\mu$.

The $V - A$ model

All previous facts can be described by:

$$\mathcal{L} = -\frac{G_F}{\sqrt{2}} J^\mu J_\mu^\dagger$$

with

$$\begin{aligned} J^\mu &= = \bar{u}\gamma^\mu(1-\gamma_5)\left[\cos\theta_C d + \sin\theta_C s\right] \\ &+ \bar{\nu}_e\gamma^\mu(1-\gamma_5)e + \bar{\nu}_\mu\gamma^\mu(1-\gamma_5)\mu \end{aligned}$$

Weak transitions proceed through a **universal interaction** involving charged-currents only.
The different strength of $\Delta S = 0$ and $\Delta S = 1$ processes parametrized by θ_C, $\sin\theta_C \equiv G^{\Delta S=1}/G_F \approx 0.22$.
Correctly describes the weak decays $\pi^+ \to \pi^0 e^+ \nu_e$, $\pi^- \to l^- \bar{\nu}_l$: strong helicity suppression in $\pi^- \to l^- \bar{\nu}_l$.

Problems of the V-A model

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- **Unitarity**: G_F is a dimensionful quantity ($[G_F] = M^{-2}$) : **cross-sections increase with energy**:

$$\sigma(\nu_\mu e^- \to \mu^- \nu_e) \approx G_F^2 s/\pi \, .$$

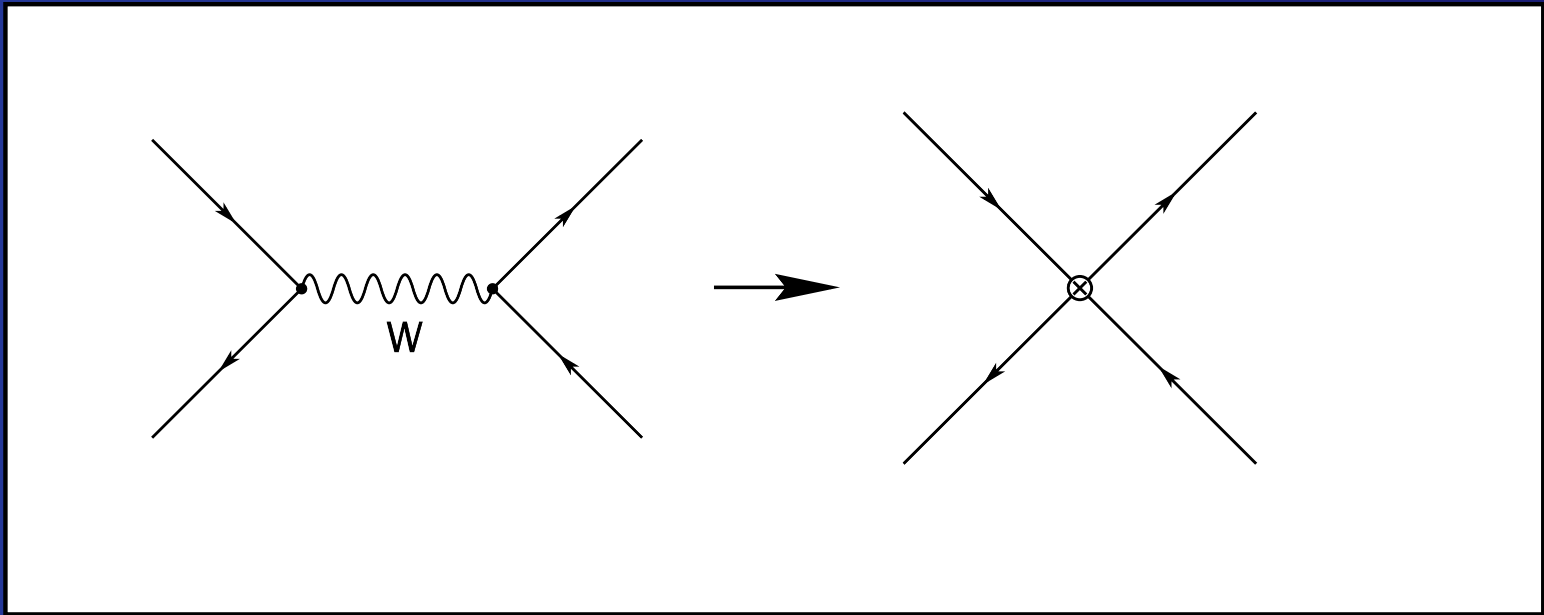
At large values of s, tree-level unitarity is violated. The unitarity bound $\sigma < 2\pi/s$ is only satisfied if $s \leq \sqrt{2}\pi/G_F \sim (617\,\text{GeV})^2$.

- **Renormalizability**: Higher-order transitions such as $\nu_\mu e^- \to \mu^- \bar{\nu}_e \to \nu_\mu e^-$ are divergent with **divergences** that cannot be absorbed in the parameters of the model. The theory is **not renormalizable**.

The successful $V - A$ model can only be a low-energy effective description of some more fundamental theory.

The IVB Hypothesis

In QED the fundamental $\gamma\bar{e}e$ interaction generates a long range 4-fermion interaction through γ-exchange.
One can try to generate four-fermion weak interactions through a similar mechanism. However, weak interactions are short range, therefore **weak bosons** should be **massive**.



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If the charged current couples to a massive spin-1 field W_μ,

$$\mathcal{L} = \frac{g}{2\sqrt{2}} \left(J^\mu W_\mu^\dagger + \text{h.c.} \right)$$

the $V - A$ interaction generated by W-exchange. At $q^2 \ll M_W^2$, the propagator reduces to a contact interaction,

$$\frac{-g_{\mu\nu} + q_\mu q_\nu / M_W^2}{q^2 - M_W^2} \quad \xrightarrow{q^2 \ll m_W^2} \quad \frac{g_{\mu\nu}}{M_W^2} \, .$$

Fourfermion interactions obtained with the identification

$$\frac{g^2}{8 M_W^2} = \frac{G_F}{\sqrt{2}} \, , \qquad g < 1 \implies M_W < 123 \, \text{GeV}$$

Better high-energy behaviour for $\nu l^- \to \nu l^-$.

Problems of the IVB

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However, the bad behaviour of the cross-section reappears in processes with external W bosons:

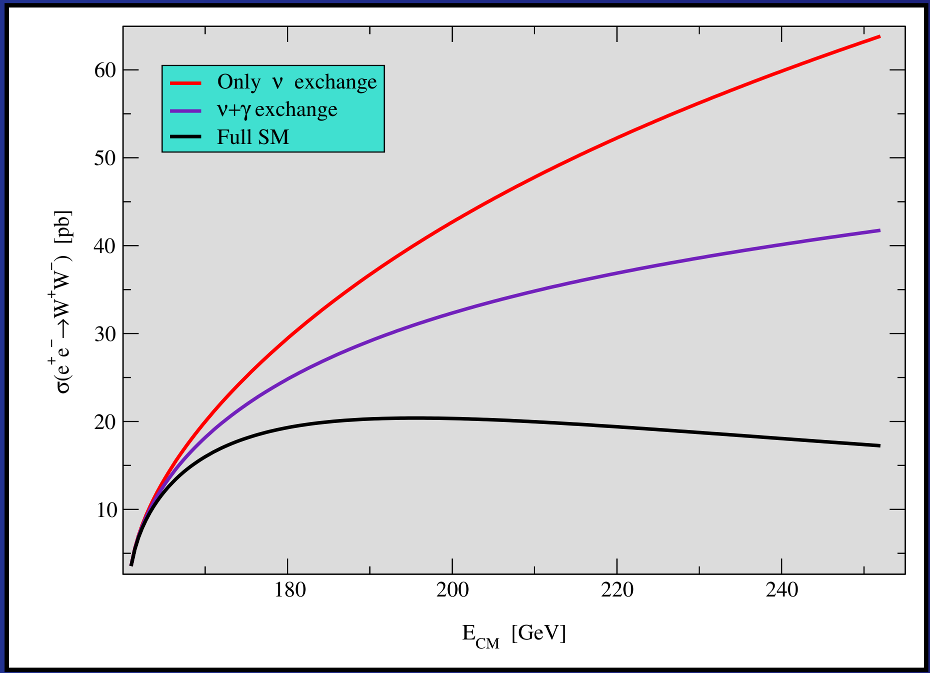
$$\sigma(\nu_e\bar{\nu}_e \to W^+W^-) \quad \overset{s\to\infty}{\propto} \quad G_F^2 s$$

$$\sigma(e^+e^- \to W^+W^-) \quad \overset{s\to\infty}{\propto} \quad G_F^2 s$$

The problem comes from the $q_\mu q_\nu/m_W^2$ piece in the sum over polarizations of the Proca fields.
This wild behavior implies that the one-loop box amplitude $T(e^+e^- \to W^+W^- \to e^+e^-)$ is badly divergent and the theory is **not renormalizable.**
Similar processes with similar diagrams exist in QED; however, **gauge symmetry**, which requires a massless photon makes those contributions harmless.

To solve the problem one needs additional diagrams. One should also consider $e^+e^- \to \gamma \to W^+W^-$, but this is not enough. (Also it does not help in $\nu_e\bar{\nu}_e \to W^+W^-$)

Something else needed: new fermions or new bosons that couple both to neutrinos and electrons and to W's.



Neutral currents

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The high-energy **cancellation** can be realized introducing an **additional neutral** intermediate **boson** Z, which couples both to neutrinos and charged leptons and W's. By cleverly choosing the Z mass and couplings, it is possible to obtain a cancellation with the s-channel contributions $e^+e^- \to Z \to W^+W^-$ and $\nu_e\bar{\nu}_e \to Z \to W^+W^-$. This idea has important implications.

The exchange of a Z boson in the t channel, should give rise to **neutral-current** processes such as $\nu_\mu e^- \to \nu_\mu e^-$ or $\nu_\mu p \to \nu_\mu p$. **Confirmed in 1973**!!

In contrast with the charged-current transitions, one finds that **flavor-changing neutral-current** processes are **very suppressed.**

Therefore, **the Z couplings** are **flavor diagonal**.

Ingredients for a theory of WI

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- **photon** γ and three massive spin-1 bosons $W^{\pm}, Z$.
- Electroweak **unification**: $g_W/2\sqrt{2} \sim g_Z/2\sqrt{2} \sim e$, i.e. $g^2/4\pi \sim 8\alpha$. Implies

$$m_W \sim \left(\frac{\sqrt{2}g^2}{8G_F}\right)^{1/2} \sim \left(\frac{4\pi\alpha\sqrt{2}}{G_F}\right)^{1/2} \sim 100\,\text{GeV}$$

- **Universality** of couplings
- The $W^{\pm}$ field couples only to **left-handed** particles
- The Z **boson** has only **flavor-diagonal couplings**
- **Lepton-number** is conserved to good accuracy
- Should allow for CP **violation**
- **Renormalizability** $\Longrightarrow$ **SSB gauge theory**

QED as a gauge theory

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Quantum field theories can have global invariances. For instance the free Dirac Lagrangian

$$\mathcal{L}_\psi = \bar{\psi}(i \not{D} - m)\psi$$

is invariant under a global phase transformation ($\alpha \equiv$const.)

$$\psi \to \psi' = e^{i\alpha Q}\psi$$

Noether theorem $\Rightarrow$ charge is conserved.
Global invariances, however require that the field is transformed exactly in the same way in the whole universe. More reasonable to think that **fundamental symmetries should be local**, with parameters depending on the position. That is the **gauge principle**.

However, the free Dirac Lagrangian is not invariant under the local gauge transformation,

$$\psi \rightarrow \psi' = e^{i\alpha(x)Q}\psi$$

since

$$\mathcal{L}_\psi \rightarrow \mathcal{L}'_\psi = \bar{\psi}\left(i\,\gamma^\mu\left(\partial_\mu + i\,Q\partial_\mu\alpha\right) - m\right)\psi$$

To preserve the local **gauge invariance** one must introduce the **gauge field** A_μ through the **minimal coupling**

$$\partial_\mu\psi \Rightarrow D_\mu\psi \equiv \left(\partial_\mu - ieQA_\mu\right)\psi$$

and require that A_μ transforms like

$$A_\mu \longrightarrow A'_\mu = A_\mu + \frac{1}{e}\partial_\mu\alpha$$

then $D_\mu \psi$ transforms nicely

$$D_\mu \psi \longrightarrow (D_\mu \psi)' \equiv e^{i\alpha(x)Q} D_\mu \psi$$

The coupling between ψ (e.g. electrons) and the gauge field A_μ (photon) arises naturally when we **promote** the global phase invariance of free Dirac Lagrangian to a local gauge symmetry.
To complete the theory we must add a kinetic term also for the gauge field. It must be quadratic in the field and gauge invariant. The only term we can build is

$$\mathcal{L}_A = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

where

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$$

is the gauge invariant electromagnetic strength tensor.
Gauge invariance **forbids mass terms** for the **gauge bosons**.
$\mathcal{L}_{\rm QED} = \mathcal{L}_A + \mathcal{L}_\psi$ is the Lagrangian of Quantum Electrodynamics (**QED**) which is **renormalizable** and has had an enormous success describing the interactions between photons and electrons.
The gauge principle provides a very simple recipe to build interacting theories. As QED these theories will be **renormalizable** and will be **universal** (particles with same quantum numbers couple with the same strength).

Non-Abelian Gauge Invariance

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Let us consider the case of N degenerate Dirac fields. The free Dirac Lagrangian is invariant under a global $U(N) = U(1) \otimes SU(N)$ transformation.
$\psi \to \psi' = U\psi$ where ψ is a N-component column of Dirac spinors and U an element of $U(N)$. Since the $U(1)$ component has already been discussed, let us consider now the $SU(N)$ part.
Following the gauge principle those symmetries must be gauged or be only approximate symmetries.
To implement the gauge principle we will require that the Lagrangian is invariant under

$$\psi \to \psi' = U(x)\,\psi$$

with

$$U \equiv \exp\left[i\,T^a \alpha^a(x)\right]$$

T^a are the generators of the group in the representation furnished by ψ and satisfy

$$\left[T^a, T^b\right] = i\,C^{abc}\,T^c$$

being C_{abc} the structure constant of the group.
As in the Abelian case, we must introduce one gauge field for each generator, and define the **covariant derivative** as

$$D_\mu \equiv \partial_\mu - igT^a A^a_\mu\,, \quad D_\mu\psi \longrightarrow (D_\mu\psi)' = U\,D_\mu\psi$$

Gauge invariance will be kept as long as

$$T^a A^a_\mu \longrightarrow T^a A'^a = U \left(T^a A^a_\mu + \frac{i}{g} \partial_\mu \right) U^{-1}$$

or, in infinitesimal form, i.e. for $U \approx 1 + i\, T^a \alpha^a(x)$,

$$A'^a_\mu = A^a_\mu + \frac{1}{g} \partial_\mu \alpha^a - C_{abc}\, \alpha^b A^c_\mu$$

Using the covariant derivative we can generalize the **strength tensor** for a non-Abelian Lie group,

$$-igT^a F^a_{\mu\nu} \equiv [D_\mu, D_\nu]$$

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\, C_{abc}\, A^b_\mu A^c_\nu$$

which transforms like

$$F^{a\,\prime}_{\mu\nu} \to F^a_{\mu\nu} - C_{abc}\alpha^b F^c_{\mu\nu}$$

Therefore, the invariant kinetic term for the gauge bosons, can be written as

$$\mathcal{L}_A = -\frac{1}{4} F^a_{\mu\nu} F^{a\,\mu\nu}$$

As in the Abelian case a **mass term for the gauge bosons is forbidden** by gauge invariance.
Note that, at difference with the Abelian case, pure **non-Abelian gauge theory is not a free theory** and contains triple and quartic self-interactions.

Right- and Left- Handed Fermions

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Dirac fields are reducible representations of the Lorentz group.

The irreducible representations are the chiral two component spinors obtained by projecting a Dirac spinor with the chirality projectors $\psi_L = P_L \psi$ and $\psi_R = P_R \psi_R$

The most general way of building of **Lorentz invariant** and **gauge invariant** theories is by using as basis for the representations left-handed and right-handed **chiral fields**.

Note, however that, parity, or other symmetries could force the fields to be combined into Dirac fields.

Note also that ordinary Dirac mass terms require the existence of the two chiralities $\bar{\psi}\psi = \bar{\psi}_R \psi_L + \bar{\psi}_L \psi_R$.

Some comments on quantization

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One can show that **non-Abelian theories are also renormalizable**. However, covariant quantization of non-Abelian theories require the introduction unphysical fields (the so-called **Fadeev-Popov ghosts**) needed necessary to keep the unitarity of the theory. They only contribute at higher orders.

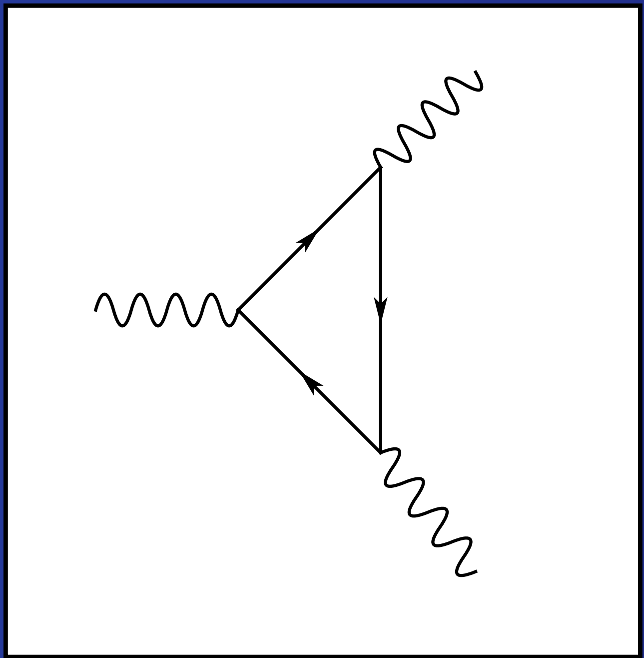
Anomalies

It can happen that a **symmetry of the classical Lagrangian** is **not a symmetry** of the **quantum theory**. In particular this can happen with symmetries that treat differently left and right-handed fields.

In general this is not a problem, and in fact this phenomenon explains the process $\pi^0 \to \gamma\gamma$.

However, if the gauge theory is broken by anomalies, this is a **big problem** since the **renormalizability** of the theory **relies on the gauge symmetry**.

Gauge theories can suffer from anomalies because **triangle diagrams** with three external gauge bosons and chiral fermions running in the loop.



One can show that the contribution to the anomaly of triangle diagrams with three gauge bosons G_a, G_b, G_c (associated to generators T^a, T^b, T^c), is proportional to

$$\mathcal{A} = \mathrm{Tr}\left(\left\{T^a, T^b\right\} T^c\right)_L - \mathrm{Tr}\left(\left\{T^a, T^b\right\} T^c\right)_R$$

This should cancel.

Spontaneous Symmetry Breaking (SSB)

Spontaneous Symmetry Breaking

Weak bosons need to be massive, but, this seems incompatible with gauge invariance and renormalizability. Fortunately, symmetry can be realized in different ways. It could happen that although the action has some symmetry the vacuum (the ground-state) does not have it.

Molecules in liquid water (or vapor) are distributed symmetrically in all directions. However if the temperature drops below 0 C, the crystals formed only have symmetry under rotations of 60 degrees. The symmetry is broken spontaneously.

A similar mechanism can occur in **quantum mechanical systems** with **infinite degrees of freedom** (quantum field theory).

An example: a **ferromagnet** where the Lagrangian describing the spin–spin interaction is **invariant** under **three-dimensional rotations**.

For $T > T_C$ the spin system is completely disordered (paramagnetic phase), and the vacuum is $SO(3)$ invariant.

For $T < T_C$ (ferromagnetic phase) a **spontaneous magnetization** of the system occurs, aligning the spins in some specific direction and the vacuum is not invariant under $SO(3)$. The **symmetry is broken to $SO(2)$**, the rotations of the whole system around the spin directions.

Let us see how it works.

Exercise: SSB of discrete symmetries

Let us take a self-interacting real field with Lagrangian,

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi - V(\phi)$$

with **potential**

$$V(\phi) = \frac{1}{2}\mu^2\phi^2 + \frac{1}{4}\lambda\phi^4$$

The Lagrangian is invariant under the discrete transformation

$$\phi \longrightarrow -\phi$$

Is the vacuum also invariant? The vacuum (ϕ_0) can be obtained by minimizing the Hamiltonian

$$\mathcal{H} = \frac{1}{2}\left[(\partial_0\phi)^2 + (\nabla\phi)^2\right] + V(\phi)$$

The minimum is found for $\phi_0 = constant$ satisfying

$$\phi_0(\mu^2 + \lambda\phi_0^2) = 0$$

Since λ should be positive to guarantee that $\mathcal{H}$ is bounded from below, the minimum depends on the sign of μ. Notice that for interacting theories μ^2 is not the physical mass and can be negative.

For $\mu^2 > 0$, we have just one minimum at $\phi_0 = 0$ and it is also invariant.

For $\mu^2 < 0$, we have two vacua states corresponding to $\phi_0^{\pm} = \pm\sqrt{-\mu^2/\lambda}$

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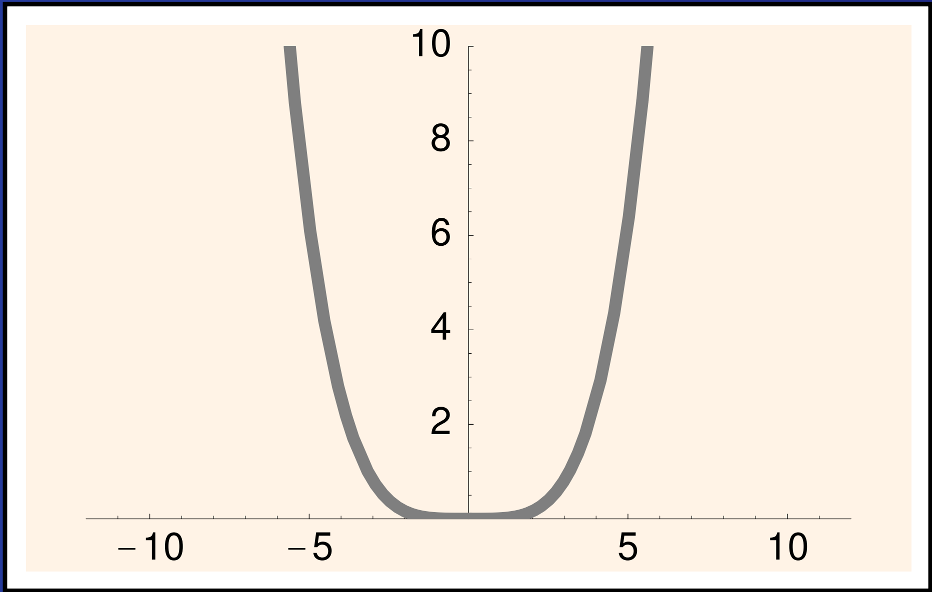

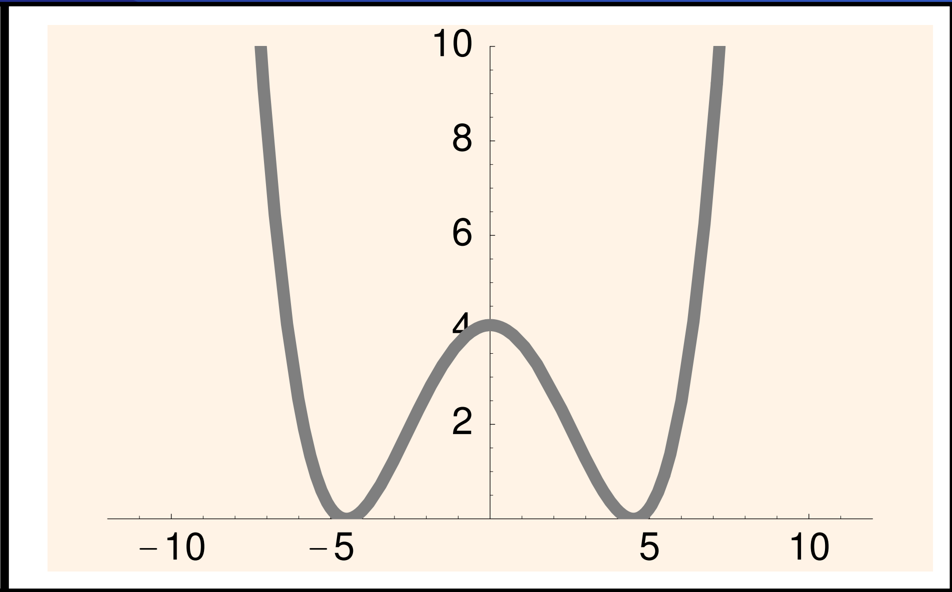

Since the Lagrangian is invariant under the symmetry, the choice between ϕ_0^+ or ϕ_0^- is irrelevant. Nevertheless, once one choice is made (e.g. $v = \phi_0^+$) the symmetry is spontaneously broken: the Lagrangian $\mathcal{L}$ is invariant but the vacuum is not.

The field ϕ is not a good starting point for perturbation theory. Perturbation theory should be defined about about the true ground-state.
We define ϕ' by shifting ϕ by the vacuum expectation value (VEV) $v = \sqrt{-\mu^2/\lambda}$,

$$\phi' \equiv \phi - v$$

The Lagrangian then becomes:

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi' \partial^\mu \phi' - \frac{1}{2}\left(\sqrt{-2\mu^2}\right)^2 \phi'^{\,2} - \lambda\, v\, \phi'^{\,3} - \frac{1}{4}\lambda \phi'^{\,4}$$

which describes a scalar field ϕ' with real and positive mass, $m_{\phi'} = \sqrt{-2\mu^2}$, but the original **symmetry seems broken**. However, the symmetry is still there **hidden** and the number of the parameters of the model is still the same.

SSB of a continuous global symmetry

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Let us consider the case of a charged self-interacting scalar field,

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - V(\phi)$$

with a potential,

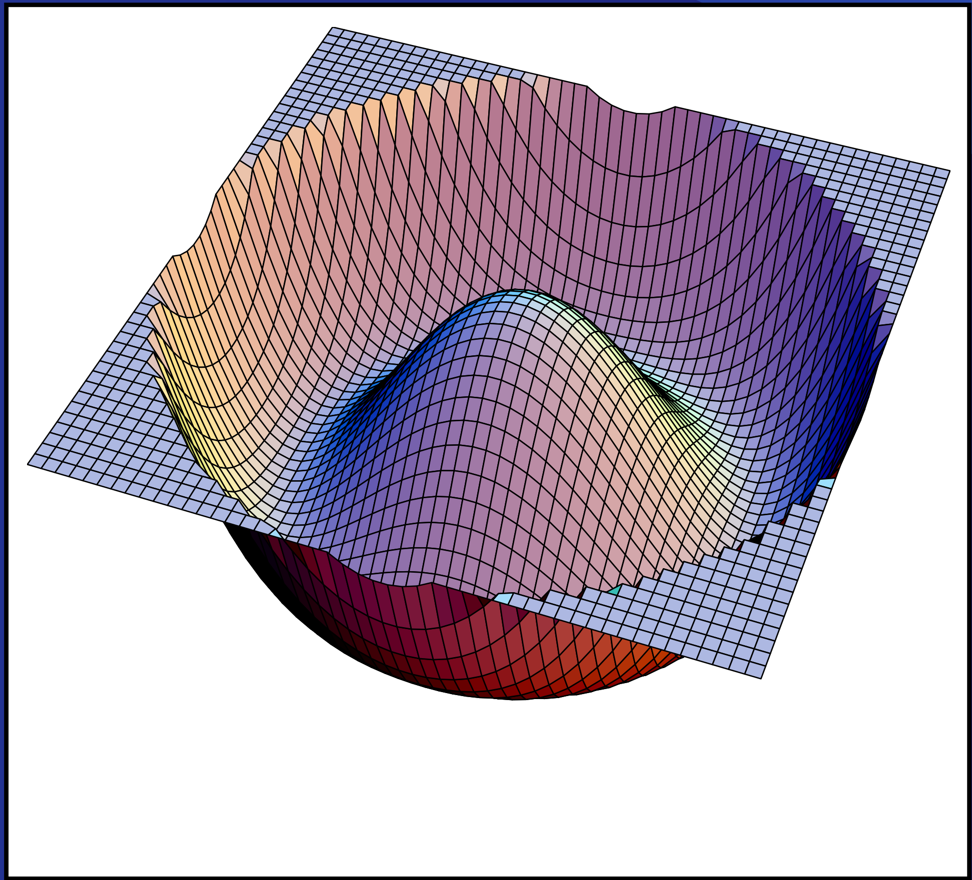
$$V(\phi) = \mu^2(\phi^\dagger \phi) + \lambda(\phi^\dagger \phi)^2$$

It is invariant under the global phase transformation

$$\phi \to \exp(i\alpha Q)\phi$$

For $\mu^2 > 0$ the minimum is at $|\phi_0| = 0\,, \;\to \phi = 0$ and we have the standard complex scalar field theory.

For $\mu^2 < 0$ the minimum is at $v = |\phi_0| = \sqrt{-\mu^2/2\lambda}$ and it is not unique. There is a continuum of degenerate states.



We are forced to choose one on the different minima.

Since all the minima are equivalent we can choose

$$\phi = \frac{(v + \phi_1' + i\phi_2')}{\sqrt{2}}$$

In terms of these new fields the Lagrangian becomes

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi_1' \partial^\mu \phi_1' - \frac{1}{2}(-2\mu^2){\phi_1'}^2 + \frac{1}{2}\partial_\mu \phi_2' \partial^\mu \phi_2' + \cdots$$

Which describes a scalar field ϕ_1' with real and positive mass squared ($-2\mu^2$) and a massless scalar boson, ϕ_2', the **Goldstone boson**.

Exercise

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With this **linear** parametrization of the fields the potential is a function of the two fields ϕ'_1 and ϕ'_2.
There is another, **non-linear**, parametrization of the field which is more physical:

$$\phi = \frac{(v + \rho(x))}{\sqrt{2}} e^{i\theta(x)/v}$$

then the potential is completely independent of $\theta(x)$. In addition the global symmetry in terms of $\theta(x)$ is just

$$\theta(x) \longrightarrow \theta(x) + \text{constant}$$

Then, the Lagrangian can only contain derivatives of $\theta(x)$ and it cannot contain a $\theta(x)$ mass term. We will see in action this parametrization when we discuss the Higgs mechanism.

Exercise (Goldstone Theorem)

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What we did is just an example of the prediction of the so called **Goldstone theorem** which states that when an exact continuous global symmetry is spontaneously broken, i.e. it is not a symmetry of the physical vacuum, the theory contains one massless scalar particle for each broken generator of the original symmetry group.
The Goldstone theorem can be proven as follows. Let us consider a Lagrangian of N_G real scalar fields ϕ_i, belonging to a N_G-dimensional vector Φ,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \Phi)(\partial^\mu \Phi) - V(\Phi)$$

Suppose that G is a continuous symmetry of the Lagrangian (for simplicity let us consider $SO(N_G)$ and that Φ transforms like (if Φ is real iT^a is real antisymmetric)

$$\delta\Phi = i\,\alpha^a\,T^a\,\Phi$$

the conserved current related to this symmetry is just

$$j_\mu^a = \left(\partial_\mu \Phi^T\right) iT^a\Phi\ , \qquad \partial^\mu j_\mu^a = 0$$

If the minimum of the potential is not zero, $\langle\Phi\rangle = \mathrm{constant} \neq 0$ and we expand around the true minimum

$$\Phi' = \Phi - \langle\Phi\rangle$$

Conservation of the current immediately implies

$$\partial^2 \left(\Phi^T T^a \langle \Phi \rangle \right) + \text{interaction terms} = 0$$

therefore the fields

$$\theta^a \equiv \Phi^T T^a \langle \Phi \rangle \ , \qquad \text{such that } T^a \langle \Phi \rangle \neq 0$$

satisfy the massless equation of motion $\Rightarrow$ **are massless**. One can also see this by looking at the scalar potential and expanding it about a true minimum of the potential. Then the mass matrix of the scalars, M is given by the second derivatives of the potential and one finds again

$$\mathrm{M}\, T^a \langle \Phi \rangle = 0$$

The Higgs Mechanism

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What if the symmetry is a local gauge symmetry?
Let us consider again the charged self-interacting scalar Lagrangian with the potential $V(\phi)$, and let us require a invariance under the local phase transformation,

$$\phi \rightarrow \exp\left[i\,\alpha(x) Q\right]\phi$$

In order to make the Lagrangian invariant, we introduce a gauge boson A_μ and the covariant derivative D_μ

$$\partial_\mu \longrightarrow D_\mu = \partial_\mu - ieQA_\mu$$

then the Lagrangian is

$$\mathcal{L} = \left(D_\mu \phi\right)^\dagger D^\mu \phi - V(\phi^\dagger \phi)$$

SSB occurs for $\mu^2 < 0$, with the vacuum $\langle|\phi|\rangle$ given as before. This time we will chose the exponential parametrization of the scalar field

$$\phi \equiv \frac{(v + \rho(x))}{\sqrt{2}} e^{i\theta(x)/v}$$

But now there is an important difference, since the symmetry is local we have that

$$\begin{aligned} \theta(x) &\rightarrow \theta'(x) = \theta(x) + Q\alpha(x)/v \\ A_\mu(x) &\rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \end{aligned}$$

leaves the Lagrangian invariant.

Without lose of generality, we can choose the gauge in such a way that $\theta(x) = 0$, removing it completely from the theory. In this gauge the Lagrangian is just

$$\mathcal{L} = \frac{1}{2} \left|\partial_\mu \rho - ieQA_\mu(v+\rho)\right|^2 - V(\frac{1}{2}(v+\rho)) - \frac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

When expanding this we immediately see that the gauge boson has obtained a mass

$$m_A^2 = e^2 Q^2 v^2$$

and will be described by a Proca field, while the $\theta(x)$ has disappeared from the spectrum. The total number of degrees of freedom, however, remains

Initial $\mathcal{L}$		Final $\mathcal{L}$	
ϕ charged scalar :	2	ρ neutral scalar :	1
A_μ massless vector :	2	A_μ massive vector :	3
	4		4

We say that the Goldstone boson has been eaten by the gauge boson to give him the longitudinal degree of freedom.

This can be generalized to non-Abelian groups:

The $N_G - N_{G'}$ Goldstone bosons will be eaten by $N_G - N_{G'}$ gauge bosons that will become massive.

The G' subgroup will remain unbroken with $N_{G'}$ massless gauge bosons.

We will see this mechanism in action when we discuss the SSB of the SM.