

Topological Sensitivity Analysis for Three-dimensional Linear Elasticity Problem

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Abstract

In this work we use the Topological-Shape Sensitivity Method to obtain the topological derivative for three-dimensional linear elasticity problems, adopting the total potential energy as the cost function and the equilibrium equation as the constraint. This method, based on classical shape sensitivity analysis, leads to a systematic procedure to compute the topological derivative. In particular, firstly we present the mechanical model, later we perform the shape derivative of the corresponding cost function and, finally, we compute the final expression for the topological derivative using the Topological-Shape Sensitivity Method and results from classical asymptotic analysis around spherical cavities.

1 Introduction

The topological derivative has been recognized as a promising tool to solve topology optimization problems (see [3], where 425 references concerning topology optimization of continuum structures are included). See also [4, 8, 24] and references therein. Nevertheless, this concept is wider. In fact, the topological derivative may also be applied to solve inverse problems and to simulate physical phenomena with changes on the configuration of the domain of the problem. In addition, extension of the topological derivative in order to include arbitrary shaped holes and its applications to Laplace, Poisson, Helmholtz, Navier, Stokes and Navier-Stokes equations were developed by Masmoudi and his co-workers and by Sokolowsky and his co-workers (see, for instance, [18] for applications of the topological derivative in the context of topology design and inverse problems).

On the other hand, although the topological derivative is extremely general, this concept may become restrictive due to mathematical difficulties involved in its calculation. However, several approaches to compute the topological derivative may be found in the literature. In particular, we proposed an alternative method based on classical shape sensitivity analysis (see [1, 13, 14, 22, 25, 26, 27] and references therein). This approach, called Topological-Shape Sensitivity Method, was already applied in the following two-dimensional problems:

- Poisson: steady-state heat conduction problem taking into account both homogeneous and non-homogeneous Neumann and Dirichlet and also Robin boundary conditions on the hole [6, 20];
- Navier: plane stress and plane strain linear elasticity [7];
- Kirchhoff: thin plate bending problem [21];

More specifically, we have respectively considered scalar second-order, vector second-order and scalar forth-order PDE two-dimensional problems. As a natural sequence of our work, therefore, in the present paper we apply the Topological-Shape Sensitivity Method to compute the topological derivative in a vector second-order PDE three-dimensional problem. In particular, we consider the three-dimensional linear elasticity problem taking the total potential energy as the cost function and the state equation as the constraint. Therefore, for the sake of completeness, in Section 2 we present a short description of the Topological-Shape Sensitivity Method. In Section 3 we use this approach to compute the topological derivative for the problem under consideration: in Section 3.1 we present the mechanical model associated to three-dimensional linear elasticity; in Section 3.2 we compute the shape derivative for this problem adopting the total potential energy as the cost function and the weak form of the state equation as the constraint and in Section 3.3, we compute the final expression for the topological derivative using classical asymptotic analysis around spherical cavities. Finally, it is important to mention that the obtained result can be applied in several engineering problems such as topology optimization of three-dimensional linear elastic structures.

2 Topological-Shape Sensitivity Method

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$. If the domain Ω is perturbed by introducing a small hole at an arbitrary point $\hat{\mathbf{x}} \in \Omega$, we have a new domain $\Omega_\varepsilon = \Omega - \overline{B}_\varepsilon$, whose boundary is denoted by $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$, where $\overline{B}_\varepsilon = B_\varepsilon \cup \partial B_\varepsilon$ is a ball of radius ε centered at point $\hat{\mathbf{x}} \in \Omega$. Therefore, we have the original domain without hole Ω and the new one Ω_ε with a small hole B_ε as shown in fig. (1). Thus, considering a cost function ψ defined in both domains, its topological derivative is given in [8] as

$$D_T(\hat{\mathbf{x}}) = \lim_{\varepsilon \rightarrow 0} \frac{\psi(\Omega_\varepsilon) - \psi(\Omega)}{f(\varepsilon)}, \quad (1)$$

where $f(\varepsilon)$ is a negative function that decreases monotonically so that $f(\varepsilon) \rightarrow 0$ with $\varepsilon \rightarrow 0^+$.

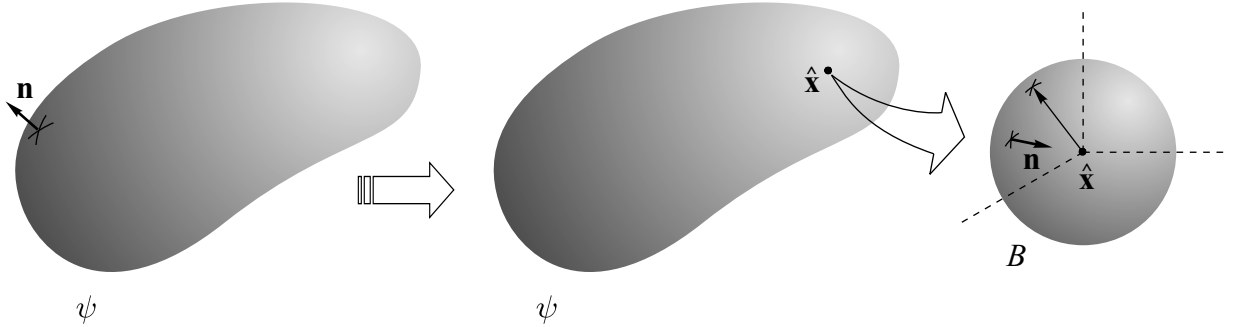


Figure 1: topological derivative concept

Recently an alternative procedure to compute the topological derivative, called Topological-Shape Sensitivity Method, have been introduced by the authors [20]. This approach makes use of the whole mathematical framework (and results) developed for shape sensitivity analysis (see, for instance, the pioneer work of Murat & Simon [17]). The main result obtained in [20] may be briefly summarized in the following Theorem (see also [6, 19]):

Theorem 1 *Let $f(\varepsilon)$ be a function chosen in order to $0 < |D_T(\hat{\mathbf{x}})| < \infty$, then the topological derivative given by eq. (1) can be written as*

$$D_T(\hat{\mathbf{x}}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{f'(\varepsilon)} \left. \frac{d}{d\tau} \psi(\Omega_\tau) \right|_{\tau=0}, \quad (2)$$

where $\tau \in \mathbb{R}^+$ is used to parameterize the domain. That is, for τ small enough, we have

$$\Omega_\tau := \{ \mathbf{x}_\tau \in \mathbb{R}^3 : \mathbf{x}_\tau = \mathbf{x} + \tau \mathbf{v}, \mathbf{x} \in \Omega_\varepsilon \} . \quad (3)$$

Therefore, $\mathbf{x}_\tau|_{\tau=0} = \mathbf{x}$ and $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon$. In addition, considering that \mathbf{n} is the outward normal unit vector (see fig. 1), then we can define the shape change velocity \mathbf{v} , which is a smooth vector field in Ω_ε assuming the following values on the boundary $\partial\Omega_\varepsilon$

$$\begin{cases} \mathbf{v} = -\mathbf{n} & \text{on } \partial B_\varepsilon \\ \mathbf{v} = \mathbf{0} & \text{on } \partial\Omega \end{cases} \quad (4)$$

and the shape sensitivity of the cost function in relation to the domain perturbation characterized by \mathbf{v} is given by

$$\left. \frac{d}{d\tau} \psi(\Omega_\tau) \right|_{\tau=0} = \lim_{\tau \rightarrow 0} \frac{\psi(\Omega_\tau) - \psi(\Omega_\varepsilon)}{\tau} . \quad (5)$$

Proof. Re-writing eq. (1) like a Taylor series expansion we have

$$\psi(\Omega_\varepsilon) = \psi(\Omega) + f(\varepsilon) D_T(\hat{\mathbf{x}}) + R(f(\varepsilon)) , \quad (6)$$

where $R(f(\varepsilon))$ contains all higher order terms than $f(\varepsilon)$, that is, it satisfies

$$R(f(\varepsilon)) : \lim_{\varepsilon \rightarrow 0} \frac{R(f(\varepsilon))}{f(\varepsilon)} = 0 . \quad (7)$$

Let us take the derivative in relation to ε in both sides of eq. (6) to obtain

$$\frac{d}{d\varepsilon} \psi(\Omega_\varepsilon) = f'(\varepsilon) D_T(\hat{\mathbf{x}}) + R'(f(\varepsilon)) f'(\varepsilon) , \quad (8)$$

where, from eq. (5), we observe, for $\tau \in \mathbb{R}^+$ small enough, that

$$\frac{d}{d\varepsilon} \psi(\Omega_\varepsilon) = \lim_{\tau \rightarrow 0} \frac{\psi(\Omega_\tau) - \psi(\Omega_\varepsilon)}{\tau} = \left. \frac{d}{d\tau} \psi(\Omega_\tau) \right|_{\tau=0} . \quad (9)$$

Considering the shape derivative of the cost function given by above expression (eq. 9) and rearranging eq. (8) we obtain

$$\frac{1}{f'(\varepsilon)} \left. \frac{d}{d\tau} \psi(\Omega_\tau) \right|_{\tau=0} = D_T(\hat{\mathbf{x}}) + R'(f(\varepsilon)) . \quad (10)$$

Finally, taking the limit $\varepsilon \rightarrow 0$ in eq. (10) and considering the definition of $R(f(\varepsilon))$ given by eq. (7), we observe that

$$\lim_{\varepsilon \rightarrow 0} R'(f(\varepsilon)) = 0 \quad \Rightarrow \quad D_T(\hat{\mathbf{x}}) = \lim_{\varepsilon \rightarrow 0} \frac{1}{f'(\varepsilon)} \left. \frac{d}{d\tau} \psi(\Omega_\tau) \right|_{\tau=0} \quad (11)$$

and we get the proof of the Theorem ■

Observe that the topological derivative given by eq. (1) can be seen as an extension of classical shape derivative, but with the mathematical difficulty concerning the lack of homeomorphism between Ω and Ω_ε . On the other hand, the above Theorem highlights that the topological derivative may be obtained by means of shape sensitivity analysis. Consequently, Topological-Shape Sensitivity Method leads to a systematic approach to compute the topological derivative of the cost function ψ considering eq. (2). In fact, the domains Ω_ε and Ω_τ have the same topology, that allow us to build an homeomorphic map between them. In addition, Ω_ε and Ω_τ can be respectively seen as the material and the spatial configurations. Therefore, in order to compute the shape derivative of the cost function (see eq. 5) we can use classical results from Continuum Mechanics like the Reynolds' transport theorem and the concept of material derivatives of spatial fields [11]. Finally, in this work we will show these features in the context of three-dimensional elasticity.

3 The topological derivative in three-dimensional linear elasticity

Now, to highlight the potentialities of the Topological-Shape Sensitivity Method, it will be applied to three-dimensional linear elasticity problems considering the total potential energy as the cost function and the equilibrium equation in its weak form as the constraint. Therefore, considering the above problem, firstly we introduce the mechanical model, later we perform the shape sensitivity of the adopted cost function with respect to the shape change of the hole and finally we compute the associated topological derivative.

3.1 Mechanical model

In this work, we consider a mechanical model restricted to small deformation and displacement and for the constitutive relation we adopt an isotropic linear elastic material. These assumptions lead to the classical three-dimensional linear elasticity theory [10]. In order to compute the topological derivative associated to this problem, we need to state the equilibrium equations in the original domain Ω (without hole) and in the new one Ω_ε (with hole).

3.1.1 Problem formulation in the original domain without hole

The mechanical model associated to the three-dimensional linear elasticity problem can be stated in its variational formulation as following: find the displacement vector field $\mathbf{u} \in \mathcal{U}$, such that

$$\int_{\Omega} \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\boldsymbol{\eta}) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta} \quad \forall \boldsymbol{\eta} \in \mathcal{V}, \quad (12)$$

where Ω represents a deformable body with boundary $\partial\Omega = \Gamma_N \cup \Gamma_D$, such that $\Gamma_N \cap \Gamma_D = \emptyset$, submitted to a set of surface forces $\bar{\mathbf{q}}$ on the Neumann boundary Γ_N and displacement constraints $\bar{\mathbf{u}}$ on the Dirichlet boundary Γ_D . Therefore, assuming that $\bar{\mathbf{q}} \in L^2(\Gamma_N)$, the admissible functions set \mathcal{U} and the admissible variations space \mathcal{V} are given, respectively, by

$$\mathcal{U} = \{\mathbf{u} \in H^1(\Omega) : \mathbf{u} = \bar{\mathbf{u}} \text{ on } \Gamma_D\}, \quad \mathcal{V} = \{\boldsymbol{\eta} \in H^1(\Omega) : \boldsymbol{\eta} = \mathbf{0} \text{ on } \Gamma_D\}. \quad (13)$$

In addition, the linearized Green deformation tensor $\mathbf{E}(\mathbf{u})$ and the Cauchy stress tensor $\mathbf{T}(\mathbf{u})$ are defined as

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2} (\nabla \mathbf{u} + \nabla \mathbf{u}^T) := \nabla^s \mathbf{u} \quad \text{and} \quad \mathbf{T}(\mathbf{u}) = \mathbf{C} \mathbf{E}(\mathbf{u}) = \mathbf{C} \nabla^s \mathbf{u}, \quad (14)$$

where $\mathbf{C} = \mathbf{C}^T$ is the elasticity tensor, that is, since \mathbf{I} and \mathbf{II} respectively are the second and forth order identity tensors, E is the Young's modulus and ν is the Poisson's ratio, we have

$$\mathbf{C} = \frac{E}{(1+\nu)(1-2\nu)} [(1-2\nu)\mathbf{II} + \nu(\mathbf{I} \otimes \mathbf{I})] \Rightarrow \mathbf{C}^{-1} = \frac{1}{E} [(1+\nu)\mathbf{II} - \nu(\mathbf{I} \otimes \mathbf{I})]. \quad (15)$$

The Euler-Lagrange equation associated to the above variational problem, eq. (12), is given by the following boundary value problem: find \mathbf{u} such that

$$\begin{cases} \operatorname{div} \mathbf{T}(\mathbf{u}) = \mathbf{0} & \text{in } \Omega \\ \mathbf{u} = \bar{\mathbf{u}} & \text{on } \Gamma_D \\ \mathbf{T}(\mathbf{u})\mathbf{n} = \bar{\mathbf{q}} & \text{on } \Gamma_N \end{cases}. \quad (16)$$

3.1.2 Problem formulation in the new domain with hole

The problem stated in the original domain Ω can also be written in the domain Ω_ε with a hole B_ε . Therefore, assuming null forces on the hole, we have the following variational problem: find the displacement vector field $\mathbf{u}_\varepsilon \in \mathcal{U}_\varepsilon$, such that

$$\int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\boldsymbol{\eta}_\varepsilon) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta}_\varepsilon \quad \forall \boldsymbol{\eta}_\varepsilon \in \mathcal{V}_\varepsilon . \quad (17)$$

where the set \mathcal{U}_ε and the space \mathcal{V}_ε are respectively defined as

$$\mathcal{U}_\varepsilon = \{ \mathbf{u}_\varepsilon \in H^1(\Omega_\varepsilon) : \mathbf{u}_\varepsilon = \bar{\mathbf{u}} \text{ on } \Gamma_D \} , \quad \mathcal{V}_\varepsilon = \{ \boldsymbol{\eta}_\varepsilon \in H^1(\Omega_\varepsilon) : \boldsymbol{\eta}_\varepsilon = \mathbf{0} \text{ on } \Gamma_D \} . \quad (18)$$

As seen before, the tensors $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)$ and $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)$ are respectively given as

$$\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \nabla^s \mathbf{u}_\varepsilon \quad \text{and} \quad \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{C} \nabla^s \mathbf{u}_\varepsilon , \quad (19)$$

where the elasticity tensor \mathbf{C} is defined in eq. (15). In accordance with the variational problem given by eq. (17), the natural boundary condition on ∂B_ε is $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{n} = \mathbf{0}$ (homogeneous Neumann condition). Therefore, the Euler-Lagrange equation associated to this new variational problem is given by the following boundary value problem: find \mathbf{u}_ε such that

$$\begin{cases} \operatorname{div} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{0} & \text{in } \Omega_\varepsilon \\ \mathbf{u}_\varepsilon = \bar{\mathbf{u}} & \text{on } \Gamma_D \\ \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{n} = \bar{\mathbf{q}} & \text{on } \Gamma_N \\ \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{n} = \mathbf{0} & \text{on } \partial B_\varepsilon \end{cases} . \quad (20)$$

3.2 Shape sensitivity analysis

Let us choose the total potential energy stored in the elastic solid under analysis as the cost function. For simplicity, we assume that the external load remains fixed during the shape change. As it is well-known, different approaches can be used to obtain the shape derivative of the cost function. However, in our particular case, as the cost function is associated with the potential of the state equation, the direct differentiation method will be adopted to compute its shape derivative. Therefore, considering the total potential energy already written in the configuration Ω_τ , defined through eq. (3), then $\psi(\Omega_\tau) := \mathcal{J}_\tau(\mathbf{u}_\tau)$ can be expressed by

$$\mathcal{J}_\tau(\mathbf{u}_\tau) = \frac{1}{2} \int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau) \cdot \mathbf{E}_\tau(\mathbf{u}_\tau) - \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \mathbf{u}_\tau , \quad (21)$$

where the tensors $\mathbf{E}_\tau(\mathbf{u}_\tau)$ and $\mathbf{T}_\tau(\mathbf{u}_\tau)$ are respectively given by

$$\mathbf{E}_\tau(\mathbf{u}_\tau) = \nabla_\tau^s \mathbf{u}_\tau \quad \text{and} \quad \mathbf{T}_\tau(\mathbf{u}_\tau) = \mathbf{C} \nabla_\tau^s \mathbf{u}_\tau , \quad (22)$$

with $\nabla_\tau(\cdot)$ used to denote

$$\nabla_\tau(\cdot) := \frac{\partial}{\partial \mathbf{x}_\tau}(\cdot) . \quad (23)$$

In addition, \mathbf{u}_τ is the solution of the variational problem defined in the configuration Ω_τ , that is: find the displacement vector field $\mathbf{u}_\tau \in \mathcal{U}_\tau$ such that

$$\int_{\Omega_\tau} \mathbf{T}_\tau(\mathbf{u}_\tau) \cdot \mathbf{E}_\tau(\boldsymbol{\eta}_\tau) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta}_\tau \quad \forall \boldsymbol{\eta}_\tau \in \mathcal{V}_\tau , \quad (24)$$

where the set \mathcal{U}_τ and the space \mathcal{V}_τ are defined as

$$\mathcal{U}_\tau = \{ \mathbf{u}_\tau \in H^1(\Omega_\tau) : \mathbf{u}_\tau = \bar{\mathbf{u}} \text{ on } \Gamma_D \} , \quad \mathcal{V}_\tau = \{ \boldsymbol{\eta}_\tau \in H^1(\Omega_\tau) : \boldsymbol{\eta}_\tau = \mathbf{0} \text{ on } \Gamma_D \} . \quad (25)$$

Observe that from the well-known terminology of Continuum Mechanics, the domains $\Omega_\tau|_{\tau=0} = \Omega_\varepsilon$ and Ω_τ can be interpreted as the material and the spatial configurations, respectively. Therefore, in order to compute the shape derivative of the cost function $\mathcal{J}_\tau(\mathbf{u}_\tau)$, at $\tau = 0$, we may use the Reynolds' transport theorem and the concept of material derivatives of spatial fields, that is [11]

$$\frac{d}{d\tau} \int_{\Omega_\tau} \varphi_\tau \Big|_{\tau=0} = \int_{\Omega_\varepsilon} (\dot{\varphi}_\tau|_{\tau=0} + \varphi_\tau|_{\tau=0} \operatorname{div} \mathbf{v}) , \quad (26)$$

where φ_τ is a spatial scalar field and $(\dot{\cdot})$ is used to denote

$$(\dot{\cdot}) := \frac{d(\cdot)}{d\tau} . \quad (27)$$

Taking into account the cost function defined through eq. (21) and assuming that the parameters E , ν , $\bar{\mathbf{u}}$, and $\bar{\mathbf{q}}$ are constants in relation to the perturbation represented by τ , we have, from eq. (26) and following Theorem 1, eqs. (3,4), that

$$\frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \Big|_{\tau=0} = \frac{1}{2} \int_{\Omega_\varepsilon} \left[\frac{d}{d\tau} (\mathbf{T}_\tau(\mathbf{u}_\tau) \cdot \mathbf{E}_\tau(\mathbf{u}_\tau)) \Big|_{\tau=0} + \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \operatorname{div} \mathbf{v} \right] - \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \dot{\mathbf{u}}_\varepsilon , \quad (28)$$

where, according to the material derivatives of spatial fields [11], we have

$$\frac{d}{d\tau} (\mathbf{T}_\tau(\mathbf{u}_\tau) \cdot \mathbf{E}_\tau(\mathbf{u}_\tau)) \Big|_{\tau=0} = 2 (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot (\nabla \mathbf{u}_\varepsilon \nabla \mathbf{v})^s) . \quad (29)$$

Substituting eq. (29) in eq. (28) we obtain

$$\begin{aligned} \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \Big|_{\tau=0} &= \int_{\Omega_\varepsilon} \left[\frac{1}{2} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) \operatorname{div} \mathbf{v} - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot (\nabla \mathbf{u}_\varepsilon \nabla \mathbf{v})^s \right] \\ &\quad + \int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \dot{\mathbf{u}}_\varepsilon . \end{aligned} \quad (30)$$

Since \mathbf{u}_ε is the solution of the variational problem given by eq. (17) and considering that $\dot{\mathbf{u}}_\varepsilon \in \mathcal{V}_\varepsilon$, the eq. (30) becomes

$$\frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \Big|_{\tau=0} = \int_{\Omega_\varepsilon} \boldsymbol{\Sigma}_\varepsilon \cdot \nabla \mathbf{v} , \quad (31)$$

where $\boldsymbol{\Sigma}_\varepsilon$ is the Eshelby energy-momentum tensor (see, for instance, [5, 26]) given in this particular case by

$$\boldsymbol{\Sigma}_\varepsilon = \frac{1}{2} (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)) \mathbf{I} - (\nabla \mathbf{u}_\varepsilon)^T \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) . \quad (32)$$

Remark 2 *It is interesting to observe that the Eshelby tensor $\boldsymbol{\Sigma}_\varepsilon$ appears as a duality pair with respect to $\nabla \mathbf{v}$, as can be seen in eq. (31). This fact allow us to interpret $\boldsymbol{\Sigma}_\varepsilon$ as the set of configurational forces [12] associated to the change in the configuration of Ω_ε characterized by $\nabla \mathbf{v}$.*

Let us compute again the shape derivative of the cost function $\mathcal{J}_\tau(\mathbf{u}_\tau)$ defined through eq. (21), at $\tau = 0$, using another version for the Reynolds' transport theorem [11], that is,

$$\frac{d}{d\tau} \int_{\Omega_\tau} \varphi_\tau \Big|_{\tau=0} = \int_{\Omega_\varepsilon} \varphi'_\tau|_{\tau=0} + \int_{\partial\Omega_\varepsilon} \varphi_\tau|_{\tau=0} (\mathbf{v} \cdot \mathbf{n}) , \quad (33)$$

where φ_τ is a spatial scalar field and $(\cdot)'$ is used to denote

$$(\cdot)' := \frac{\partial(\cdot)}{\partial\tau} = \frac{d(\cdot)}{d\tau} \Big|_{\mathbf{x}_\tau \text{ fixed}} . \quad (34)$$

Which results in

$$\left. \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \right|_{\tau=0} = \frac{1}{2} \int_{\partial\Omega_\varepsilon} (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)) (\mathbf{v} \cdot \mathbf{n}) + \frac{1}{2} \int_{\Omega_\varepsilon} \frac{\partial}{\partial \tau} (\mathbf{T}_\tau(\mathbf{u}_\tau) \cdot \mathbf{E}_\tau(\mathbf{u}_\tau)) \Big|_{\tau=0} - \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \dot{\mathbf{u}}_\varepsilon, \quad (35)$$

where $\dot{\mathbf{u}}_\varepsilon$ can be written as [11]

$$\dot{\mathbf{u}}_\varepsilon = \mathbf{u}'_\varepsilon + (\nabla \mathbf{u}_\varepsilon) \mathbf{v} \quad \Rightarrow \quad \mathbf{u}'_\varepsilon = \dot{\mathbf{u}}_\varepsilon - (\nabla \mathbf{u}_\varepsilon) \mathbf{v}. \quad (36)$$

Taking into account the notation introduced through eq. (34) and from eq. (36), we have

$$\begin{aligned} \left. \frac{\partial}{\partial \tau} (\mathbf{T}_\tau(\mathbf{u}_\tau) \cdot \mathbf{E}_\tau(\mathbf{u}_\tau)) \right|_{\tau=0} &= 2\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}'_\varepsilon) \\ &= 2(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\varphi_\varepsilon)), \end{aligned} \quad (37)$$

where

$$\varphi_\varepsilon = (\nabla \mathbf{u}_\varepsilon) \mathbf{v} \quad \Rightarrow \quad \mathbf{E}_\varepsilon(\varphi_\varepsilon) = \nabla^s \varphi_\varepsilon. \quad (38)$$

Substituting eq. (37) in eq. (35) we obtain

$$\begin{aligned} \left. \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \right|_{\tau=0} &= \frac{1}{2} \int_{\partial\Omega_\varepsilon} (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)) (\mathbf{v} \cdot \mathbf{n}) - \int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\varphi_\varepsilon) \\ &\quad + \int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\dot{\mathbf{u}}_\varepsilon) - \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \dot{\mathbf{u}}_\varepsilon \\ &= \frac{1}{2} \int_{\partial\Omega_\varepsilon} (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)) (\mathbf{n} \cdot \mathbf{v}) - \int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\varphi_\varepsilon), \end{aligned} \quad (39)$$

since $\dot{\mathbf{u}}_\varepsilon \in \mathcal{V}_\varepsilon$ and \mathbf{u}_ε is the solution of eq. (17). In addition, we observe that

$$\int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\varphi_\varepsilon) = \int_{\partial\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \varphi_\varepsilon \cdot \mathbf{n} - \int_{\Omega_\varepsilon} \operatorname{div}(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)) \cdot \varphi_\varepsilon. \quad (40)$$

Considering this last result (eq. 40) in eq. (39) and taking into account again that \mathbf{u}_ε is the solution of eq. (20), we have

$$\begin{aligned} \left. \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \right|_{\tau=0} &= \frac{1}{2} \int_{\partial\Omega_\varepsilon} (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)) (\mathbf{v} \cdot \mathbf{n}) - \int_{\partial\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \varphi_\varepsilon \cdot \mathbf{n} \\ &= \int_{\partial\Omega_\varepsilon} \left[\frac{1}{2} (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon)) \mathbf{I} - (\nabla \mathbf{u}_\varepsilon)^T \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \right] \mathbf{n} \cdot \mathbf{v} \\ &= \int_{\partial\Omega_\varepsilon} \boldsymbol{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{v}, \end{aligned} \quad (41)$$

remembering that $\boldsymbol{\Sigma}_\varepsilon$ and φ_ε are respectively given by eq. (32) and eq. (38).

On the other hand, taking into account eq. (31) and considering the tensorial relation

$$\operatorname{div}(\boldsymbol{\Sigma}_\varepsilon^T \mathbf{v}) = \boldsymbol{\Sigma}_\varepsilon \cdot \nabla \mathbf{v} + \operatorname{div} \boldsymbol{\Sigma}_\varepsilon \cdot \mathbf{v}, \quad (42)$$

we can apply the divergence theorem to obtain

$$\left. \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \right|_{\tau=0} = \int_{\partial\Omega_\varepsilon} \boldsymbol{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{v} - \int_{\Omega_\varepsilon} \operatorname{div} \boldsymbol{\Sigma}_\varepsilon \cdot \mathbf{v}. \quad (43)$$

Thus, from eqs. (43,41) we observe that the Eshelby tensor has null divergence. In fact, since \mathbf{v} is an arbitrary velocity field, then from the fundamental theorem of the calculus of variations it is straightforward to verify that

$$\int_{\Omega_\varepsilon} \operatorname{div} \boldsymbol{\Sigma}_\varepsilon \cdot \mathbf{v} = 0 \quad \forall \mathbf{v} \quad \Leftrightarrow \quad \operatorname{div} \boldsymbol{\Sigma}_\varepsilon = \mathbf{0} \quad (44)$$

and the shape derivative of the cost function $\mathcal{J}_\tau(\mathbf{u}_\tau)$ defined through eq. (21), at $\tau = 0$, becomes an integral defined on the boundary $\partial\Omega_\varepsilon$, that is,

$$\left. \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \right|_{\tau=0} = \int_{\partial\Omega_\varepsilon} \boldsymbol{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{v} . \quad (45)$$

In other words, if the velocity field \mathbf{v} is smooth enough in the domain Ω_ε , then the shape sensitivity of the problem only depends on the definition of this field on the boundary $\partial\Omega_\varepsilon$.

3.3 Topological sensitivity analysis

In order to compute the topological derivative using the Topological-Shape Sensitivity Method, we need to substitute eq. (45) in the result of Theorem 1 (eq. 2). Therefore, from the definition of the velocity field (eq. 4) and considering the shape derivative of the cost function (eq. 45), we have that

$$\left. \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \right|_{\tau=0} = - \int_{\partial B_\varepsilon} \boldsymbol{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{n} , \quad (46)$$

where

$$\boldsymbol{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{n} = \frac{1}{2} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) - \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{n} \cdot (\nabla \mathbf{u}_\varepsilon) \mathbf{n} . \quad (47)$$

In addition, taking into account homogeneous Neumann boundary condition on the hole, we have, from eq. (20), that $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{n} = \mathbf{0}$ on ∂B_ε , therefore

$$\left. \frac{d}{d\tau} \mathcal{J}_\tau(\mathbf{u}_\tau) \right|_{\tau=0} = - \frac{1}{2} \int_{\partial B_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) . \quad (48)$$

Finally, substituting eq. (48) in the result of the Theorem 1 (eq. 2), the topological derivative becomes

$$D_T(\hat{\mathbf{x}}) = - \frac{1}{2} \lim_{\varepsilon \rightarrow 0} \frac{1}{f'(\varepsilon)} \int_{\partial B_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) . \quad (49)$$

Considering the inverse of the constitutive relation $\mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \mathbf{C}^{-1} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)$ (see eq. 15), then the integrand of eq. (49) may be expressed as a function of the stress tensor as following

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\mathbf{u}_\varepsilon) = \frac{1}{E} \left[(1 + \nu) \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) - \nu (\text{tr} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))^2 \right] . \quad (50)$$

Let us introduce a spherical coordinate system (r, θ, φ) centered in $\hat{\mathbf{x}}$ (see fig. 2), then the stress tensor $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) = (\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon))^T$, when defined on the boundary ∂B_ε , can be decomposed as

$$\begin{aligned} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)|_{\partial B_\varepsilon} &= T_\varepsilon^{rr}(\mathbf{e}_r \otimes \mathbf{e}_r) + T_\varepsilon^{r\theta}(\mathbf{e}_r \otimes \mathbf{e}_\theta) + T_\varepsilon^{r\varphi}(\mathbf{e}_r \otimes \mathbf{e}_\varphi) \\ &\quad + T_\varepsilon^{\theta r}(\mathbf{e}_\theta \otimes \mathbf{e}_r) + T_\varepsilon^{\theta\theta}(\mathbf{e}_\theta \otimes \mathbf{e}_\theta) + T_\varepsilon^{\theta\varphi}(\mathbf{e}_\theta \otimes \mathbf{e}_\varphi) \\ &\quad + T_\varepsilon^{\varphi r}(\mathbf{e}_\varphi \otimes \mathbf{e}_r) + T_\varepsilon^{\varphi\theta}(\mathbf{e}_\varphi \otimes \mathbf{e}_\theta) + T_\varepsilon^{\varphi\varphi}(\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi) , \end{aligned} \quad (51)$$

where \mathbf{e}_r , \mathbf{e}_θ and \mathbf{e}_φ are the basis of the spherical coordinate system such that

$$\mathbf{e}_r \cdot \mathbf{e}_r = \mathbf{e}_\theta \cdot \mathbf{e}_\theta = \mathbf{e}_\varphi \cdot \mathbf{e}_\varphi = 1 \quad \text{and} \quad \mathbf{e}_r \cdot \mathbf{e}_\theta = \mathbf{e}_r \cdot \mathbf{e}_\varphi = \mathbf{e}_\theta \cdot \mathbf{e}_\varphi = 0 . \quad (52)$$

Since we have homogeneous Neumann boundary condition on ∂B_ε , then

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{n} = \mathbf{0} \quad \Rightarrow \quad \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{e}_r = \mathbf{0} \quad \text{on} \quad \partial B_\varepsilon . \quad (53)$$

From the decomposition of the stress tensor shown in eq. (51) and taking into account eqs. (52,53), we observe that

$$\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \mathbf{e}_r = T_\varepsilon^{rr} \mathbf{e}_r + T_\varepsilon^{r\theta} \mathbf{e}_\theta + T_\varepsilon^{r\varphi} \mathbf{e}_\varphi = \mathbf{0} \quad \Rightarrow \quad T_\varepsilon^{rr} = T_\varepsilon^{r\theta} = T_\varepsilon^{r\varphi} = 0 . \quad (54)$$

Substituting eqs. (51,54) into eq. (50), the topological derivative given by eq. (49) may be written in terms of the components of the stress tensor in spherical coordinate, as following

$$\begin{aligned} D_T(\hat{\mathbf{x}}) &= -\frac{1}{2E} \lim_{\varepsilon \rightarrow 0} \frac{1}{f'(\varepsilon)} \int_{\partial B_\varepsilon} d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi}) \\ &= -\frac{1}{2E} \lim_{\varepsilon \rightarrow 0} \frac{1}{f'(\varepsilon)} \int_0^{2\pi} \left(\int_0^\pi d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi}) \varepsilon^2 \sin \theta d\theta \right) d\varphi, \end{aligned} \quad (55)$$

where

$$d_T(T_\varepsilon^{\theta\theta}, T_\varepsilon^{\theta\varphi}, T_\varepsilon^{\varphi\varphi}) = (T_\varepsilon^{\theta\theta})^2 + (T_\varepsilon^{\varphi\varphi})^2 - 2\nu T_\varepsilon^{\theta\theta} T_\varepsilon^{\varphi\varphi} + 2(1+\nu)(T_\varepsilon^{\theta\varphi})^2. \quad (56)$$

Now, it is enough to calculate the limit $\varepsilon \rightarrow 0$ in the eq. (55) to obtain the final expression of the topological derivative. Thus, an asymptotic analysis [15] shall be performed in order to know the behavior of the solution $T_\varepsilon^{\theta\theta}$, $T_\varepsilon^{\theta\varphi}$ and $T_\varepsilon^{\varphi\varphi}$ when $\varepsilon \rightarrow 0$. This behavior may be obtained from the analytical solution for a stress distribution around a spherical void in a three-dimensional elastic body [23], which is given, for any $\delta > 0$ and at $r = \varepsilon$, by (see Appendix A)

$$\begin{aligned} T_\varepsilon^{\theta\theta} \Big|_{\partial B_\varepsilon} &= \frac{3}{4} \frac{1}{7-5\nu} \{ \sigma_1(\mathbf{u}) [3 - 5(1-2\nu) \cos 2\varphi + 10 \cos 2\theta \sin^2 \varphi] \\ &\quad + \sigma_2(\mathbf{u}) [3 + 5(1-2\nu) \cos 2\varphi + 10 \cos 2\theta \cos^2 \varphi] \\ &\quad + \sigma_3(\mathbf{u}) [2(4-5\nu) - 10 \cos 2\theta] \} + \mathcal{O}(\varepsilon^{1-\delta}), \end{aligned} \quad (57)$$

$$T_\varepsilon^{\theta\varphi} \Big|_{\partial B_\varepsilon} = \frac{15}{2} \frac{1-\nu}{7-5\nu} (\sigma_1(\mathbf{u}) - \sigma_2(\mathbf{u})) \cos \theta \sin 2\varphi + \mathcal{O}(\varepsilon^{1-\delta}), \quad (58)$$

$$\begin{aligned} T_\varepsilon^{\varphi\varphi} \Big|_{\partial B_\varepsilon} &= \frac{3}{4} \frac{1}{7-5\nu} \{ \sigma_1(\mathbf{u}) [8 - 5\nu + 5(2-\nu) \cos 2\varphi + 10\nu \cos 2\theta \sin^2 \varphi] \\ &\quad + \sigma_2(\mathbf{u}) [8 - 5\nu - 5(2-\nu) \cos 2\varphi + 10\nu \cos 2\theta \cos^2 \varphi] \\ &\quad - 2\sigma_3(\mathbf{u}) (1 + 5\nu \cos 2\theta) \} + \mathcal{O}(\varepsilon^{1-\delta}), \end{aligned} \quad (59)$$

where $\sigma_1(\mathbf{u})$, $\sigma_2(\mathbf{u})$ and $\sigma_3(\mathbf{u})$ are the principal

stress values of the tensor $\mathbf{T}(\mathbf{u})$, associated to the original domain without hole Ω (see eq. 12), evaluated in the point $\hat{\mathbf{x}} \in \Omega$, that is $\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}}$.

Substituting the asymptotic expansion given by eqs. (57,58,59) in eq. (55) we observe that function $f(\varepsilon)$ must be chosen such that

$$f'(\varepsilon) = -|\partial B_\varepsilon| = -4\pi\varepsilon^2 \quad \Rightarrow \quad f(\varepsilon) = -|B_\varepsilon| = -\frac{4}{3}\pi\varepsilon^3 \quad (60)$$

in order to take the limit $\varepsilon \rightarrow 0$ in eq. (55).

Therefore, from this choice of function $f(\varepsilon)$ shown in eq. (60), the final expression for the topological derivative becomes a scalar function that depends on the solution \mathbf{u} associated to the original domain Ω (without hole), that is (see also [9, 16]):

- in terms of the principal stress values $\sigma_1(\mathbf{u})$, $\sigma_2(\mathbf{u})$ and $\sigma_3(\mathbf{u})$ of tensor $\mathbf{T}(\mathbf{u})$

$$D_T(\hat{\mathbf{x}}) = \frac{3}{4E} \frac{1-\nu}{7-5\nu} [10(1+\nu)S_1(\mathbf{u}) - (1+5\nu)S_2(\mathbf{u})], \quad (61)$$

where $S_1(\mathbf{u})$ and $S_2(\mathbf{u})$ are respectively given by

$$S_1(\mathbf{u}) = \sigma_1(\mathbf{u})^2 + \sigma_2(\mathbf{u})^2 + \sigma_3(\mathbf{u})^2 \quad \text{and} \quad S_2(\mathbf{u}) = (\sigma_1(\mathbf{u}) + \sigma_2(\mathbf{u}) + \sigma_3(\mathbf{u}))^2; \quad (62)$$

- in terms of the stress tensor $\mathbf{T}(\mathbf{u})$

$$D_T(\hat{\mathbf{x}}) = \frac{3}{4E} \frac{1-\nu}{7-5\nu} [10(1+\nu)\mathbf{T}(\mathbf{u}) \cdot \mathbf{T}(\mathbf{u}) - (1+5\nu)(\text{tr} \mathbf{T}(\mathbf{u}))^2]; \quad (63)$$

- in terms of the stress $\mathbf{T}(\mathbf{u})$ and strain $\mathbf{E}(\mathbf{u})$ tensors

$$D_T(\hat{\mathbf{x}}) = \frac{3}{4} \frac{1-\nu}{7-5\nu} \left[10\mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\mathbf{u}) - \frac{1-5\nu}{1-2\nu} \text{tr}\mathbf{T}(\mathbf{u}) \text{tr}\mathbf{E}(\mathbf{u}) \right] ; \quad (64)$$

which was obtained from a simple manipulation considering the constitutive relation given by eq. (14). See also eq. (15).

Remark 3 *It is interesting to observe that if we take $\nu = 1/5$ in eq. (64), the final expression for the topological derivative in terms of $\mathbf{T}(\mathbf{u})$ and $\mathbf{E}(\mathbf{u})$ becomes*

$$D_T(\hat{\mathbf{x}}) = \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\mathbf{u}) . \quad (65)$$

4 Conclusions

In this work, we have computed the topological derivative in three-dimensional linear elasticity taking the total potential energy as the cost function and the state equation in its weak form as the constraint. The relationship between shape and topological derivatives was formally established in Theorem 1, leading to the Topological-Shape Sensitivity Method. Therefore, results from classical shape sensitivity analysis could be used to compute the topological derivative in a systematic way. In particular, we have obtained the explicit formula for the topological derivative for the problem under consideration given by eqs. (61,63,64), whose result can be applied in several engineering problems such as topology optimization of three-dimensional linear elastic structures (see, for instance, [2]).

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A Asymptotic Analysis

In this appendix we present the analytical solution for the stress distribution around a spherical cavity in a three-dimensional linear elastic body, whose result was used to perform the asymptotic analysis in relation to the parameter ε in Section 3.3. Therefore, let us introduce a spherical coordinate system (r, θ, φ) centered in $\hat{\mathbf{x}}$, as shown in fig. 2.

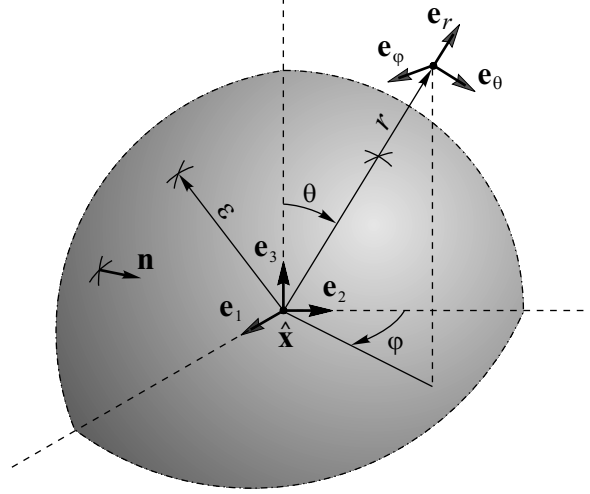


Figure 2: spherical coordinate system (r, θ, φ) positioned in the center $\hat{\mathbf{x}}$ of the ball B_ε .

Then, the stress distribution around the spherical cavity B_ε is given, for any $\delta > 0$, by

$$\begin{aligned}
 T_\varepsilon^{rr} &= T_1^{rr} + T_2^{rr} + T_3^{rr} + \mathcal{O}(\varepsilon^{1-\delta}), \\
 T_\varepsilon^{r\theta} &= T_1^{r\theta} + T_2^{r\theta} + T_3^{r\theta} + \mathcal{O}(\varepsilon^{1-\delta}), \\
 T_\varepsilon^{r\varphi} &= T_1^{r\varphi} + T_2^{r\varphi} + T_3^{r\varphi} + \mathcal{O}(\varepsilon^{1-\delta}), \\
 T_\varepsilon^{\theta\theta} &= T_1^{\theta\theta} + T_2^{\theta\theta} + T_3^{\theta\theta} + \mathcal{O}(\varepsilon^{1-\delta}), \\
 T_\varepsilon^{\theta\varphi} &= T_1^{\theta\varphi} + T_2^{\theta\varphi} + T_3^{\theta\varphi} + \mathcal{O}(\varepsilon^{1-\delta}), \\
 T_\varepsilon^{\varphi\varphi} &= T_1^{\varphi\varphi} + T_2^{\varphi\varphi} + T_3^{\varphi\varphi} + \mathcal{O}(\varepsilon^{1-\delta}),
 \end{aligned} \tag{66}$$

where T_i^{rr} , $T_i^{r\theta}$, $T_i^{r\varphi}$, $T_i^{\theta\theta}$, $T_i^{\theta\varphi}$ and $T_i^{\varphi\varphi}$, for $i = 1, 2, 3$, are written, as:

- for $i = 1$

$$T_1^{rr} = \frac{\sigma_1}{14 - 10\nu} \left[12 \left(\frac{\varepsilon^3}{r^3} - \frac{\varepsilon^5}{r^5} \right) + \left(14 - 10\nu - 10(5 - \nu) \frac{\varepsilon^3}{r^3} + 36 \frac{\varepsilon^5}{r^5} \right) \sin^2 \theta \sin^2 \varphi \right], \quad (67)$$

$$T_1^{r\theta} = \frac{\sigma_1}{14 - 10\nu} \left[7 - 5\nu + 5(1 + \nu) \frac{\varepsilon^3}{r^3} - 12 \frac{\varepsilon^5}{r^5} \right] \sin 2\theta \sin^2 \varphi, \quad (68)$$

$$T_1^{r\varphi} = \frac{\sigma_1}{14 - 10\nu} \left[7 - 5\nu + 5(1 + \nu) \frac{\varepsilon^3}{r^3} - 12 \frac{\varepsilon^5}{r^5} \right] \sin \theta \sin 2\varphi, \quad (69)$$

$$T_1^{\theta\theta} = \frac{\sigma_1}{56 - 40\nu} \left[14 - 10\nu + (1 + 10\nu) \frac{\varepsilon^3}{r^3} + 3 \frac{\varepsilon^5}{r^5} - \left(14 - 10\nu + 25(1 - 2\nu) \frac{\varepsilon^3}{r^3} - 9 \frac{\varepsilon^5}{r^5} \right) \cos 2\varphi \right. \\ \left. + \left(28 - 20\nu - 10(1 - 2\nu) \frac{\varepsilon^3}{r^3} + 42 \frac{\varepsilon^5}{r^5} \right) \cos 2\theta \sin^2 \varphi \right], \quad (70)$$

$$T_1^{\theta\varphi} = \frac{\sigma_1}{14 - 10\nu} \left[7 - 5\nu + 5(1 - 2\nu) \frac{\varepsilon^3}{r^3} + 3 \frac{\varepsilon^5}{r^5} \right] \cos \theta \sin 2\varphi, \quad (71)$$

$$T_1^{\varphi\varphi} = \frac{\sigma_1}{56 - 40\nu} \left[28 - 20\nu + (11 - 10\nu) \frac{\varepsilon^3}{r^3} + 9 \frac{\varepsilon^5}{r^5} + \left(28 - 20\nu + 5(1 - 2\nu) \frac{\varepsilon^3}{r^3} + 27 \frac{\varepsilon^5}{r^5} \right) \cos 2\varphi \right. \\ \left. - 30 \left((1 - 2\nu) \frac{\varepsilon^3}{r^3} - \frac{\varepsilon^5}{r^5} \right) \cos 2\theta \sin^2 \varphi \right], \quad (72)$$

- for $i = 2$

$$T_2^{rr} = \frac{\sigma_2}{14 - 10\nu} \left[12 \left(\frac{\varepsilon^3}{r^3} - \frac{\varepsilon^5}{r^5} \right) + \left(14 - 10\nu - 10(5 - \nu) \frac{\varepsilon^3}{r^3} + 36 \frac{\varepsilon^5}{r^5} \right) \sin^2 \theta \cos^2 \varphi \right], \quad (73)$$

$$T_2^{r\theta} = \frac{\sigma_2}{14 - 10\nu} \left[7 - 5\nu + 5(1 + \nu) \frac{\varepsilon^3}{r^3} - 12 \frac{\varepsilon^5}{r^5} \right] \cos^2 \varphi \sin 2\theta, \quad (74)$$

$$T_2^{r\varphi} = \frac{-\sigma_2}{14 - 10\nu} \left[7 - 5\nu + 5(1 + \nu) \frac{\varepsilon^3}{r^3} - 12 \frac{\varepsilon^5}{r^5} \right] \sin \theta \sin 2\varphi, \quad (75)$$

$$T_2^{\theta\theta} = \frac{\sigma_2}{56 - 40\nu} \left[14 - 10\nu + (1 + 10\nu) \frac{\varepsilon^3}{r^3} + 3 \frac{\varepsilon^5}{r^5} + \left(14 - 10\nu + 25(1 - 2\nu) \frac{\varepsilon^3}{r^3} - 9 \frac{\varepsilon^5}{r^5} \right) \cos 2\varphi \right. \\ \left. + \left(28 - 20\nu - 10(1 - 2\nu) \frac{\varepsilon^3}{r^3} + 42 \frac{\varepsilon^5}{r^5} \right) \cos 2\theta \cos^2 \varphi \right], \quad (76)$$

$$T_2^{\theta\varphi} = \frac{-\sigma_2}{14 - 10\nu} \left[7 - 5\nu + 5(1 - 2\nu) \frac{\varepsilon^3}{r^3} + 3 \frac{\varepsilon^5}{r^5} \right] \cos \theta \sin 2\varphi, \quad (77)$$

$$T_2^{\varphi\varphi} = \frac{\sigma_2}{56 - 40\nu} \left[28 - 20\nu + (11 - 10\nu) \frac{\varepsilon^3}{r^3} + 9 \frac{\varepsilon^5}{r^5} - \left(28 - 20\nu + 5(1 - 2\nu) \frac{\varepsilon^3}{r^3} + 27 \frac{\varepsilon^5}{r^5} \right) \cos 2\varphi \right. \\ \left. - 30 \left((1 - 2\nu) \frac{\varepsilon^3}{r^3} - \frac{\varepsilon^5}{r^5} \right) \cos 2\theta \cos^2 \varphi \right], \quad (78)$$

- for $i = 3$

$$T_3^{rr} = \frac{\sigma_3}{14 - 10\nu} \left[14 - 10\nu - (38 - 10\nu) \frac{\varepsilon^3}{r^3} + 24 \frac{\varepsilon^5}{r^5} - \left(14 - 10\nu - 10(5 - \nu) \frac{\varepsilon^3}{r^3} + 36 \frac{\varepsilon^5}{r^5} \right) \sin^2 \theta \right] , \quad (79)$$

$$T_3^{r\theta} = \frac{-\sigma_3}{14 - 10\nu} \left[14 - 10\nu + 10(1 + \nu) \frac{\varepsilon^3}{r^3} - 24 \frac{\varepsilon^5}{r^5} \right] \cos \theta \sin \theta , \quad (80)$$

$$T_3^{r\varphi} = 0 , \quad (81)$$

$$T_3^{\theta\theta} = \frac{\sigma_3}{14 - 10\nu} \left[(9 - 15\nu) \frac{\varepsilon^3}{r^3} - 12 \frac{\varepsilon^5}{r^5} + \left(14 - 10\nu - 5(1 - 2\nu) \frac{\varepsilon^3}{r^3} + 21 \frac{\varepsilon^5}{r^5} \right) \sin^2 \theta \right] , \quad (82)$$

$$T_3^{\theta\varphi} = 0 , \quad (83)$$

$$T_3^{\varphi\varphi} = \frac{\sigma_3}{14 - 10\nu} \left[(9 - 15\nu) \frac{\varepsilon^3}{r^3} - 12 \frac{\varepsilon^5}{r^5} - 15 \left((1 - 2\nu) \frac{\varepsilon^3}{r^3} - \frac{\varepsilon^5}{r^5} \right) \sin^2 \theta \right] , \quad (84)$$

where σ_1 , σ_2 and σ_3 are the principal stress values of the tensor $\mathbf{T}(\mathbf{u})$, associated to the original domain without hole Ω , evaluated in the point $\hat{\mathbf{x}} \in \Omega$, that is $\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}}$. In other words, the tensor $\mathbf{T}(\mathbf{u})$ was diagonalized in the following way

$$\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}} = \sum_{i=1}^3 \sigma_i (\mathbf{e}_i \otimes \mathbf{e}_i) , \quad (85)$$

where σ_i is the eigen-value associated to the \mathbf{e}_i eigen-vector of the tensor $\mathbf{T}(\mathbf{u})|_{\hat{\mathbf{x}}}$.

Remark 4 *It is important to mention that the stress distribution for $i = 1, 2$ was obtained from a rotation of the stress distribution for $i = 3$. In addition, the derivation of this last result (for $i = 3$) can be found in [23], for instance.*