Asymptotic behavior of nonautonomous reaction-diffusion equations

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Benasque, 2005
The problem

We study nonautonomous parabolic equations of the form

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\begin{aligned}
u_t - \Delta u &= f(t, x, u) \quad \text{in } \Omega, \quad t > s \\
u &= 0, \quad \text{on } \Gamma = \partial \Omega \\
u(s) &= u_0 \geq 0.
\end{aligned}
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- \( f : \mathbb{R} \times \Omega \times \mathbb{R} \rightarrow \mathbb{R} \), Typically logistic nonlinearity:

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f(t, x, u) = m(t, x)u - n(t, x)u^\rho
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with \( n(t, x) \geq 0, \rho > 1 \).

- the initial data is in \( C(\overline{\Omega}) \)

- solutions \( u_f(t, s, x, u_0) \) defined for all \( t > s \).
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Part I

Some approaches to asymptotic behavior
The autonomous case: $f = f(x, u)$

- **Semigroup:** $X ⊋ u_0 \mapsto u_f(t, u_0) = S(t)u_0 \in X$
- **Smoothing and estimates:** compactness

$$\{S(t)u_0, \quad t \geq 1\} \text{ is compact}$$

- **$\omega$–limit sets:**

$$\omega(u_0) = \{v_0 \in X, \exists t_n \to \infty, S(t_n)u_0 \to v_0\}$$

If $B \subset X$

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- The global attractor: $\mathcal{A} \subset X$ compact, such that

$$S(t)\mathcal{A} = \mathcal{A}, \quad t \geq 0$$

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[Hale, Temam, Ladyzhenskaya .....]
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- Denote $H(f) = \text{cl}\{f_\tau(\cdot, \cdot, \cdot), \ \tau \in \mathbb{R}\}$ which is **compact** where
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  \textbf{skew product flow}
The general case: pullback attraction

For a given time $t$ and a state $u_0$
the evolution that started some time before

The state $\omega$ pullback attracts the state $u_0$: $\lim_{s \to -\infty} u_f(t, s, \cdot; u_0) = \omega$
The general case: pullback attraction

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The pullback attractor is a family

$$\mathcal{A} = \{\mathcal{A}(t)\}_{t \in \mathbb{R}}$$

such that for each $t \in \mathbb{R}$ and $B \subset X$ bounded

$$\lim_{s \to -\infty} \text{dist}(u(t, s; B), \mathcal{A}(t)) = 0$$
Theorem Assume $f$ satisfies

$$u f(t, x, u) \leq C(t, x)u^2 + D(t, x)|u| \quad \forall u \in \mathbb{R}$$

and $U_{\Delta + C}(t, s)$ is exponentially stable. Then there exist $\varphi_m(t) \leq \varphi_M(t)$ minimal and maximal complete trajectories, such that

1. Any complete trajectory $\psi$ satisfies

$$\varphi_m(t) \leq \psi(t) \leq \varphi_M(t);$$

2. $$\varphi_m(t) \leq \liminf_{s \to -\infty} u(t, s; v_0) \leq \limsup_{s \to -\infty} u(t, s; v_0) \leq \varphi_M(t)$$

uniformly in $\Omega$ for $v_0 \in B \subset X.$
[Robinson, Vidal-López, R-B, (2005)]

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Moreover

3. $\varphi_m(t)$ is stable from below in the pullback sense and $\varphi_M(t)$ is stable from above

4. there exists a pullback attractor and

$$\mathcal{A}(t) \subset [\varphi_m(t), \varphi_M(t)], \quad \varphi_m(t), \varphi_M(t) \in \mathcal{A}(t);$$

5. the interval $[\varphi_m(t), \varphi_M(t)]$ is positively invariant
The autonomous case

The periodic case

The almost-periodic case

The general case

Asymptotic behavior of nonautonomous reaction-diffusion equations
Part II

Positive solutions
The autonomous case: \( f = f(x, u) \)

Assume \( f(x, u) \) continuous in \( u \geq 0 \)

**Theorem (Brezis, Oswald)**

\[
\frac{f(x, u)}{u} \text{ is nonincreasing in } u \geq 0.
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Then there exists at most a positive solution.
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**Theorem**  
(Brezis, Oswald)

Assume for $0 \leq u \leq \delta$, $f(x, u) \geq -C_\delta u$

\[ f(\cdot, u) \in L^\infty(\Omega), \quad f(x, u) \leq C(u + 1) \]

\[ a_0(x) = \limsup_{u \to 0^+} \frac{f(x, u)}{u} \quad \text{and} \quad a_\infty(x) = \liminf_{u \to +\infty} \frac{f(x, u)}{u}. \]

Assume

\[ \lambda_1(-\Delta - a_0) < 0 < \lambda_1(-\Delta - a_\infty). \]

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*Assume*

$$\lambda_1(-\Delta - a_0) < 0 < \lambda_1(-\Delta - a_\infty).$$

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The proof is based on energy arguments using

\[ \lambda_1(-\Delta - a) = \inf_{v \in H^1_0(\Omega)} \frac{\int_{\Omega} |\nabla v|^2 - \int_{\Omega, v \neq 0} a(x)|v|^2}{\int_{\Omega} |v|^2} \]

\[ V(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 - \int_{\Omega} F(x, u) \]

\[ F(x, u) = \int_0^u f(x, r) \, dr \]

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The periodic case: \( f(t + T, x, u) = f(t, x, u) \)

For given \( t_0 \in \mathbb{R} \) one can use the Poincaré map

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• Inherits strong compactness and positivity properties.
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then \( P \) has a unique positive globally attractive fixed point.

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  \[ S(t)(u_0, g) = (u_g(t, \cdot, u_0), g_t) \quad t \geq 0 \]

  **skew product flow**
The almost periodic case

- If

\[
\frac{f(t, x, u)}{u}
\]

is nonincreasing in \( u \geq 0 \).

then \( S \) has a unique positive globally attractive fixed point.

The special positive solution

The trivial state and the special solution
At each time the state $u_S(t)$ is the important one because it is the pullback attractor.
But also $u_S$ attracts forwards.

---

$u = 0$

time
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A. Rodríguez-Bernal

Asymptotic behavior of nonautonomous reaction-diffusion equations
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A result with restrictions

[Langa, Robinson, Suarez, Int.J.Bif.Chaos (2005)]

\[ f(t, x, u) = \lambda u - n(x, t)u^\rho \]

\[ \lambda > \lambda_1^D(\Omega) \]

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Warning for uniqueness: autonomous case

A positive equilibrium: $u_E$ and a connection from 0 to $u_E$
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Hence there are two complete positive trajectories
Uniqueness of CBNDS: complete bounded nondegenerate solution

\[ \varphi_0, \varphi_1 \in C^1_0(\Omega), \quad 0 < \varphi_0 \leq u_S(t) \leq \varphi_1, \quad t << -1 \]
Theorem  (ARB-A.Vidal–López, 2005)

Assume

\[ f(t, x, u) = m(x, t)u - n(x, t)u^p, \quad n(x, t) \geq 0 \]

\[ n(x, t) \geq a_0 > 0 \quad \text{in} \quad \bar{\Omega} \times \mathbb{R} \quad (\text{can be weakened}) \]

\[ m \in L^\infty(\mathbb{R}, L^p(\Omega)) \quad \text{with} \quad p > N \]

for \( t << -1 \)

\[ m(x, t) \geq M(x) \quad n(x, t) \leq N(x) \]

such that

\[ f_0(x) = M(x)u - N(x)u^p \leq f(t, x, u) \]

is of “Brezis-Oswald” type. Then there exists a unique CBPND trajectory

\[ 0 \leq u_s \in C_b(\mathbb{R}, C^0(\bar{\Omega})) \]
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\[ 0 \leq u_s \in C_b(\mathbb{R}, C_0(\overline{\Omega})). \]
Definition

i) If $X$ a Banach space, the family $T(t, s) \in \mathcal{L}(X)$, is an evolution operator
   
   a) $T(t, t) = I$ for all $t \in \mathbb{R}$,
   
   b) $T(t, s)T(s, r)u = T(t, r)u$ for all $r \leq s \leq t$, $u \in X$, and
   
   c) $u \mapsto T(t, r)u$ is continuous in $X$ for $t > r$.

ii) $T(t, s)$ is exponentially stable of exponent $\beta > 0$ if for some $M > 0$

\[
\|T(t, s)\|_{\mathcal{L}(X)} \leq Me^{-\beta(t-s)} \quad \text{for all } t > s.
\]