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Propagation, Observation, and Control of Waves Approximated by Finite Difference Methods*

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Abstract. This paper surveys several topics related to the observation and control of wave propagation phenomena modeled by finite difference methods. The main focus is on the property of *observability*, corresponding to the question of whether the total energy of solutions can be estimated from partial measurements on a subregion of the domain or boundary. The mathematically equivalent property of *controllability* corresponds to the question of whether wave propagation behavior can be controlled using forcing terms on that subregion, as is often desired in engineering applications. Observability/controllability of the continuous wave equation is well understood for the scalar linear constant coefficient case that is the focus of this paper. However, when the wave equation is discretized by finite difference methods, the control for the discretized model does not necessarily yield a good approximation to the control for the original continuous problem. In other words, the classical convergence (consistency + stability) property of a numerical scheme does not suffice to guarantee its suitability for providing good approximations to the controls that might be needed in applications. Observability/controllability may be lost under numerical discretization as the mesh size tends to zero due to the existence of high-frequency spurious solutions for which the group velocity vanishes. This phenomenon is analyzed and several remedies are suggested, including filtering, Tychonoff regularization, multigrid methods, and mixed finite element methods.

We also briefly discuss these issues for the heat, beam, and Schrödinger equations to illustrate that diffusive and dispersive effects may help to retain the observability/controllability properties at the discrete level. We conclude with a list of open problems and future subjects for research.

Key words. waves, finite difference approximation, propagation, observation, control, heat and beam equations

AMS subject classifications. 65M06, 35L05, 93B07, 93B05

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1. Introduction.

This article analyzes numerical methods for approximating the controllability and observability of wave-like equations. These properties can be summarized by the following questions:

- *Observability.* Can waves satisfying a wave equation and suitable boundary conditions be fully reconstructed from measurements on a subregion of the domain or boundary during a given time interval? More precisely, we will

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focus on the question of whether the whole energy of the solution can be estimated in terms of the energy measured on a subregion during a given time interval.

- *Controllability.* Can solutions be driven to a given state at a given final time by means of a control acting on the system on that subregion?

These properties are equivalent in the appropriate functional setting (see, e.g., [67, 68, 117]).

Here we examine whether discrete numerical approximations preserve the observability/controllability properties of continuous wave problems in the sense that the continuous properties are recovered as the mesh interval tends to zero. It is well known that numerical wave approximations yield dispersion phenomena and *spurious*[1] high-frequency oscillations [105, 101]. The propagation speed of these nonphysical waves can be characterized by the *group velocity*,[2] which may converge to zero when the oscillation wavelength is of the order of the mesh interval and the latter tends to zero. As a consequence, the time needed to uniformly (with respect to the mesh size) observe (or control) the numerical waves from a subset of the domain or boundary may tend to infinity as the mesh is refined. Hence, *controlling a discrete version of a continuous wave model is often a bad way of controlling the continuous wave model itself.*

In essence, the numerical discretization and observation/control do not commute:

$$\begin{array}{c}[Continuous\ Model + Observation/Control] + Numerics \\ \neq \\ [Continuous\ Model + Numerics] + Observation/Control\end{array} \tag{1.1}$$

Our primary objective is to explain this pathological fact, which holds despite the good control properties of the underlying continuous wave equation and the convergence of the numerical scheme approximating the PDE. In pursuing this goal, we build on work by Glowinski, Li, and Lions [41], Glowinski [38], and Asch and Lebeau [2], among others. After diagnosing the problem pertaining to high-frequency spurious modes, we will mention several approaches for restoring numerical observation and control properties using filtering, Tychonoff regularization, multigrid methods, and mixed finite elements.

It is striking that the instabilities we shall encounter are catastrophic in nature. Indeed the divergence rate of the controls is not polynomial but exponential in the number of mesh nodes. Hence, stability cannot be reestablished simply by modifying the observed quantities or by relaxing the regularity of the controls by a finite number of derivatives.

Strictly speaking, we are concerned with the problem of *exact controllability*, in which the goal is to drive the solution of an evolution problem to a given final state exactly in a given time. It is in this setting where numerical high-frequency waves may lead to a pathological lack of convergence. This difficulty does not arise if the control problem is relaxed to an approximate or optimal control problem [119]. However,

[1] *Spurious* is used to designate any component of the numerical solution that does not correspond to a solution of the underlying PDE. In the context of the wave equation, spurious modes occur at high frequencies. Hence, the existence of these oscillations is compatible with the convergence (in the classical sense) of the numerical scheme, which holds for fixed initial data.

[2] In a numerical setting, it is important to distinguish between phase and group velocities. Phase velocity characterizes the propagation of individual monochromatic waves, while group velocity characterizes the propagation of wave packets (see, e.g., [101] and section 4.3.)

even if one is interested in these weaker control problems, the fact that the exact controllability problem embodies the limiting behavior of these problems *raises a serious warning regarding the instability of numerical controls for wave-like processes.*

The implications of this discussion are considerable because expression (1.1) relates the two main alternatives for numerically computing controls to PDEs.[3] The former "continuous" approach, if fully and rigorously developed, provides convergent algorithms that produce good numerical approximations of the true control of the continuous PDE [41]. The latter "discrete" approach consists in first discretizing the continuous model and then computing the control of the discrete system as an approximation to the continuous control. This approach is generally more straightforward since problem-specific PDE control theory analysis is not required. *But this second black-box procedure may diverge.*[4]

The examination of black-box methods for observability/controllability is relevant to wide-ranging problem in control theory, optimal design, and inverse problems. Specific examples (see [34, 66]) involving wave-like phenomena include noise reduction [4], structural control in response to aerodynamic [95] or seismic [93, 87] forces, laser control of chemical reactions [10] or waves in crystals [1], biomechanical control [20], and control of gravity waves in data assimilation [98]. We also refer to the SIAM Reports [34], [81], or, for more historical and engineering-oriented applications, to [66].

Our analysis relies mainly on the Fourier decomposition of solutions and classical results on the theory of nonharmonic Fourier series. We shall also briefly explain how the tools of discrete Wigner measures (in the spirit of Gérard [35] and Lions and Paul [70]) have been applied to these problems [72, 73] following previous developments by Trefethen [101]. These methods allow us to discover the numerical counterpart of the so-called geometric control condition[5] (GCC), which is a sharp sufficient condition for the controllability of the wave equation.

The paper is organized as follows. Section 2 recalls the basic ingredients of the finite-dimensional theory we will employ throughout the paper. Section 3 then discusses observability/controllability for the constant coefficient one-dimensional (1D) wave equation. The main results on the lack of observability/controllability of finite difference semidiscretizations are then presented in section 4, which also examines the use of filtering to restore uniform controllability. Section 5 summarizes the relevant features of the multidimensional wave equation, and section 6 analyzes observability/controllability for finite difference semidiscretizations of the two-dimensional (2D) wave equation in a square. In section 7 we discuss several additional methods for curing high-frequency pathologies. Section 8 examines finite difference semidiscretizations of the heat, beam, and Schrödinger equations to illustrate that dissipative and dispersive processes can improve the observability/controllability of discrete approximations. Section 9 concludes with further comments and a list of important open problems.

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[3]See [113, 117] for surveys of the state-of-the-art in the controllability of PDEs and the books of Lee and Markus [64], Sontag [99], and Fattorini [29] for broader introductions to control theory for finite- and infinite-dimensional systems.

[4]There are, however, some other situations in which it works. We refer to [14] for the analysis of finite element approximations of elliptic optimal control problems and to [23] for an optimal shape design problem for the Laplace operator.

[5]Bardos, Lebeau, and Rauch [7] show that the wave equation is exactly controllable in time T with controls in a given subdomain if all rays of geometric optics enter the control subregion in that time.

2. Preliminaries on Finite-Dimensional Systems. Most of this article is devoted to analyzing the wave equation and its numerical approximations. Numerical approximation schemes and, more precisely, those that are semidiscrete (discrete in space and continuous in time) yield finite-dimensional systems of ODEs. There is by now an extensive literature on the control of finite-dimensional systems, and the problem is completely understood for linear ones (see [64, 99]).

As we have mentioned above, the problem of convergence of controls as the mesh size in the numerical approximation tends to zero is very closely related to passing to the limit as the dimension of finite-dimensional systems tends to infinity. The latter topic is widely open, and this article may be considered as a contribution in this direction.

In this section we briefly summarize the most basic material on finite-dimensional systems that will be used throughout this article (we refer to [77] for more details).

Consider the finite-dimensional system of dimension N:

$$x' + Ax = Bv, \quad 0 \le t \le T; \quad x(0) = x_0, \tag{2.1}$$

where x is the N-dimensional state and v is the M-dimensional control, with $M \le N$.

Here A is an $N \times N$ matrix with constant real coefficients and B is an $N \times M$ matrix. The matrix A determines the dynamics of the system and the matrix B models the way M controls act on it.

In practice, it is desirable to control the N components of the system with a low number of controls, and the best would be to do it by a single one, in which case $M = 1$.

System (2.1) is said to be *controllable* in time T when every initial datum $x_0 \in \mathbf{R}^N$ can be driven to any final datum x_1 in $\mathbf{R}^N$ in time T. There is a necessary and sufficient condition for controllability which is purely algebraic in nature. It is the so-called *Kalman condition*: System (2.1) is controllable in some time $T > 0$ if and only if

$$\mathrm{rank}[B, AB, \ldots, A^{N-1}B] = N. \tag{2.2}$$

There is a direct proof of this result which uses the representation of solutions of (2.1) by means of the variations of constants formula. However, the methods we shall develop along this article rely more on the dual (but completely equivalent!) problem of observability of the adjoint system that we discuss now.

Consider the *adjoint system*

$$-\varphi' + A^*\varphi = 0, \quad 0 \le t \le T; \quad \varphi(T) = \varphi_0. \tag{2.3}$$

THEOREM 2.1. *System* (2.1) *is controllable in time T if and only if the adjoint system* (2.3) *is* observable *in time T, i.e., if there exists a constant $C > 0$ such that, for every solution φ of* (2.3),

$$|\varphi_0|^2 \le C \int_0^T |B^*\varphi|^2 dt. \tag{2.4}$$

Both properties hold in all time T if and only if the Kalman rank condition (2.2) *is satisfied.*

REMARK 2.1. *The equivalence between the controllability of the state equation and the observability of the adjoint one is one of the most classical ingredients of* *the controllability theory of finite-dimensional systems (see, for instance, Theorem* 1.10.2 *in* [55]*). In general, observability refers to the possibility of recovering the full solution by means of some partial measurements or observations. Here one is allowed to measure the output* $B\varphi$ *during the time interval* $[0,T]$ *and wishes to recover complete information on the initial datum* $\varphi(0)$*. Since in finite-dimensions all norms are equivalent, this is equivalent to the observability inequality* (2.4)*.*

Sketch of the proof. We shall simply recall the proof of the fact that observability implies controllability. It is the main property we shall use throughout the paper. Our proof provides a constructive method for building controls.

We proceed in several steps.

Step 1. *Construction of controls as minimizers of a quadratic functional.*

Assume (2.4) holds and consider the quadratic functional $J : \mathbf{R}^N \to \mathbf{R}$:

$$J(\varphi_0) = \frac{1}{2}\int_0^T |B^*\varphi(t)|^2 dt - \langle x_1, \varphi_0\rangle + \langle x_0, \varphi(0)\rangle. \tag{2.5}$$

If $\tilde{\varphi}_0$ is a minimizer for J, since $DJ(\tilde{\varphi}_0) = 0$, then the control $v = B^*\tilde{\varphi}$, where $\tilde{\varphi}$ is the solution of (2.3) with that datum at time $t = T$, is such that the solution x of (2.1) satisfies the control requirement $x(T) = x_1$. Thus, it is sufficient to minimize the functional J. We apply the direct method of the calculus of variations (DMCV). The functional J being continuous, quadratic, and nonnegative, since we are in finite space dimensions, it is sufficient to prove its coercivity, which holds if and only if the Kalman condition is satisfied. Indeed, when (2.4) holds, the following variant holds as well,[6] with possibly a different constant $C > 0$:

$$|\varphi_0|^2 + |\varphi(0)|^2 \le C\int_0^T |B^*\varphi|^2 dt, \tag{2.6}$$

and the coercivity of J follows.

Step 2. *Equivalence between the observability inequality* (2.6) *and the Kalman condition.*

Since we are in finite-dimensions and all norms are equivalent, (2.6) is equivalent to the uniqueness property: *Does the fact that* $B^*\varphi$ *vanish for all* $0 \le t \le T$ *imply that* $\varphi \equiv 0$*?*

Taking into account that solutions φ are analytic in time, $B^*\varphi$ vanishes if and only if all the derivatives of $B^*\varphi$ of any order at time $t = T$ vanish. Since $\varphi = e^{A^*(t-T)}\varphi_0$ this is equivalent to $B^*[A^*]^k\varphi_0 \equiv 0$ for all $k \ge 0$. But, according to the Cayley–Hamilton theorem, this holds if and only if it is satisfied for all $k = 0, \dots, N-1$. Therefore $B\varphi \equiv 0$ is equivalent to $\varphi_0 \equiv 0$ if and only if rank $[B^*, B^*A^*, \dots, B^*[A^*]^{N-1}] = N$, which is obviously equivalent to (2.2).

Remark 2.2. *The property of observability of the adjoint system* (2.3) *is equivalent to the inequality* (2.4) *because of the linear character of the system. In general, the problem of observability can be formulated as that of determining uniquely the adjoint state everywhere in terms of partial measurements.*

We emphasize that in the finite-dimensional context under consideration the observability inequality (2.4) *is completely equivalent to* (2.6)*. In other words, it is totally equivalent to formulate the problem of estimating the initial or final data of*

[6] Both inequalities are equivalent. This is so since $\varphi(t) = e^{A^*(t-T)}\varphi_0$ and the operator $e^{A^*(t-T)}$ is bounded and invertible.

the adjoint system. This is no longer true for time irreversible PDEs, as we shall see when discussing the heat equation in section 8.2.

REMARK 2.3. *This proof of controllability also yields explicit bounds on the controls. Indeed, since the functional* $J \leq 0$ *at the minimizer, and in view of the observability inequality* (2.6)*, it follows that*

$$||v|| \leq 2\sqrt{C}[|x_0|^2 + |x_1|^2]^{1/2}, \tag{2.7}$$

C *being the same constant as in* (2.6)*. Therefore, we see that the observability constant is, up to a multiplicative factor, the norm of the control map associating to the data the control of minimal norm.*

The reverse is also true. Assume for instance that the system is controllable and that we have the bound

$$||v||_{L^2(0,T)} \leq C^*||x_1||$$

for the initial datum $x_0 = 0$ *and for all final targets* x_1*. Then, multiplying the state equation satisfied by* x *by the adjoint state* φ *and integrating by parts, it follows that*

$$\int_0^T \langle v, B^*\varphi\rangle = -\langle x_1, \varphi_0\rangle.$$

These two facts imply (2.4) *with* $C = (C^*)^2$.

REMARK 2.4. *It is important to note that in this finite-dimensional context, the time* T *plays no role. In particular, whether a system is controllable (or its adjoint observable) is independent of the time* T *of control. Note that the situation is totally different for the wave equation. There, due to the finite velocity of propagation, the time needed to control/observe waves from the boundary needs to be large enough, of the order of the size of the ratio between the size of the domain and velocity of propagation.*

In fact, the main task to be undertaken in order to pass to the limit in numerical approximations of control problems for wave equations as the mesh size tends to zero is to explain why, even though at the finite-dimensional level the control time T *is irrelevant, it may play a key role for PDEs.*

3. The Constant Coefficient Wave Equation.

3.1. Problem Formulation. Let us first consider the constant coefficient 1D wave equation:

$$\begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ u(0,t) = u(1,t) = 0, & 0 < t < T, \\ u(x,0) = u^0(x),\ u_t(x,0) = u^1(x), & 0 < x < 1. \end{cases} \tag{3.1}$$

In (3.1) $u = u(x,t)$ describes the displacement of a vibrating string occupying $(0,1)$.

The energy of solutions of (3.1) is conserved in time, i.e.,

$$E(t) = \frac{1}{2}\int_0^1 \left[|u_x(x,t)|^2 + |u_t(x,t)|^2\right] dx = E(0) \quad \forall 0 \leq t \leq T. \tag{3.2}$$

The problem of boundary observability of (3.1) can be formulated as follows: *To give sufficient conditions on* T *such that there exists* $C(T) > 0$ *for which the following inequality holds for all solutions of* (3.1)*:*

$$E(0) \leq C(T)\int_0^T |u_x(1,t)|^2\, dt. \tag{3.3}$$

Inequality (3.3), when it holds, guarantees that the total energy of solutions can be "observed" from the boundary measurement on the extreme $x = 1$. The best constant $C(T)$ in (3.3) is the so-called *observability constant*.[7]

As in finite space dimensions, the observability problem above is equivalent to the following boundary controllability property: For any $\left(y^0, y^1\right) \in L^2(0,1) \times H^{-1}(0,1)$ there exists $v \in L^2(0,T)$ such that the solution of the controlled wave equation

$$\left\{\begin{array}{ll} y_{tt} - y_{xx} = 0, & 0 < x < 1,\ 0 < t < T, \\ y(0,t) = 0;\ y(1,t) = v(t), & 0 < t < T, \\ y(x,0) = y^0(x),\ y_t(x,0) = y^1(x), & 0 < x < 1, \end{array}\right. \tag{3.4}$$

satisfies

$$y(x,T) = y_t(x,T) = 0, \quad 0 < x < 1. \tag{3.5}$$

Let us explain below why controllability is a consequence of (3.3) by the minimization method we developed in the previous section in the finite-dimensional setting, which yields the control of the minimal $L^2(0,T)$-norm.[8]

Given $\left(y^0, y^1\right) \in L^2(0,1) \times H^{-1}(0,1)$, the control $v \in L^2(0,T)$ is

$$v(t) = u_x^*(1,t), \tag{3.6}$$

where u^* is the solution of (3.1) corresponding to initial data $(u^{0,*}, u^{1,*}) \in H_0^1(0,1) \times L^2(0,1)$ minimizing the functional[9]

$$J((u^0, u^1)) = \frac{1}{2}\int_0^T |u_x(1,t)|^2 dt + \int_0^1 y^0 u^1 dx - \int_0^1 y^1 u^0 dx \tag{3.7}$$

in the space $H_0^1(0,1) \times L^2(0,1)$.

Note that J is convex. The continuity of J in $H_0^1(0,1) \times L^2(0,1)$ is guaranteed by the fact that the solutions of (3.1) satisfy $u_x(1,t) \in L^2(0,T)$ (a fact that holds also for the Dirichlet problem for the wave equation in several space dimensions; see [57, 67, 68]). More, precisely, for all $T > 0$ there exists a constant $C_*(T) > 0$ such that, for all solution of (3.1),

$$\int_0^T \left[|u_x(0,t)|^2 + |u_x(1,t)|^2\right] dt \leq C_*(T)E(0). \tag{3.8}$$

Thus, to apply the DMCV and to prove the existence of a minimizer for J, it is sufficient to prove that it is coercive. This is guaranteed by the observability inequality (3.3).

Let us see that the minimum of J provides the control. The functional J is of class C^1. Consequently, the gradient of J at the minimizer vanishes:

$$\begin{aligned} \langle DJ((u^{0,*}, u^{1,*})), (w^0, w^1)\rangle &= \int_0^T u_x^*(1,t) w_x(1,t) dt \\ &+ \int_0^1 y^0 w^1 dx - \langle y^1, w^0\rangle_{H^{-1}\times H_0^1} = 0 \end{aligned} \tag{3.9}$$

[7] Inequality (3.3) is just an example of a variety of similar observability problems: (a) one could observe the energy concentrated on the extreme $x = 0$ or in the two extremes $x = 0$ and 1 simultaneously; (b) the $L^2(0,T)$-norm of $u_x(1,t)$ could be replaced by some other norm; (c) one could also observe the energy concentrated in a subinterval (α, β) of $(0,1)$, etc.

[8] We refer to Lions [67, 68] for a systematic analysis of the equivalence between controllability and observability through the so-called Hilbert uniqueness method (HUM).

[9] The integral $\int_0^1 y^1 u^0 dx$ represents the duality $\langle y^1, u^0\rangle_{H^{-1}\times H_0^1}$.

for all $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, where w stands for the solution of (3.1) with initial data (w^0, w^1). By choosing the control as in (3.6) this identity yields

$$\int_0^T v(t) w_x(1,t)dt + \int_0^1 y^0 w^1 dx - \langle y^1, w^0 \rangle_{H^{-1}\times H_0^1} = 0. \tag{3.10}$$

On the other hand, multiplying in (3.4) by w and integrating by parts, we get

$$\begin{aligned} &\int_0^T v(t) w_x(1,t)dt + \int_0^1 y^0 w^1 dx - \langle y^1, w^0 \rangle_{H^{-1}\times H_0^1} \\ &\quad - \int_0^1 y(T) w_t(T) dx + \langle y_t(T), w(T) \rangle_{H^{-1}\times H_0^1} = 0. \end{aligned} \tag{3.11}$$

Combining these two identities we get $\int_0^1 y(T)w_t(T)dx - \langle y_t(T), w(T)\rangle_{H^{-1}\times H_0^1} = 0$ for all $(w^0, w^1) \in H_0^1(0,1) \times L^2(0,1)$, which is equivalent to the exact controllability condition (3.5).

This argument shows that *observability implies controllability*. The reverse is also true. If controllability holds, then the linear map that to all initial data $(y^0, y^1) \in L^2(0,1)\times H^{-1}(0,1)$ of the state equation (3.4) associates the control v of the minimal $L^2(0,T)$-norm is bounded. Multiplying the state equation (3.4) with that control by u, solution of (3.1), and using (3.5), we obtain

$$\int_0^T v(t) u_x(1,t)dt + \int_0^1 y^0 u^1 dx - \langle y^1, u^0 \rangle_{H^{-1}\times H_0^1} = 0. \tag{3.12}$$

Consequently,

$$\begin{aligned} \left| \int_0^1 [y^0 u^1 - y^1 u^0] dx \right| = \left| \int_0^T v(t) u_x(1,t) dt \right| &\le ||v||_{L^2(0,T)} ||u_x(1,t)||_{L^2(0,T)} \\ &\le C ||(y^0, y^1)||_{L^2(0,1)\times H^{-1}(0,1)} ||u_x(1,t)||_{L^2(0,T)} \end{aligned} \tag{3.13}$$

for all $(y^0, y^1) \in L^2(0,1)\times H^{-1}(0,1)$, which implies the observability inequality (3.3).

Throughout this paper we shall mainly focus on the problem of observability. However, in view of the equivalence above, all the results we present have immediate consequences for controllability. The most important ones will also be stated. Note, however, that controllability is not the only application of the observability inequalities, which are also of systematic use in the context of inverse problems (Isakov [52]). We shall discuss this issue briefly in open problem 9 in section 9.2.

3.2. Observability. The following holds.

PROPOSITION 3.1. *For any $T \ge 2$, system* (3.1) *is observable. In other words, for any $T \ge 2$ there exists $C(T) > 0$ such that* (3.3) *holds for any solution of* (3.1). *Conversely, if $T < 2$,* (3.1) *is not observable, or, equivalently,*

$$\sup_{u \text{ solution of } (3.1)} \left[\frac{E(0)}{\int_0^T |\, u_x(1,t)\,|^2 \,dt} \right] = \infty. \tag{3.14}$$

The proof of observability for $T \ge 2$ can be carried out in several ways, including Fourier series, multipliers (Komornik [57]; Lions [67, 68]), Carleman inequalities

(Zhang [108]), and microlocal[10] tools (Bardos, Lebeau, and Rauch [7]; Burq and Gérard [12]). Let us explain how it can be proved using Fourier series. Solutions of (3.1) can be written in the form

$$u = \sum_{k\geq 1} \left(a_k \cos(k\pi t) + \frac{b_k}{k\pi} \sin(k\pi t) \right) \sin(k\pi x), \tag{3.15}$$

$$u^0(x) = \sum_{k\geq 1} a_k \sin(k\pi x), \quad u^1(x) = \sum_{k\geq 1} b_k \sin(k\pi x).$$

It follows that $E(0) = \frac{1}{4} \sum_{k\geq 1} \left[a_k^2 k^2 \pi^2 + b_k^2 \right]$.

On the other hand, $u_x(1,t) = \sum_{k\geq 1} (-1)^k \left[k\pi a_k \sin(k\pi t) + b_k \cos(k\pi t) \right]$. Using the orthogonality properties of $\sin(k\pi t)$ and $\cos(k\pi t)$ in $L^2(0,2)$, it follows that $\int_0^2 |u_x(1,t)|^2 \, dt = \sum_{k\geq 1} \left(\pi^2 k^2 a_k^2 + b_k^2 \right)$. The two identities above show that the observability inequality holds when $T = 2$ and therefore for any $T > 2$ as well. In fact, in this particular case, we have

$$E(0) = \frac{1}{4} \int_0^2 |u_x(1,t)|^2 \, dt. \tag{3.16}$$

On the other hand, for $T < 2$ the observability inequality does not hold. Indeed, suppose that $T \leq 2 - 2\delta$ with $\delta > 0$. Solve

$$u_{tt} - u_{xx} = 0, \quad 0 < x < 1, \, 0 < t < T; \quad u(0,t) = u(1,t) = 0, \quad 0 < t < T, \tag{3.17}$$

with data at time $t = T/2$ with support in the subinterval $(0, \delta)$. This solution is such that $u_x(1,t) = 0$ for $\delta < t < T - \delta$ since the segment $x = 1$, $t \in (\delta, T - \delta)$ remains outside the domain of influence of the space segment $t = T/2$, $x \in (0, \delta)$ (see Figure 1).

Note that the observability time ($T = 2$) is twice the length of the string. This is due to the fact that an initial disturbance concentrated near $x = 1$ may propagate to the left (in the space variable) as t increases and only reach the extreme $x = 1$ of the interval after bouncing at the left extreme $x = 0$ (as described in Figure 1).

As we have seen, in one dimension and with constant coefficients, the observability inequality is easy to understand. The same results are true for sufficiently smooth coefficients (BV-regularity suffices). However, when the coefficients are simply Hölder continuous, these properties may fail, thereby contradicting an initial intuition (see [18]).

4. 1D Finite Difference Semidiscretizations.

4.1. Orientation. In section 3 we showed how the observability/controllability problem for the constant coefficient wave equation can be solved by Fourier series expansions. We now address the problem of the continuous dependence of the observability constant $C(T)$ in (3.3) with respect to finite difference space semidiscretizations

[10] Microlocal analysis deals, roughly speaking, with the possibility of localizing functions and its singularities not only in the physical space but also in the frequency domain. Localization in the frequency domain may be done according to the size of frequencies but also to sectors in the euclidean space in which they belong to. This allows introducing the notion of microlocal regularity; see, for instance, [47].

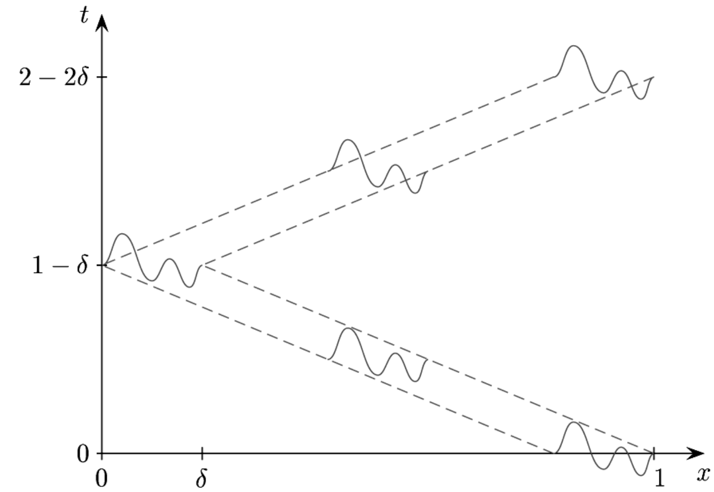

Fig. 1 *Wave localized at $t = 0$ near the endpoint $x = 1$ that propagates with velocity 1 to the left, bounces at $x = 0$, and reaches $x = 1$ again in a time of the order of 2.*

as the parameter h of the discretization tends to zero. This problem arises naturally in the numerical implementation of the controllability and observability properties of the continuous wave equation but is of independent interest in the analysis of discrete models for vibrations.

There are several important facts and results that deserve emphasis and that we shall discuss below:

- The observability constant for the semidiscrete model tends to infinity for any T as $h \to 0$. This is related to the fact that the velocity of propagation of solutions tends to zero as $h \to 0$ and the wavelength of solutions is of the same order as the size of the mesh.
- As a consequence of this fact and of the Banach–Steinhaus theorem, there are initial data for the wave equation for which the controls of the semidiscrete models diverge. This proves that one cannot simply rely on the classical convergence (consistency + stability) analysis of the underlying numerical schemes to design algorithms for computing the controls.
- The observability constant may be made uniform if the high frequencies are filtered in an appropriate manner.

4.2. Finite Difference Approximations. Given $N \in \mathbf{N}$ we define $h = 1/(N + 1) > 0$. We consider the mesh $\{x_j = jh,\ j = 0, \ldots, N+1\}$ which divides $[0,1]$ into $N+1$ subintervals $I_j = [x_j, x_{j+1}]$, $j = 0, \ldots, N$.

Consider the following finite difference approximation of the wave equation (3.1):

$$\begin{cases} u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T,\ j = 1, \ldots, N, \\ u_j(t) = 0, & j = 0,\ N+1,\ 0 < t < T, \\ u_j(0) = u_j^0,\ u_j'(0) = u_j^1, & j = 1, \ldots, N, \end{cases} \tag{4.1}$$

which is a coupled system of N linear differential equations of second order. In it the function $u_j(t)$ provides an approximation of $u(x_j, t)$ for all $j = 1, \ldots, N$, u being the solution of the continuous wave equation (3.1). The conditions $u_0 = u_{N+1} = 0$

take account of the homogeneous Dirichlet boundary conditions, and the second order differentiation with respect to x has been replaced by the three-point finite difference.

We shall use a vector notation to simplify the expressions. In particular, the column vector

$$\vec{u}(t) = \begin{pmatrix} u_1(t) \\ \vdots \\ u_N(t) \end{pmatrix} \tag{4.2}$$

will represent the whole set of unknowns of the system. Introducing the matrix

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 & -1 & 0 & 0 \\ -1 & \ddots & \ddots & 0 \\ 0 & \ddots & \ddots & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}, \tag{4.3}$$

the system (4.1) reads as follows:

$$\vec{u}''(t) + A_h \vec{u}(t) = 0, \quad 0 < t < T; \qquad \vec{u}(0) = \vec{u}^0,\ \vec{u}'(0) = \vec{u}^1. \tag{4.4}$$

The solution $\vec{u}$ of (4.4) depends also on h, but we shall denote it by $\vec{u}$ for simplicity.

The energy of the solutions of (4.1) is

$$E_h(t) = \frac{h}{2} \sum_{j=0}^{N} \left[\mid u_j' \mid^2 + \left| \frac{u_{j+1} - u_j}{h} \right|^2 \right], \tag{4.5}$$

and it is constant in time. It is also a natural discretization of the continuous energy (3.2).

The problem of observability of system (4.1) can be formulated as follows: *To find $T > 0$ and $C_h(T) > 0$ such that*

$$E_h(0) \le C_h(T) \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \tag{4.6}$$

holds for all solutions of (4.1).

Observe that $\mid u_N/h \mid^2$ is a natural approximation[11] of $\mid u_x(1,t) \mid^2$ for the solution of the continuous system (3.1). Indeed $u_x(1,t) \sim [u_{N+1}(t) - u_N(t)]/h$ and, taking into account that $u_{N+1} = 0$, it follows that $u_x(1,t) \sim -u_N(t)/h$.

System (4.1) is finite-dimensional. Therefore, if observability holds for some $T > 0$, then it holds for all $T > 0$, as we have seen in section 3.

We are interested mainly in the uniformity of the constant $C_h(T)$ as $h \to 0$. If $C_h(T)$ remains bounded as $h \to 0$, we say that system (4.1) is *uniformly observable* as $h \to 0$. Taking into account that the observability of the limit system (3.1) holds only for $T \ge 2$, it would be natural to expect $T \ge 2$ to be a necessary condition for the uniform observability of (4.1). This is indeed the case but, as we shall see, the condition $T \ge 2$ is far from being sufficient. In fact, *uniform observability fails for all* $T > 0$. In order to explain this fact it is convenient to analyze the spectrum of (4.1).

[11] Here and in what follows u_N refers to the Nth component of the solution $\vec{u}$ of the semidiscrete system, which obviously depends also on h.

Let us consider the eigenvalue problem

$$-\left[w_{j+1}+w_{j-1}-2w_j\right]/h^2=\lambda w_j, \quad j=1,\dots,N; \qquad w_0=w_{N+1}=0. \tag{4.7}$$

The spectrum can be computed explicitly in this case (Isaacson and Keller [51]):

$$\lambda_k^h=\frac{4}{h^2}\sin^2\left(\frac{k\pi h}{2}\right), \quad k=1,\dots,N, \tag{4.8}$$

and the corresponding eigenvectors are

$$\vec{w}_k^h=\left(w_{k,1},\dots,w_{k,N}\right)^T : w_{k,j}=\sin(k\pi jh), \quad k,j=1,\dots,N. \tag{4.9}$$

Obviously, $\lambda_k^h\to\lambda_k=k^2\pi^2$, as $h\to 0$ for each $k\geq 1$, $\lambda_k=k^2\pi^2$ being the kth eigenvalue of the continuous wave equation (3.1). On the other hand we see that the eigenvectors $\vec{w}_k^h$ of the discrete system (4.7) coincide with the restriction to the meshpoints of the eigenfunctions $w_k(x)=\sin(k\pi x)$ of the continuous wave equation (3.1).[12]

According to (4.8) we have $\sqrt{\lambda_k^h}=\frac{2}{h}\sin\left(\frac{k\pi h}{2}\right)$, and therefore, in a first approximation, we have

$$\left|\sqrt{\lambda_k^h}-k\pi\right|\sim\frac{k^3\pi^3h^2}{24}. \tag{4.10}$$

This indicates that the spectral convergence is uniform only in the range $k\ll h^{-2/3}$. Thus, one cannot solve the problem of uniform observability for the semidiscrete system (4.1) as a consequence of the observability property of the continuous wave equation and a perturbation argument with respect to h.

4.3. Nonuniform Observability. The following identity holds (see [48, 49]).

LEMMA 4.1. *For any $h>0$ and any eigenvector of* (4.7) *associated with the eigenvalue λ,*

$$h\sum_{j=0}^{N}\left|\frac{w_{j+1}-w_j}{h}\right|^2=\frac{2}{4-\lambda h^2}\left|\frac{w_N}{h}\right|^2. \tag{4.11}$$

We now observe that the largest eigenvalue λ_N^h of (4.7) is such that $\lambda_N^h h^2\to 4$ as $h\to 0$ and note the following result on nonuniform observability.

THEOREM 4.2. *For any $T>0$ it follows that, as $h\to 0$,*

$$\sup_{u \text{ solution of } (4.1)}\left[\frac{E_h(0)}{\int_0^T \mid u_N/h\mid^2\,dt}\right]\to\infty. \tag{4.12}$$

Proof of Theorem 4.2. We consider solutions of (4.1) of the form $\vec{u}^h=\cos(\sqrt{\lambda_N^h}t)\vec{w}_N^h$, where λ_N^h and $\vec{w}_N^h$ are the Nth eigenvalue and eigenvector of (4.7), respectively. We have

$$E_h(0)=\frac{h}{2}\sum_{j=0}^{N}\left|\frac{w_{N,j+1}^h-w_{N,j}^h}{h}\right|^2 \tag{4.13}$$

[12] This is a nongeneric fact that occurs only for the constant coefficient 1D problem with uniform meshes.

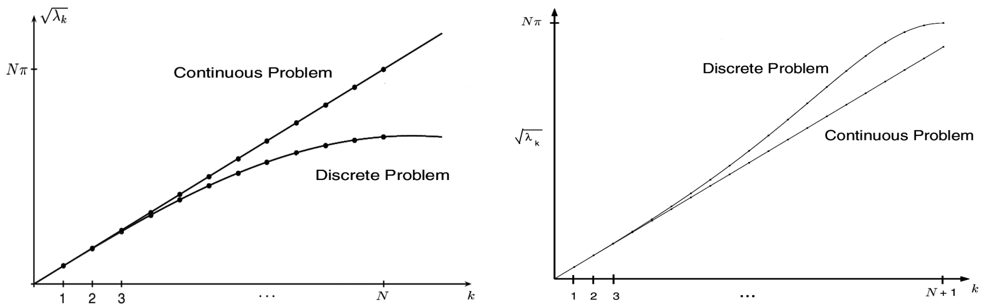

Fig. 2 *Left: Square roots of the eigenvalues in the continuous and discrete cases (finite difference semidiscretization). The gaps are clearly independent of k in the continuous case and of order h for large k in the discrete one. Right: Dispersion diagram for the piecewise linear finite element space semidiscretization versus the continuous wave equation.*

and

$$\int_0^T \left|\frac{u_N^h}{h}\right|^2 dt = \left|\frac{w_{N,N}^h}{h}\right|^2 \int_0^T \cos^2\left(\sqrt{\lambda_N^h}\,t\right) dt. \tag{4.14}$$

Taking into account that $\lambda_N^h \to \infty$ as $h \to 0$, it follows that

$$\int_0^T \cos^2\left(\sqrt{\lambda_N^h}\,t\right) dt \to T/2 \quad \text{as } h \to 0. \tag{4.15}$$

By combining (4.11), (4.13), (4.14), and (4.15), (4.12) follows immediately.

It is important to note that the solution we have used in the proof of this theorem is not the only impediment for the uniform observability inequality to hold.

Indeed, let us consider the following solution of the semidiscrete system (4.1), constituted by the last two eigenvectors:

$$\vec{u} = \frac{1}{\sqrt{\lambda_N}}\left[\exp(i\sqrt{\lambda_N}t)\vec{w}_N - \exp(i\sqrt{\lambda_{N-1}}t)\vec{w}_{N-1}\right]. \tag{4.16}$$

This solution is a simple superposition of two monochromatic semidiscrete waves corresponding to the last two eigenfrequencies of the system. The total energy of this solution is of the order 1 (because each of both components has been normalized in the energy norm and the eigenvectors are orthogonal one to each other). However, the trace of its discrete normal derivative is of the order of h in $L^2(0,T)$. This is due to two facts.

- First, the trace of the discrete normal derivative of each eigenvector is very small compared to its total energy.
- Second, and more important, the gap between $\sqrt{\lambda_N}$ and $\sqrt{\lambda_{N-1}}$ is of the order of h, as is shown in Figure 2, left. This wave packet then has a group velocity of the order of h.

  By Taylor expansion, the difference between the two time-dependent complex exponentials $\exp(i\sqrt{\lambda_N}t)$ and $\exp(i\sqrt{\lambda_{N-1}}t)$ is of the order Th, and we need a time T of the order of $1/h$ to guarantee an observation independent of h. This fact is represented in Figure 3.

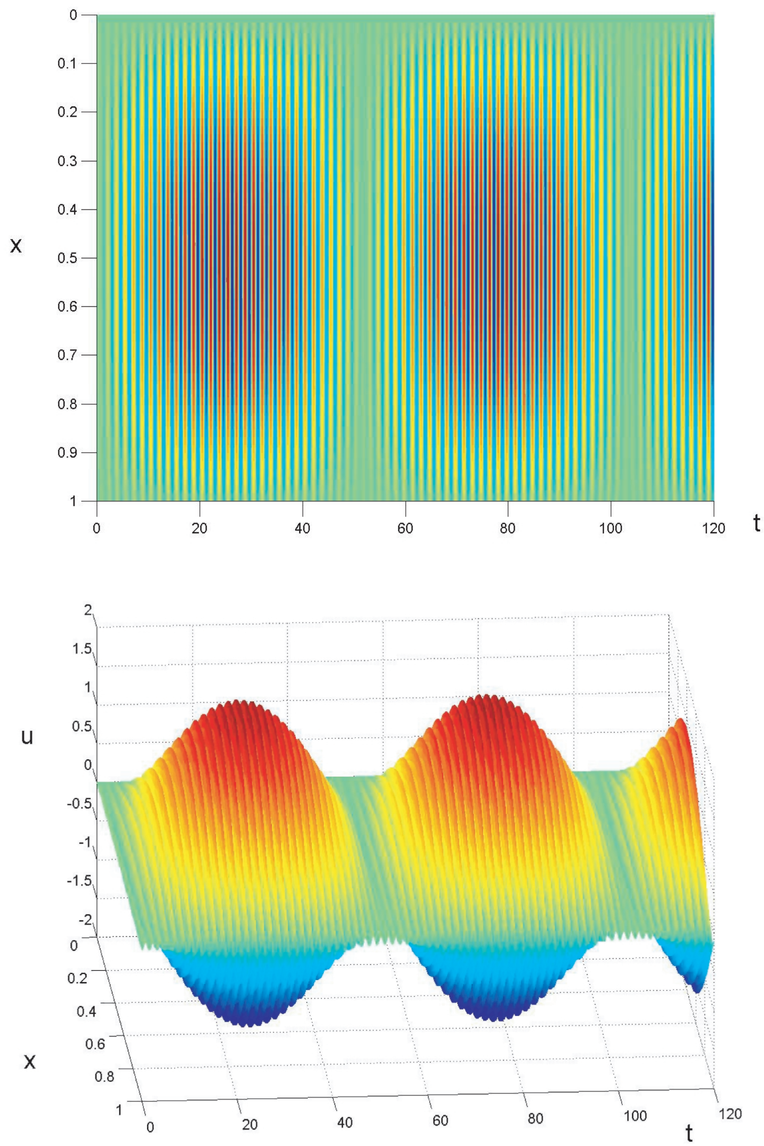

Fig. 3 *Time evolution of solution* (4.16) *for* $h = 1/61$ *(*$N = 60$*) and* $0 \leq t \leq 120$*. It is clear that, according to the figure, the solution seems to exhibit a time-periodicity property with period* τ *of the order of* $\tau \sim 50$*. Note, however, that all solutions of the wave equation are time-periodic of period* 2*. In the figure it is also clear that fronts propagate in space at velocity of the order of* $1/50$*. This is in agreement with the prediction of the theory in the sense that high-frequency wave packets travel at a group velocity of the order of* h*.*

This idea of building wave packets may be used to show that the observability constant has to blow up at infinite order as $h \to 0$. To do this it is sufficient to proceed as above but combining an increasing number of eigenfrequencies. Actually, Micu in [75] proved that the constant $C_h(T)$ blows up exponentially by means of a careful analysis of the biorthogonal sequences to the family of exponentials $\{\exp(i\sqrt{\lambda_k}t)\}_{k=1,\dots,N}$ as $h \to 0$.

All these high-frequency pathologies are in fact very closely related to the notion of group velocity. We refer to [105, 101] for an in-depth analysis of this notion that we discuss briefly in the context of this example.

Since the eigenvectors $\vec{w}_k$ are sinusoidal functions (see (4.9)) the solutions of the semidiscrete system may be written as linear combinations of complex exponentials (in space-time): $\exp[\pm ik\pi[\frac{\sqrt{\lambda_k}}{k\pi}t - x]]$.

In view of this, we see that each monochromatic wave propagates at a speed

$$\frac{\sqrt{\lambda_k}}{k\pi} = \frac{2\sin(k\pi h/2)}{k\pi h} = \frac{\omega_h(\xi)}{\xi}\bigg|_{\{\xi=k\pi h\}} = c_h(\xi)\bigg|_{\{\xi=k\pi h\}}, \tag{4.17}$$

with $\omega_h(\xi) = 2\sin(\xi/2)$. This is the so-called *phase velocity.* The velocity of propagation of monochromatic semidiscrete waves (4.17) turns out to be bounded above and below by positive constants, independently of h, i.e., $0 < \alpha \leq c_h(\xi) \leq \beta < \infty$ for all $h > 0, \xi \in [0, \pi]$. Note that $[0, \pi]$ is the relevant range of frequencies. Indeed, $\xi = j\pi h$ and $j = 1, \dots, N$, $Nh = 1 - h$.

But wave packets may travel at a different speed because of the cancellation phenomena we discussed above. The corresponding speed for those semidiscrete wave packets is given by the derivative of $\omega_h(\cdot)$ (see [101]). At high frequencies ($j \sim N$) the derivative of $\omega_h(\xi)$ at $\xi = N\pi h = \pi(1-h)$ is of the order of h and therefore the wave packet propagates with velocity of the order of h.

Note that the fact that this group velocity is of the order of h is equivalent[13] to the fact that the gap between $\sqrt{\lambda_{N-1}}$ and $\sqrt{\lambda_N}$ is of order h.

According to this analysis, *the group velocity being bounded below is a necessary condition for the uniform observability inequality to hold. Moreover, this is equivalent to a uniform spectral gap condition.*

The convergence property of the numerical scheme guarantees only that the group velocity is correct for low-frequency wave packets.[14] The negative results we have mentioned above are a new reading of well-known pathologies of finite difference schemes for the wave equation.

The careful analysis of this negative example is useful to design possible remedies, i.e., to modify the numerical scheme in order to reestablish the uniform observability inequality. The first remedy is very natural: To cut off the high frequencies or, in other words, to ignore the high-frequency components of the numerical solutions. This method is analyzed in the following section.

4.4. Fourier Filtering. Filtering works as soon as we deal with solutions where the only Fourier components are those corresponding to the eigenvalues $\lambda \leq \gamma h^{-2}$ with $0 < \gamma < 4$ or with indices $0 < j < \delta h^{-1}$ with $0 < \delta < 1$, and the observability

[13] Defining group velocity as the derivative of ω_h, i.e., of the curve in the dispersion diagram (see Figure 2), is a natural consequence of the classical properties of the superposition of linear harmonic oscillators with close but not identical phases (see [24]). There is a one-to-one correspondence between the group velocity and the spectral gap which may be viewed as a discrete derivative of this diagram. In particular, when the group velocity decreases, the gap between consecutive eigenvalues also decreases.

[14] Note that in Figure 2, both for finite differences and elements, the semidiscrete and continuous curves are tangent at low frequencies. This is in agreement with the convergence property of the numerical algorithm under consideration and with the fact that low-frequency wave packets travel essentially with the velocity of the continuous model.

inequality becomes uniform. Note that these classes of solutions correspond to taking projections of the complete solutions by cutting off all frequencies with $\gamma h^{-2} < \lambda < 4h^{-2}$.

It is important to observe that the high-frequency pathologies cannot be avoided by simply taking, for instance, a different approximation of the discrete normal derivative since the fact that the group velocity vanishes is due to the scheme itself and, therefore, cannot be compensated by suitable boundary measurements.

The following classical result due to Ingham in the theory of nonharmonic Fourier series (see Ingham [50] and Young [106]) is useful for proving the uniform observability of filtered solutions.

INGHAM'S THEOREM. *Let $\{\mu_k\}_{k\in\mathbf{Z}}$ be a sequence of real numbers such that $\mu_{k+1} - \mu_k \geq \gamma > 0$ for all $k \in \mathbf{Z}$. Then for any $T > 2\pi/\gamma$ there exists a positive constant $C(T,\gamma) > 0$ such that*

$$\frac{1}{C(T,\gamma)} \sum_{k\in\mathbf{Z}} | a_k |^2 \leq \int_0^T \left| \sum_{k\in\mathbf{Z}} a_k e^{i\mu_k t} \right|^2 dt \leq C(T,\gamma) \sum_{k\in\mathbf{Z}} | a_k |^2 \tag{4.18}$$

for all sequences of complex numbers $\{a_k\} \in \ell^2$.

REMARK 4.1. *Ingham's inequality can be viewed as a generalization of the orthogonality property of trigonometric functions we used to prove the observability of the 1D wave equation in section 3.*

Ingham's inequality allows showing that, as soon as the gap condition is satisfied, there is uniform observability provided the time is large enough.

All these facts confirm that a suitable cutoff or filtering of the spurious numerical high frequencies may be a cure for these pathologies. Let us now describe the basic *Fourier filtering mechanism.* We recall that solutions of (4.1) can be developed in Fourier series as follows: $\vec{u} = \sum_{k=1}^{N}(a_k \cos(\sqrt{\lambda_k^h} t) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin(\sqrt{\lambda_k^h} t))\vec{w}_k^h$, where a_k, b_k are the Fourier coefficients of the initial data, i.e., $\vec{u}^0 = \sum_{k=1}^{N} a_k \vec{w}_k^h$, $\vec{u}^1 = \sum_{k=1}^{N} b_k \vec{w}_k^h$.

Given $0 < \delta < 1$, we introduce the following classes of solutions of (4.1):

$$\mathcal{C}_\delta(h) = \left\{ \vec{u} \text{ sol. of (4.1) s.t. } \vec{u} = \sum_{k=1}^{[\delta/h]} \left(a_k \cos\left(\sqrt{\lambda_k^h} t\right) + \frac{b_k}{\sqrt{\lambda_k^h}} \sin\left(\sqrt{\lambda_k^h} t\right)\right) \vec{w}_k^h \right\}, \tag{4.19}$$

in which the high frequencies corresponding to the indices $j > [\delta(N+1)]$ have been cut off. As a consequence of Ingham's inequality and the analysis of the gap of the spectra of the semidiscrete systems we have the following result.[15]

THEOREM 4.3 (see [48, 49]). *For any $\delta > 0$ there exists $T(\delta) > 0$ such that for all $T > T(\delta)$ there exists $C = C(T,\delta) > 0$ such that*

$$\frac{1}{C} E_h(0) \leq \int_0^T \left| \frac{u_N(t)}{h} \right|^2 dt \leq C E_h(0) \tag{4.20}$$

for every solution u of (4.1) in the class $\mathcal{C}_\delta(h)$ and for all $h > 0$. Moreover, the minimal time $T(\delta)$ for which (4.20) holds is such that $T(\delta) \to 2$ as $\delta \to 0$ and $T(\delta) \to \infty$ as $\delta \to 1$.

REMARK 4.2. *Theorem 4.3 guarantees the uniform observability in each class $\mathcal{C}_\delta(h)$ for all $0 < \delta < 1$, provided the time T is larger than $T(\delta)$.*

---

[15] These results may also be obtained using discrete multiplier techniques [48, 49].

The last statement in the theorem shows that when the filtering parameter δ *tends to zero, i.e., when the solutions under consideration contain fewer and fewer frequencies, the time for uniform observability converges to* $T = 2$*, which is the corresponding one for the continuous equation. This is in agreement with the observation that the group velocity of the low-frequency semidiscrete waves coincides with the velocity of propagation in the continuous model.*

By contrast, when the filtering parameter increases, i.e., when the solutions under consideration contain more and more frequencies, the time of uniform control tends to infinity. This is in agreement and explains further the negative result showing that, in the absence of filtering, there is no finite time T *for which the uniform observability inequality holds.*

The proof of Theorem 4.3 *below provides an explicit estimate on the minimal observability time in the class* $\mathcal{C}_\delta(h)$*:* $T(\delta) = 2/\cos(\pi\delta/2)$*.*

REMARK 4.3. *In the context of the numerical computation of the boundary control for the wave equation the need of an appropriate filtering of the high frequencies was observed by Glowinski* [38] *and further investigated numerically by Asch and Lebeau in* [2]*.*

4.5. Conclusion and Controllability Results. We have shown that the uniform observability property of the finite difference approximations (4.1) fails for any $T > 0$ due to high-frequency wave packets with zero group velocity. On the other hand, we have proved that by filtering the high frequencies or, in other words, considering solutions in the classes $\mathcal{C}_\delta(h)$ with $0 < \delta < 1$, the uniform observability holds in a minimal time $T(\delta)$ that satisfies $T(\delta) \to \infty$ as $\delta \to 1$, $T(\delta) \to 2$ as $\delta \to 0$.

Observe that, as $\delta \to 0$, we recover the minimal observability time $T = 2$ of the continuous wave equation (3.1). This allows us to obtain, for all $T > 2$, the observability property of the continuous wave equation (3.1) as the limit $h \to 0$ of uniform observability inequalities for the semidiscrete systems (4.1). Indeed, given any $T > 2$ there exists $\delta > 0$ such that $T > T(\delta)$ and, consequently, by filtering the high frequencies corresponding to the indices $k > \delta N$, the uniform observability in time T is guaranteed.

Before describing the consequences of these results in the context of controllability, it is important to distinguish two notions on the controllability of any evolution system, regardless of whether it is finite- or infinite-dimensional:

- to control exactly to zero the whole solution for initial data in a given subspace;
- to control the projection of the solution over a given subspace for all initial data.

In the present case, the controlled state equation under consideration is as follows:

$$
(4.21) \qquad \begin{cases} y_j'' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = 0, & 0 < t < T,\ j = 1, \ldots, N, \\ y_0(t) = 0;\ y_{N+1}(t) = v(t), & 0 < t < T, \\ y_j(0) = y_j^0,\ y_j'(0) = y_j^1, & j = 1, \ldots, N. \end{cases}
$$

As we shall see below, for any given $T > 0$ and initial data $(\vec{y}^0, \vec{y}^1)$, there exists a control $v_h \in L^2(0, T)$ such that

$$
(4.22) \qquad \vec{y}(T) = \vec{y}'(T) = 0.
$$

However, this does not mean that the controls will be bounded as h tends to zero. In fact they are not, even if $T \geq 2$, as one could predict from the previous results on observability.

We have the following main results:

- Taking into account that for all $h > 0$ the Kalman rank condition is satisfied, for all $T > 0$ and all $h > 0$ the semidiscrete system (4.21) is controllable. In other words, for all $T > 0$, $h > 0$ and initial data $(\vec{y}^0, \vec{y}^1)$, there exists $v \in L^2(0,T)$ such that the solution $\vec{y}$ of (4.21) satisfies (4.22). Moreover, the control v of the minimal $L^2(0,T)$-norm can be built as in section 3 by minimizing the functional

$$J_h((\vec{u}^0, \vec{u}^1)) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0 \tag{4.23}$$

  over the space of all initial data $(\vec{u}^0, \vec{u}^1)$ for the adjoint semidiscrete system (4.1).

  This strictly convex and continuous functional is coercive thanks to the observability inequality (4.6) and, consequently, has a unique minimizer. The control we are looking for is then

$$v_h(t) = u_N^*(t)/h, \quad 0 < t < T, \tag{4.24}$$

  where $\vec{u}^*$ is the solution of the semidiscrete adjoint system (4.1), corresponding to the initial data $(\vec{u}^{0,*}, \vec{u}^{1,*})$ minimizing the functional J_h.

  In view of Remark 2.2 and the observability inequality (4.6), we deduce that

$$||v_h||_{L^2(0,T)} \leq 2\sqrt{C_h(T)}||(y^0, y^1)||_{*,h}, \tag{4.25}$$

  where $||\cdot||_{*,h}$ denotes the norm

$$||(y^0, y^1)||_{*,h} = \sup_{(u_j^0, u_j^1)_{j=1,\ldots,N}} \left[\left|h\sum_{j=1}^N y_j^0 u_j^1 - h\sum_{j=1}^N y_j^1 u_j^0\right| \Big/ E_h^{1/2}(u^0, u^1)\right]. \tag{4.26}$$

  This norm converges as $h \to 0$ to the norm in $L^2(0,1) \times H^{-1}(0,1)$. It can also be written in terms of the Fourier coefficients and becomes a weighted euclidean norm whose weights are uniformly (with respect to h) equivalent to those of the continuous $L^2 \times H^{-1}$-norm.
- For all $T > 0$ the constant $C_h(T)$ diverges as $h \to 0$. This shows that there are initial data for the wave equation in $L^2(0,1) \times H^{-1}(0,1)$ such that the controls of the semidiscrete systems $v_h = v_h(t)$ diverge as $h \to 0$. There are different ways of making this result precise. For instance, given initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system, we can consider in the semidiscrete control system (4.21) the initial data that take the same Fourier coefficients as (y^0, y^1) for the indices $j = 1, \ldots, N$. It then follows by the Banach–Steinhaus theorem that, because of the divergence of the observability constant $C_h(T)$, there is necessarily some initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the continuous system such that the corresponding controls v_h for the semidiscrete system diverge in $L^2(0,T)$ as $h \to 0$. Indeed, assume that for any initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$, the controls v_h remain uniformly bounded in $L^2(0,T)$ as $h \to 0$. Then, according to the uniform boundedness principle, we would deduce that the maps that associate the controls v_h to the initial data are also uniformly bounded. But, according

to Remark 2.2, this implies the uniform boundedness of the observability constant $C_h(T)$, which we know blows up.

This lack of convergence is in fact easy to understand. As we have shown above, the semidiscrete system generates a lot of spurious high-frequency oscillations. The control of the semidiscrete system has to take these into account. When doing this it gets further and further away from the true control of the continuous wave equation, as the numerical experiments in the following section illustrate.

- The observability inequality is uniform in the class of filtered solutions $\mathcal{C}_\delta(h)$ for $T > T(\delta)$. As a consequence of this, one can control uniformly the projection of the solutions of the semidiscretized systems over subspaces in which the high frequencies have been filtered. More precisely, if the control requirement (4.22) is weakened to

$$\pi_\delta \vec{y}(T) = \pi_\delta \vec{y}\,'(T) = 0, \tag{4.27}$$

where π_δ denotes the projection of the solution of the semidiscrete system (4.21) over the subspace of the eigenfrequencies involved in the filtered space $\mathcal{C}_\delta(h)$, then the corresponding control remains uniformly bounded as $h \to 0$, provided $T > T(\delta)$. The control that produces (4.27) can be obtained by minimizing the functional J_h in (4.23) over the subspace $\mathcal{C}_\delta(h)$. Note that the uniform (with respect to h) coercivity of this functional and, consequently, the uniform bound on the controls hold as a consequence of the uniform observability inequality.

- Moreover, one may recover the controllability property of the continuous wave equation as a limit of this partial controllability results since, as $h \to 0$, the projections π_δ end up covering the whole range of frequencies.

THEOREM 4.4. *Assume that $T > T(\delta)$ as in Theorem 4.3 for some $0 < \delta < 1$. Fix initial data $(y^0, y^1) \in L^2(0,1) \times H^{-1}(0,1)$ for the wave equation. For any $h > 0$ consider the initial data (y^0_h, y^1_h) whose first N Fourier coefficients coincide with those of (y^0, y^1).*

Then

(a) *the corresponding controls v_h in the semidiscrete system* (4.21) *satisfying* (4.27) *are bounded in $L^2(0,T)$;*

(b) *the controls v_h converge as $h \to 0$ to a control $v \in L^2(0,T)$ of the minimal $L^2(0,T)$-norm of the wave equation* (3.4) *such that* (3.5) *holds.*

It is important to note that the time of control depends on the filtering parameter δ in the projections π_δ. But, as we mentioned above, for any $T > 2$ there is a $\delta \in (0,1)$ for which $T > T(\delta)$ and such that the uniform (with respect to h) results apply.

We refer to [65] for the details of the proof of a similar convergence result in the context of the space semidiscretization by finite differences of the beam equation. The proof combines standard arguments in Γ-convergence [28] and the numerical analysis of PDEs.

REMARK 4.4. *In* [48] *and* [49]*, in one space dimension, similar results were proved for the finite element space semidiscretization of the wave equation* (3.1) *as well. This is in agreement with the plot of the dispersion diagram in Figure* 4*, right. This time the discrete spectrum and, consequently, the dispersion diagram lie above the one corresponding to the continuous wave equation. However, the group velocity of high-frequency numerical solutions vanishes again. This is easily seen on the slope of the discrete dispersion curve.*

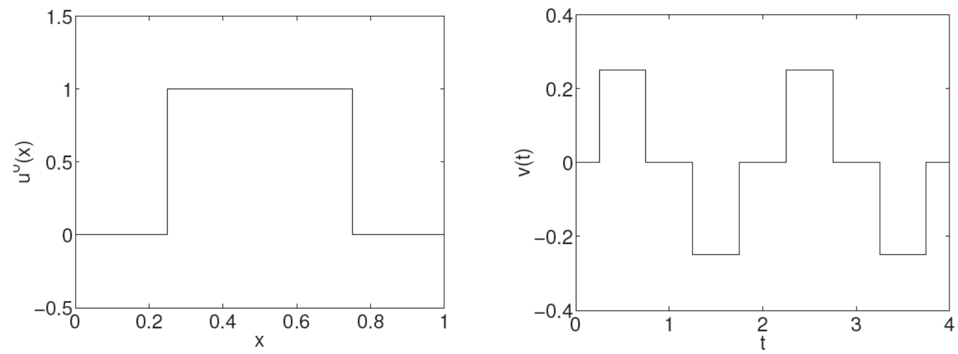

Fig. 4 *Plot of the initial datum to be controlled for the string occupying the space interval* $0 < x < 1$ *(left) and of the time evolution of the exact control for the wave equation in time* $T = 4$ *with this initial data (right).*

4.6. Numerical Experiments. In this section we briefly illustrate by some simple but convincing numerical experiments the theory provided along this section. These experiments have been developed by Rasmussen [91] using MATLAB.

We consider the wave equation in the space interval $(0, 1)$ with control time $T = 4$.

We address the case of continuous and piecewise constant initial data y^0 of the form in Figure 4 (left) together with $y^1 \equiv 0$. In this simple situation and when the control time $T = 4$ the Dirichlet control can be computed at the endpoint $x = 1$ explicitly (Figure 4, right). This can be done using Fourier series and the time periodicity of solutions of the adjoint system (with time period $= 2$) which guarantees complete time orthogonality of the different Fourier components when $T = 4$.

Obviously the time $T = 4$ is sufficient for exact controllability to hold, the minimal control time being $T = 2$.

We now consider the finite difference semidiscrete approximation of the wave equation by finite differences. First of all, we ignore all discussion of the present section concerning the need for filtering. Thus, we merely compute the exact control of the semidiscrete system (4.21). This is done as follows. As described in section 4.5, the control is characterized through the minimization of the functional J_h in (4.23) over the space of all solutions of the adjoint equation (3.1). This allows writing the control v_h as the solution of an equation of the form $\Lambda_h(v_h) = \{-y^1, y^0\}$, where $\{y^0, y^1\}$ is the initial datum to be controlled.

The operator Λ_h can be computed by first solving the adjoint system and then the state equation with the normal derivative of the adjoint state as boundary datum and starting from equilibrium at time $t = T$ (see [67, 68]). Of course, in practice, we do not deal with the continuous adjoint equation but rather with a fully discrete approximation. We simply take the centered discretization in time with time step $\Delta t = 0.99\,\Delta x$ $(\Delta x = h)$, which, of course, guarantees the convergence of the scheme and the fact that our computations yield results which are very close to the semidiscrete case. Applying this procedure to the initial datum under consideration, we get the exact control.

In Figure 5 we show the evolution of the control of the discrete problem as the number of meshpoints N increases, or, equivalently, when the mesh size h tends to zero. We see that when $N = 20$, a low number of meshpoints, the control captures essentially the form of the continuous control in Figure 4 (right) but with some extra

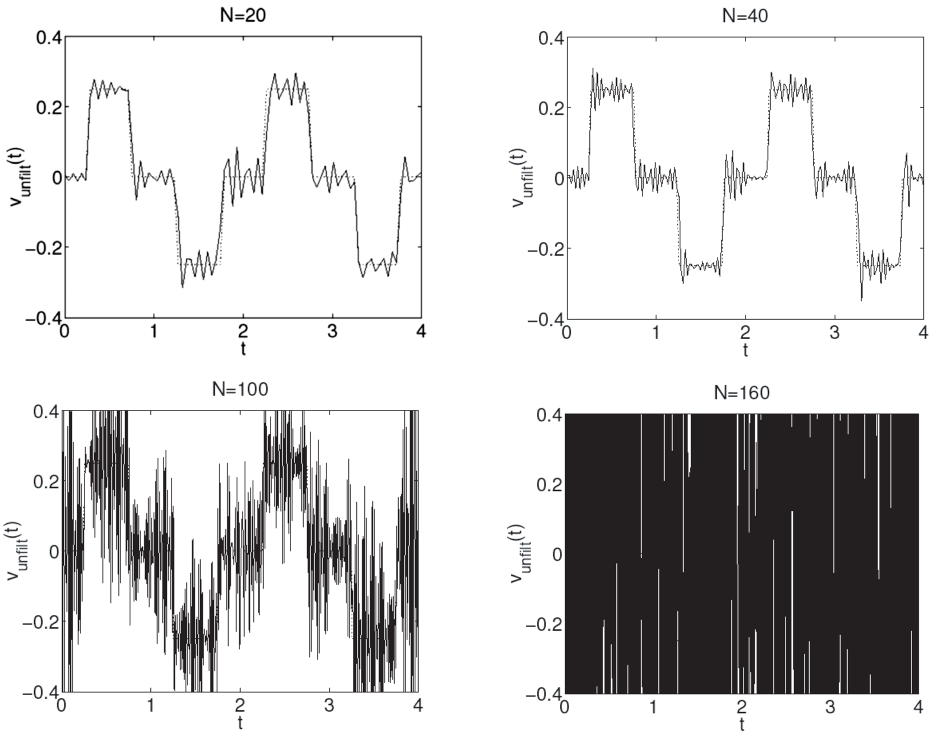

Fig. 5 *Divergent evolution of the control, in the absence of filtering, when the number N of meshpoints increases.*

unwanted oscillations. The situation is very similar when $N = 40$. But when $N = 100$ we see that these oscillations become wild, and for $N = 160$ the dynamics of the control is completely chaotic. This is a good example of lack of convergence in the absence of filtering and confirms the predictions of the theory.

We do the same experiment but now with filtering parameter $= 0.6$, which has been chosen in order to guarantee the uniform observability of the filtered solutions of the adjoint semidiscrete and fully discrete (with $\Delta t = 0.99\Delta x$) schemes in time $T = 4$ and, consequently, the convergence of controls as $h \to 0$ as well. The decay of the error as the number of meshpoints N increases, or, equivalently, when $h \to 0$, is obvious in the figure.

The control for the filtered problem is obtained by restricting and inverting the operator Λ_h above to the solutions of the adjoint system that involve only the Fourier components that remain after filtering. Theory predicts convergence of controls in $L^2(0,T)$. The numerical experiments we draw in Figure 6 confirm this fact. These figures exhibit a mild Gibbs phenomenon which is compatible with L^2-convergence.

5. The Multidimensional Wave Equation. In several space dimensions the observability problem for the wave equation is much more complex and cannot be solved using Fourier series. The velocity of propagation is still one for all solutions but energy propagates along bicharacteristic rays.

However, before going further let us give the precise definition of *bicharacteristic ray*. Consider the wave equation with a scalar, positive, and smooth variable

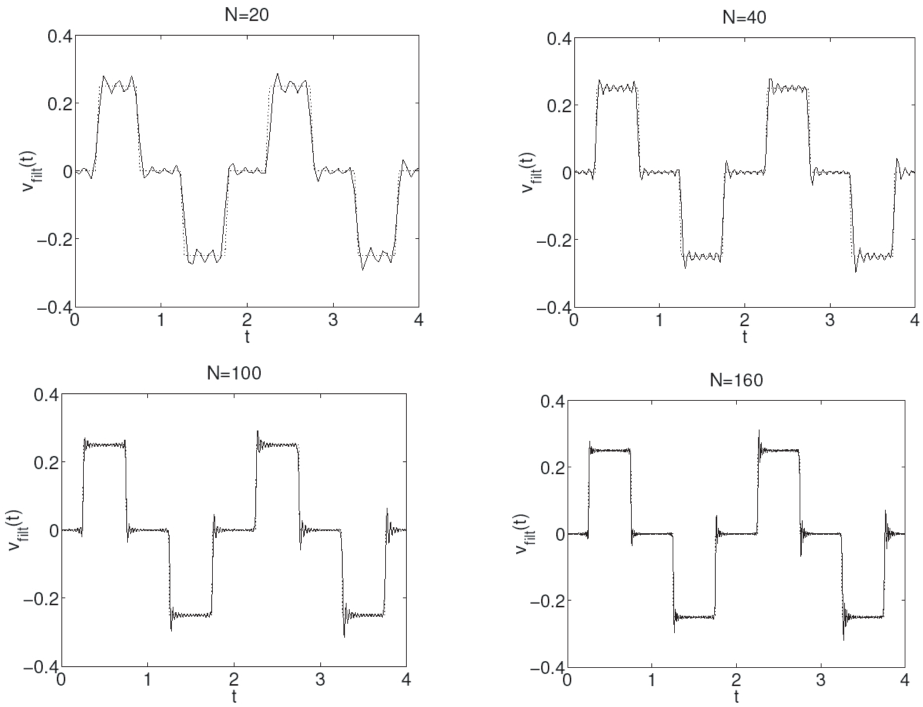

Fig. 6 *Convergent evolution of the control, with filtering parameter* $= 0.6$, *when the number* N *of meshpoints increases.*

coefficient $a = a(x)$:

$$u_{tt} - \operatorname{div}(a(x)\nabla u) = 0. \tag{5.1}$$

Bicharacteristic rays solve the Hamiltonian system

$$\begin{cases} x'(s) = -a(x)\xi;\ t'(s) = \ \tau, \\ \xi'(s) = \nabla a(x)|\xi|^2;\ \tau'(s) = 0. \end{cases} \tag{5.2}$$

Rays describe the microlocal propagation of energy. The projections of the bicharacteristic rays in the (x, t) variables are the rays of geometric optics that play a fundamental role in the analysis of the observation and control properties through the GCC. As time evolves, the rays move in the physical space according to the solutions of (5.2). Moreover, the direction in the Fourier space (ξ, τ) in which the energy of solutions is concentrated as they propagate is given precisely by the projection of the bicharacteristic ray in the (ξ, τ) variables. When the coefficient $a = a(x)$ is constant, the ray is a straight line and carries the energy outward, which is always concentrated in the same direction in the Fourier space, as expected. But for variable coefficients the dynamics is more complex. This Hamiltonian system describes the dynamics of rays in the interior of the domain where the equation is satisfied. When rays reach the boundary they are reflected according to the laws of geometric optics.[16]

When the coefficient $a = a(x)$ varies in space, the dynamics of this system may be quite complex and can lead to some unexpected behavior [74].

[16]Note, however, that tangent rays may be diffractive or even enter the boundary. We refer to [7] for a deeper discussion of these issues.

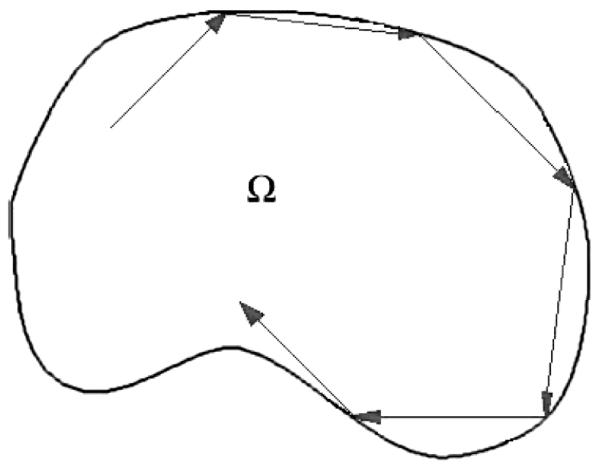

Fig. 7 *Ray that propagates inside the domain Ω following straight lines that are reflected on the boundary according to the laws of geometric optics.*

Let us now address the control problem for smooth domains[17] in the constant coefficient case.

Let Ω be a bounded domain of $\mathbf{R}^n, n \geq 1$, with boundary Γ of class C^2, let ω be an open and nonempty subset of Ω, and let $T > 0$. Consider the linear controlled wave equation in the cylinder $Q = \Omega \times (0,T)$:

$$\begin{cases} y_{tt} - \Delta y = f 1_\omega & \text{in } \ Q, \\ y = 0 & \text{on } \ \Sigma, \\ y(x,0) = y^0(x), y_t(x,0) = y^1(x) & \text{in } \ \Omega. \end{cases} \tag{5.3}$$

In (5.1) Σ represents the lateral boundary of the cylinder Q, i.e., $\Sigma = \Gamma \times (0,T), 1_\omega$ is the characteristic function of the set ω, $y = y(x,t)$ is the state, and $f = f(x,t)$ is the control variable. Since f is multiplied by 1_ω the action of the control is localized in ω.

When $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$ and $f \in L^2(Q)$, the system (5.1) has a unique solution $y \in C\left([0,T]; H_0^1(\Omega)\right) \cap C^1\left([0,T]; L^2(\Omega)\right)$.

The problem of *controllability*, generally speaking, is as follows: *Given* $(y^0, y^1) \in H_0^1(\Omega) \times L^2(\Omega)$, *find* $f \in L^2(Q)$ *such that the solution of system* (5.1) *satisfies*

$$y(T) \equiv y_t(T) \equiv 0. \tag{5.4}$$

The method of section 3, the so-called HUM, shows that the exact controllability property is equivalent to the following *observability inequality*:

$$\left|\left|(u^0, u^1)\right|\right|^2_{L^2(\Omega) \times H^{-1}(\Omega)} \leq C \int_0^T \int_\omega u^2 dx dt \tag{5.5}$$

for every solution of the adjoint uncontrolled system

$$\begin{cases} u_{tt} - \Delta u = 0 & \text{in } \ Q, \\ u = 0 & \text{on } \ \Sigma, \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x) & \text{in } \ \Omega. \end{cases} \tag{5.6}$$

[17] We refer to Grisvard [42] for a discussion of these problems in the context of nonsmooth domains.

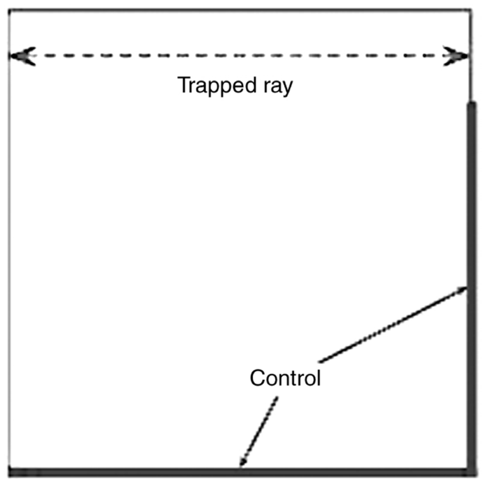

Fig. 8 *A geometric configuration in which the GCC is not satisfied, whatever $T > 0$ is. The domain where waves evolve is a square. The control is located on a subset of two adjacent sides of the boundary, leaving a small vertical subsegment uncontrolled. There is a horizontal line that constitutes a ray that bounces back and forth for all time perpendicularly on two points of the vertical boundaries where the control does not act.*

Roughly speaking, the observability inequality holds if and only if the GCC is satisfied (see, for instance, Bardos, Lebeau, and Rauch [7] and Burq and Gérard [12]). For instance, when the domain is a ball, the subset of the boundary where the control is being applied needs to contain a point of each diameter. Otherwise, if a diameter skips the control region, it may support solutions that are not observed (see Ralston [88]). In the case of the square domain Ω, observability/controllability fail if the control is supported on a set which is strictly smaller than two adjacent sides, as shown in Figure 8.

Several remarks are in order.

Remark 5.1.

(a) *Since we are dealing with solutions of the wave equation, for the GCC to hold, the control time T has to be sufficiently large due to the finite speed of propagation, the trivial case $\omega = \Omega$ being the exception. However, the time being large enough does not suffice, since the control subdomain ω needs to satisfy a geometric condition for the GCC to be fulfilled in finite time. Figure* 8 *provides an example of this fact.*

(b) *Most of the literature on the controllability of the wave equation has been written in the framework of the* boundary control *problem discussed in the previous section. The control problems formulated above for* (5.1) *are usually referred to as* internal controllability *problems since the control acts on the subset ω of Ω. The latter is easier to deal with since it avoids considering nonhomogeneous boundary conditions, in which case solutions have to be defined in the sense of transposition* [67, 68].

Let us now discuss what is known about (5.5):

(a) Using multiplier techniques, Ho [46] proved that if one considers subsets of Γ of the form $\Gamma(x^0) = \left\{x \in \Gamma : (x - x^0) \cdot n(x) > 0\right\}$ for some $x^0 \in \mathbf{R}^n$ (we denote by $n(x)$ the outward unit normal to Ω in $x \in \Gamma$ and by $\cdot$ the scalar product in $\mathbf{R}^n$) and if $T > 0$ is large enough, the following boundary observability inequality holds:

$$||\left(u^0,u^1\right)||^2_{H^1_0(\Omega)\times L^2(\Omega)} \leq C\int_0^T\int_{\Gamma(x^0)}\left|\frac{\partial u}{\partial n}\right|^2 d\sigma dt \tag{5.7}$$

for all $\left(u^0,u^1\right)\in H^1_0(\Omega)\times L^2(\Omega)$, which is the observability inequality that is required to solve the boundary controllability problem.
Later, (5.7) was proved in [67, 68] for any $T > T(x^0) = 2 \parallel x - x^0 \parallel_{L^\infty(\Omega)}$. This is the optimal observability time that one may derive by means of this multiplier (see Osses [86] for other variants).
Proceeding as in [67], one can easily prove that (5.7) implies (5.5) when ω is a neighborhood of $\Gamma(x^0)$ in Ω, i.e., $\omega = \Omega\cap\Theta$, where Θ is a neighborhood of $\Gamma(x^0)$ in $\mathbf{R}^n$, with $T > 2 \parallel x-x^0 \parallel_{L^\infty(\Omega\setminus\omega)}$. In particular, exact controllability holds when ω is a neighborhood of the boundary of Ω.

(b) Bardos, Lebeau, and Rauch [7] proved that, in the class of C^∞ domains, the observability inequality (5.5) holds if and only if the pair (ω, T) satisfies the GCC in Ω: Every ray of geometric optics that propagates in Ω and is reflected on its boundary Γ intersects ω in time less than T.
This result was proved by means of microlocal analysis. Recently the microlocal approach was greatly simplified by Burq [11] by using the microlocal defect measures introduced by Gérard [35] in the context of homogenization and kinetic equations. In [11] the GCC was shown to be sufficient for exact controllability for domains Ω of class C^3 and equations with C^2 coefficients. The result for variable coefficients is the same: The observability inequality and, thus, the exact controllability property hold if and only if all rays of geometric optics intersect the control region before the control time. However, it is important to note that, although in the constant coefficient equation all rays are straight lines, in the variable coefficient case this is no longer the case, which makes it harder to have an easy intuition about the GCC.

6. Space Discretizations of the 2D Wave Equations. In this section we briefly discuss the results in [114] on the space finite difference semidiscretizations of the 2D wave equation in the square $\Omega = (0,\pi)\times(0,\pi)$ of $\mathbf{R}^2$:

$$\begin{cases} u_{tt}-\Delta u = 0 & \text{in} \quad Q=\Omega\times(0,T),\\ u=0 & \text{on} \quad \Sigma=\partial\Omega\times(0,T),\\ u(x,0)=u^0(x),\ u_t(x,0)=u^1(x) & \text{in} \quad \Omega. \end{cases} \tag{6.1}$$

Obviously, the fact that classical finite differences provide divergent results for 1D problems in what concerns observability and controllability indicates that the same should be true in two dimensions as well. This is indeed the case. However, the multidimensional case exhibits some new features and deserves additional analysis, in particular in what concerns filtering techniques. Given $\left(u^0,u^1\right)\in H^1_0(\Omega)\times L^2(\Omega)$, system (6.1) admits a unique solution $u\in C\left([0,T];H^1_0(\Omega)\right)\cap C^1\left([0,T];L^2(\Omega)\right)$. Moreover, the energy

$$E(t)=\frac{1}{2}\int_\Omega\left[\mid u_t(x,t)\mid^2+\mid\nabla u(x,t)\mid^2\right]dx \tag{6.2}$$

remains constant, i.e.,

$$E(t)=E(0) \quad \forall 0<t<T. \tag{6.3}$$

Let Γ_0 denote a subset of the boundary of Ω constituted by two consecutive sides, for instance,

$$\Gamma_0 = \{(x_1,\pi) : x_1 \in (0,\pi)\} \cup \{(\pi, x_2) : x_2 \in (0,\pi)\}\,. \tag{6.4}$$

It is well known (see [67, 68]) that for $T > 2\sqrt{2}\pi$ there exists $C(T) > 0$ such that

$$E(0) \leq C(T) \int_0^T \int_{\Gamma_0} \left| \frac{\partial u}{\partial n} \right|^2 d\sigma dt \tag{6.5}$$

holds for every finite-energy solution of (6.1). In (6.5), n denotes the outward unit normal to Ω, $\partial \cdot / \partial n$ the normal derivative, and $d\sigma$ the surface measure.

We can now address the standard five-point finite difference space semidiscretization scheme for the 2D wave equation.

As in one dimension we may perform a complete description of both the continuous solutions and those of the semidiscrete system in terms of Fourier series. One can then deduce the following:

- The semidiscrete system is observable for all time T and mesh size h;
- The observability constant $C_h(T)$ blows up as h tends to 0 because of the spurious high-frequency numerical solutions.
- The uniform (with respect to h) observability property may be reestablished by a suitable filtering of the high frequencies. There is, however, an important difference at this level with respect to the 1D case that we mention now.

The upper bound on the spectrum of the semidiscrete system in two dimensions is $8/h^2$. However, in two dimensions it is not sufficient to filter by a constant $0 < \gamma < 8$, i.e., to consider solutions that do not contain the contribution of the high frequencies $\lambda > \gamma\, h^{-2}$, to guarantee uniform observability.

In fact, one has to filter by means of a constant $0 < \gamma < 4$. This is due to the existence of solutions corresponding to high-frequency oscillations in one direction and very slow oscillations in the other. Roughly speaking, one needs to filter efficiently in both space directions, and this requires taking $\gamma < 4$ (see [114]).

In order to better understand the necessity of filtering and getting sharp observability times it is convenient to adopt the approach of [72, 73] based on the use of discrete Wigner measures. The symbol of the semidiscrete system for solutions of wavelength h is

$$\tau^2 - 4\left(\sin^2(\xi_1/2) + \sin^2(\xi_2/2)\right) \tag{6.6}$$

and can be easily obtained as in the von Neumann analysis of the stability of numerical schemes by taking the Fourier transform of the semidiscrete equation: the continuous one in time and the discrete one in space.[18]

Note that in the symbol in (6.6) the parameter h disappears. This is due to the fact that we are analyzing the propagation of waves of wavelength of the order of h.

The bicharacteristic rays are then defined as follows:

$$\begin{cases} x_j'(s) = -2\sin(\xi_j/2)\cos(\xi_j/2) = -\sin(\xi_j), \quad j = 1,2, \\ t'(s) = \tau, \\ \xi_j'(s) = 0, \quad j = 1,2, \\ \tau'(s) = 0. \end{cases} \tag{6.7}$$

[18] This argument can be easily adapted to the case where the numerical approximation scheme is discrete in both space and time by taking discrete Fourier transforms in both variables.

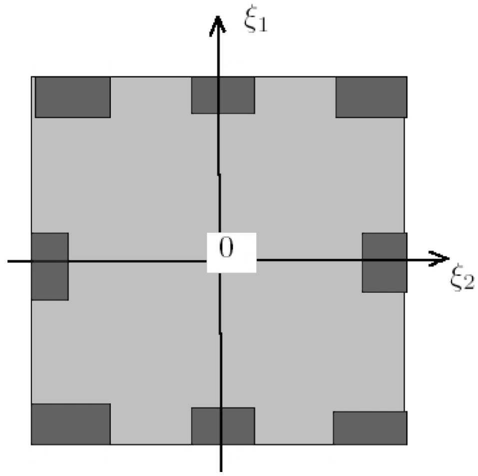

Fig. 9 *This figure represents the zones in the frequency space that need to be filtered out in order to guarantee a uniform minimal velocity of propagation of rays as $h \to 0$. When the filtering excludes the areas within the eight small neighborhoods of the distinguished points on the boundary of the frequency cell, the velocity of propagation of rays is uniform. Obviously the minimal velocity depends on the size of these patches that have been removed by filtering and, consequently, so does the observation/control time.*

It is interesting to note that the rays are straight lines, as for the constant coefficient wave equation, since the coefficients of the equation and the numerical discretization are both constant. We see, however, that in (6.7) both the direction and the velocity of propagation change with respect to those of the continuous wave equation.

Let us now consider initial data for this Hamiltonian system with the following particular structure: x_0 is any point in the domain Ω, the initial time $t_0 = 0$, and the initial microlocal direction (τ^*, ξ^*) is such that

$$(\tau^*)^2 = 4\left(\sin^2(\xi_1^*/2) + \sin^2(\xi_2^*/2)\right). \tag{6.8}$$

Note that the last condition is compatible with the choice $\xi_1^* = 0$ and $\xi_2^* = \pi$ together with $\tau^* = 2$. Thus, let us consider the initial microlocal direction $\xi_2^* = \pi$ and $\tau^* = 2$. In this case the ray remains constant in time, $x(t) = x_0$, since, according to the first equation in (6.7), x_j' vanishes both for $j = 1$ and $j = 2$. Thus, the projection of the ray over the space x does not move as time evolves. This ray never reaches the exterior boundary $\partial\Omega$ where the equation evolves and excludes the possibility of having a uniform boundary observability property. More precisely, this construction allows one to show that, as $h \to 0$, there exists a sequence of solutions of the semidiscrete problem whose energy is concentrated in any finite time interval $0 \le t \le T$ as much as one wishes in a neighborhood of the point x_0.

Note that this example corresponds to the case of very slow oscillations in the space variable x_1 and very rapid ones in the x_2-direction, and it can be ruled out, precisely, by taking the filtering parameter $\gamma < 4$. In view of the structure of the Hamiltonian system, it is clear that one can be more precise when choosing the space of filtered solutions. Indeed, it is sufficient to exclude by filtering the rays that do not propagate at all to guarantee the existence of a minimal velocity of propagation (see

Figure 9).[19]

All the results we have presented in this section have their counterpart in the context of controllability, which are close analogues of those developed previously in the 1D case.

7. Other Remedies for High-Frequency Pathologies. In the previous sections we have described the high-frequency spurious oscillations that arise in finite difference space semidiscretizations of the wave equation and how they produce divergence of the controls as the mesh size tends to zero. We have also shown that there is a remedy for this, which consists in filtering the high frequencies by truncating the Fourier series. However, this method, which is natural from a theoretical point of view, can be hard to implement in numerical simulations. Indeed, solving the semidiscrete system provides the nodal values of the solution. One then needs to compute its Fourier coefficients and, once this is done, to recalculate the nodal values of the filtered/truncated solution. Therefore, it is convenient to explore other ways of avoiding these high-frequency pathologies that do not require going back and forth from the physical space to the frequency one. Here we shall briefly discuss other cures that have been proposed in the literature.

7.1. Tychonoff Regularization. Glowinski, Li, and Lions [41] proposed a Tychonoff regularization technique that allows one to recover the uniform (with respect to the mesh size) coercivity of the functional that one must minimize to get the controls in the HUM approach. The method was tested to be efficient in numerical experiments. Here we give a sketch of the proof of convergence in the particular case under consideration, which is new, to our knowledge.

Let us recall that the lack of uniform observability makes the functionals (4.23) not uniformly coercive, as we mentioned in section 4.5. As a consequence of this, for some initial data, the controls v_h diverge as $h \to 0$. In order to avoid this lack of uniform coercivity, the functional J_h can be reinforced by means of a Tychonoff regularization procedure.[20] Consider the new functional

$$
\begin{aligned}
J_h^*((u_j^0, u_j^1)_{j=1,\ldots,N}) = \frac{1}{2}\int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt + h^3 \sum_{j=0}^{N} \int_0^T \left|\frac{u'_{j+1}-u'_j}{h}\right|^2 dt \\
+ h\sum_{j=1}^{N} y_j^0 u_j^1 - h\sum_{j=1}^{N} y_j^1 u_j^0.
\end{aligned}
\tag{7.1}
$$

This functional is coercive when $T > 2$ and, more importantly, its coercivity is uniform

[19]Roughly speaking, this suffices for the observability inequality to hold uniformly in h for a sufficiently large time [72, 73]. This ray approach makes it possible to obtain the optimal uniform observability time depending on the class of filtered solutions under consideration. The optimal time is simply that needed by all characteristic rays entering in the class of filtered solutions to reach the controlled region. It is in fact the discrete version of the GCC for the continuous wave equation. Moreover, if the filtering is done so that the wavelength of the solutions under consideration is of an order strictly less than h, then one recovers the classical observability result for the constant coefficient continuous wave equation with the optimal observability time.

[20]This functional is a variant of the one proposed in [41], where the added term was $h^2||(\vec{u}^0,\vec{u}^1)||^2_{H^2\times H^1}$ instead of $h^3\sum_{j=0}^N\int_0^T\left|\frac{u'_{j+1}-u'_j}{h}\right|^2 dt$. Both terms have the same scales, so that both are negligible at low frequencies but are of the order of the energy for the high ones. This is due to the fact that for the solutions of wave-like equations the H^1-norm of u_t is of the order of the H^2-norm of u. The one introduced in (7.1) arises naturally in view of (7.2,) but the same arguments could be used to justify the convergence of the one proposed in [41].

in h. This is a consequence of the following observability inequality (see [100]):

$$E_h(0) \leq C(T)\Big[\int_0^T \Big|\frac{u_N(t)}{h}\Big|^2 dt + h^3 \sum_{j=0}^{N} \int_0^T \Big|\frac{u'_{j+1}-u'_j}{h}\Big|^2 dt\Big]. \tag{7.2}$$

This inequality holds for all $T > 2$ for a suitable $C(T) > 0$ which is independent of h and of the solution of the semidiscrete problem (4.1) under consideration.

Note that in (7.2) we have the extra term

$$h^3 \sum_{j=0}^{N} \int_0^T \Big|\frac{u'_{j+1}-u'_j}{h}\Big|^2 dt, \tag{7.3}$$

which has also been used in the regularization of the functional J_h^* in (7.1). By inspection of the solutions of (4.1) in separated variables it is easy to understand why this added term is a suitable one to reestablish the uniform observability property. Indeed, consider the solution of the semidiscrete system $u = \exp(\pm i\sqrt{\lambda_j} t) w_j$. The extra term we have added is of the order of $h^2\lambda_j E_h(0)$. Obviously this term is negligible as $h \to 0$ for the low-frequency solutions (for j fixed) but becomes relevant for the high-frequency ones when $\lambda_j \sim 1/h^2$. Accordingly, when inequality (4.6) fails, i.e., for the high-frequency solutions, the extra term in (7.2) reestablishes the uniform character of the estimate with respect to h. It is important to emphasize that both terms are needed for (7.2) to hold. Indeed, (7.3) by itself does not suffice since its contribution vanishes as $h \to 0$ for the low-frequency solutions.

As we said above, this uniform observability inequality guarantees the uniform boundedness of the minima of J_h^* and the corresponding controls. But there is an important price to pay. The control that J_h^* yields not only is at the boundary but also is distributed everywhere in the interior of the domain. The corresponding control system reads as follows:

$$\begin{cases} y''_j - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = h^2 g'_{h,j}, & 0 < t < T,\ j = 1, \dots, N, \\ y_0(t) = 0;\ y_{N+1}(t) = v_h(t), & 0 < t < T, \\ y_j(0) = y_j^0,\ y'_j(0) = y_j^1, & j = 1, \dots, N. \end{cases} \tag{7.4}$$

And the controlled state satisfies $\vec{y}_h(T) \equiv \vec{y}'_h(T) \equiv 0$. In this case, roughly speaking, when the initial data are fixed independently of h (for instance, we consider initial data in $L^2(0,1) \times H^{-1}(0,1)$ and we choose those in (7.4) as the corresponding Fourier truncation) then there exist controls $v_h \in L^2(0,T)$ and g_h such that the solution of (7.4) reaches equilibrium at time T with the following uniform bounds:

$$v_h \text{ is uniformly bounded in } L^2(0,T), \tag{7.5}$$

$$||(A_h)^{-1/2}\vec{g}_h||_h \text{ is uniformly bounded in } L^2(0,T), \tag{7.6}$$

where A_h is the matrix in (4.3), and $||\cdot||_h$ stands for the standard euclidean norm

$$||\overrightarrow{f_h}||_h = \Big[h \sum_{j=1}^{N} |f_{h,j}|^2\Big]^{1/2}. \tag{7.7}$$

These bounds on the controls can be obtained directly from the coercivity property of the functional J_h^* we minimize, which is a consequence of the uniform observability inequality (7.2). The roles that the two controls play are of different natures: The internal control $h^2 g_h'$ takes care of the high-frequency spurious oscillations, and the boundary control deals with the low-frequency components. In fact, it can be shown that, as $h \to 0$, the boundary control v_h converges to the control v of (3.4) in $L^2(0,T)$. In this sense, the limit of the control system (7.4) is the boundary control problem for the wave equation. To better understand this fact it is important to observe that, due to the h^2 multiplicative factor on the internal control, its effect vanishes in the limit. Indeed, in view of the uniform bound (7.6), roughly speaking,[21] the internal control is of the order of h^2 in the space $H^{-1}(0,T;H^{-1}(0,1))$ and therefore tends to zero in the distributional sense. The fact that the natural space for the internal control is $H^{-1}(0,T;H^{-1}(0,1))$ comes from the nature of the regularizing term introduced in the functional J_h^*. Indeed, its continuous counterpart is $\int_0^T \int_0^1 |\nabla u_t|^2 dxdt$ and it can be seen that, by duality, it produces controls of the form $\partial_t \partial_x(f)$ with $f \in L^2((0,1)\times(0,T))$. The discrete internal control reproduces this structure.

The control $h^2 g_{h,j}'$ is bounded in L^2 with respect to both space and time. This is due to two facts: (a) the norm of the operator $(A_h)^{1/2}$ is of order $1/h$, and (b) taking one time derivative produces multiplicative factors of order $\sqrt{\lambda}$ for the solutions in separated variables. Since the maximum of the square roots of the eigenvalues at the discrete level is of order $1/h$, this yields a contribution of order $1/h$ too. These two contributions are balanced by the multiplicative factor h^2. Now recall that the natural space for the controlled trajectories is $L^\infty(0,T;L^2(0,1)) \cap W^{1,\infty}(0,T;H^{-1}(0,1))$ at the continuous level, with the corresponding counterpart for the discrete one. However, the right-hand side terms in L^2 for the wave equation produce finite energy solutions in $L^\infty(0,T;H^1(0,1)) \cap W^{1,\infty}(0,T;L^2(0,1))$. Thus, the added internal control produces only a compact correction on the solution at the level of the space $L^\infty(0,T;L^2(0,1)) \cap W^{1,\infty}(0,T;H^{-1}(0,1))$. As a consequence of this one can show, for instance, that, using only boundary controls, one can reach states at time T that weakly (resp., strongly) converge to zero as $h \to 0$ in $H^1(0,1)\times L^2(0,1)$ (resp., $L^2(0,1)\times H^{-1}(0,1)$).

Summarizing, we may say that a Tychonoff regularization procedure may allow controlling the semidiscrete system uniformly at the price of adding an extra internal control but in such a way that the boundary controls converge to the boundary control for the continuous wave equation. Consequently, in practice, one can ignore the internal control this procedure gives and keep only the boundary one that, even though it does not exactly control the numerical approximation scheme it does converge to the right control of the wave equation. Thus, the method is efficient for computing approximations of the boundary control for the wave equation as the numerical experiments in [41] confirm.

7.2. A Two-Grid Algorithm. Glowinski and Li in [40] introduced a two-grid algorithm that also makes it possible to compute efficiently the control of the continuous model. The method was further developed by Glowinski in [38].

The relevance and impact of using two grids can be easily understood in view of the above analysis of the 1D semidiscrete model. In section 4 we have seen that all the eigenvalues of the semidiscrete system satisfy $\lambda \le 4/h^2$. We have also seen that the observability inequality becomes uniform when one considers solutions involving eigenvectors corresponding to eigenvalues $\lambda \le 4\gamma/h^2$, with $\gamma < 1$. Glowinski's algo-

[21] To make this more precise we should introduce Sobolev spaces of negative order at the discrete level as in (4.26). This can be done using Fourier series representations or extension operators from the discrete grid to the continuous space variable.

rithm is based on the idea of using two grids: one with step size h and a coarser one of size $2h$. In the coarser mesh the eigenvalues obey the sharp bound $\lambda \leq 1/h^2$. Thus, the oscillations in the coarse mesh that correspond to the largest eigenvalues $\lambda \sim 1/h^2$ are associated to eigenvalues in the class of filtered solutions with parameter $\gamma = 1/4$ in the finer mesh. Formally, this corresponds to a situation where the observability inequality is uniform for $T > 2/\cos(\pi/8)$.

This explains the efficiency of the two-grid algorithm for computing the control of the continuous wave equation.

This method was introduced by Glowinski [38] in the context of the full finite difference and finite element discretizations in two dimensions. It was then further developed in the framework of finite differences by Asch and Lebeau in [2], where the GCC for the wave equation in different geometries was tested numerically.

The convergence of this method has recently been proved rigorously in [85] for finite difference and finite element semidiscrete approximations in one space dimension. It was also proved that the sharp time for the convergence of the algorithm is $T = 4$, twice the minimal time needed for the control of the continuous wave equation.

In practice, the two-grid algorithm works as follows: One minimizes J_h over the subspace of data obtained by extending the slowly oscillating data given over the coarse mesh to the fine one by interpolation. This gives a sequence of bounded (as h tends to zero) controls. The controls, for h fixed, provide a partial controllability result in the sense that only a projection of solutions of the controlled system over the coarse grid vanishes. But the limit of these controls as h tends to zero is an exact control for the wave equation. Consequently, the two-grid algorithm is a good method for getting numerical approximations of the control of the wave equation.

The key point in the proof of this result in [85] is a uniform (with respect to h) observability inequality for the adjoint system over the subspace of slowly varying interpolated data.

7.3. Mixed Finite Elements. Let us now discuss a different approach that is somewhat simpler than the previous ones. It consists in using mixed finite element methods rather than finite differences or standard finite elements. As we have seen, these two require some filtering—Tychonoff regularization or multigrid techniques—to provide convergent methods for the computation of controls. The advantage of the mixed finite elements, as we shall see, is that they may converge without the need of any extra filtering or corrections.

First of all, it is important to emphasize that the analysis we have developed in section 4 for the finite difference space semidiscretization of the 1D wave equation can be carried out with minor changes for finite element semidiscretizations as well. In particular, due to the high-frequency spurious oscillations, uniform observability does not hold [49]. It is thus natural to consider mixed finite element methods. This idea was introduced by Banks, Ito, and Wang [5] in the context of boundary stabilization of the wave equation. Here we adapt that approach to the analysis of controllability and observability. A variant of this method was introduced in [39].

The starting point is writing the adjoint wave equation (3.1) in the system form $u_t = v$, $v_t = u_{xx}$. We now use two different Galerkin bases for the approximation of u and v. Since u lies in H_0^1, we use classical piecewise linear finite elements, and for v piecewise constant ones.

In these bases, and after some work which is needed to handle the fact that the left- and right-hand side terms of the equations in this system do not have the same

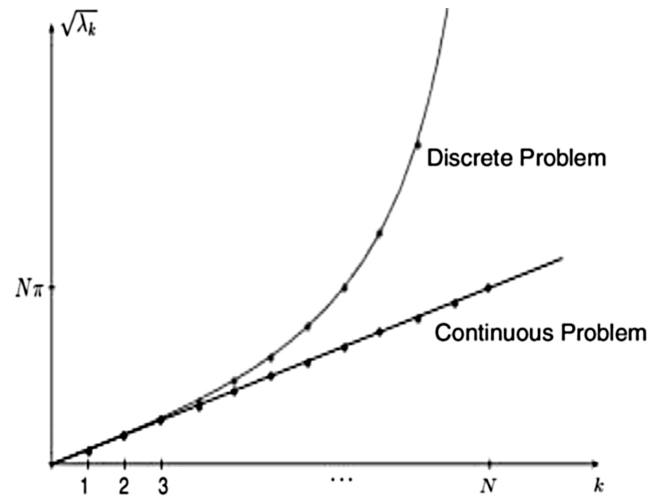

Fig. 10 *Square roots of the eigenvalues in the continuous and discrete cases with mixed finite elements (compare with Figure 2).*

regularity, one is led to the following semidiscrete system:

$$(7.8)\quad \begin{cases} \frac{1}{4}\left[u''_{j+1}+u''_{j-1}+2u''_j\right]=\frac{1}{h^2}\left[u_{j+1}+u_{j-1}-2u_j\right], & 0<t<T,\ j=1,\ldots,N,\\ u_j(t)=0, & j=0,\ N+1,\\ u_j(0)=u_j^0,\ u'_j(0)=u_j^1, & j=1,\ldots,N. \end{cases}$$

This system is a good approximation of the wave equation and converges in classical terms. Moreover, the spectrum of the mass and stiffness matrices involved in this scheme can be computed explicitly and the eigenvectors are those of (4.9), i.e., the restriction of the sinusoidal eigenfunctions of the Laplacian to the mesh points. The eigenvalues are now

$$(7.9)\qquad \lambda_k=\frac{4}{h^2}\tan^2(k\pi h/2),\quad k=1,\ldots,N.$$

For this spectrum the gap between the square roots of consecutive eigenvalues is uniformly bounded from below, and in fact tends to infinity for the highest frequencies as $h\to 0$ (Figure 10). According to this, and applying Ingham's inequality, the uniform observability property may be proved (see [16]). Note, however, that one cannot expect (4.6) to hold since it is not even uniform for the eigenvectors. One gets instead that, for all $T>2$, there exists $C(T)>0$ such that

$$(7.10)\qquad E_h(0)\le C(T)\int_0^T\left[\left|\frac{u_N(t)}{h}\right|^2+h^2\left|\frac{u'_N(t)}{h}\right|^2\right]dt$$

for every solution of (7.8) and for all $h>0$. As a consequence, the corresponding systems are also uniformly controllable and the controls converge as $h\to 0$. These results may be extended to suitable 2D mixed finite element schemes.

One of the drawbacks of this method is that the CFL stability condition that is required when dealing with fully discrete approximations based on this method is stronger than for classical finite difference or finite element methods because of the sparsity of the spectrum. This is why one has to be more careful in the choice of the time-discretization scheme (see [16, 78]).

The mixed finite element method above may also be viewed as a suitable modification of the classical finite difference scheme in which the mass matrix has been worked out to improve the dispersion diagram of the scheme. This is a classical idea (see, for instance, Krenk [58]).

8. Other Models. The controllability properties of wave-like equations are rather unstable under numerical discretizations. In this section we show that the dissipative and dispersive effects that the heat and Schrödinger equations introduce, respectively, do reestablish the stability.

8.1. Finite Difference Space Semidiscretizations of the Heat Equation. As we mentioned in the introduction, the dissipative character of the models may help to reestablish stability. But mild dissipation does not suffice. That is, for instance, the case for the dissipative wave equation $u_{tt} - \Delta u + k u_t = 0$ that, under the change of variables $v = e^{-kt/2} u$, can be transformed into the wave equation plus potential $v_{tt} - \Delta v - \frac{k^2}{4} v = 0$. In the latter, the presence of the zero order potential introduces a compact perturbation of the d'Alembertian and does not change the dynamics of the system in what concerns the problems of observability and controllability under consideration. Therefore, the presence of the damping term in the equation for u introduces, roughly, a decay rate[22] in time of the order of $e^{-kt/2}$ but does not change the properties of the system in what concerns control/observation. As we shall see in this section the strong damping that the heat equation introduces helps much more.

The convergence of numerical schemes for control problems associated with parabolic equations has been extensively studied in the literature (see, e.g., [56, 94, 102]). But this has been done mainly in the context of optimal control and very little is known about the controllability issues that we address now.

Let us consider the following 1D heat equation with control acting at the boundary point $x = L$:

$$\text{(8.1)} \qquad \begin{cases} y_t - y_{xx} = 0, & 0 < x < L,\ 0 < t < T, \\ y(0,t) = 0,\ y(L,t) = v(t), & 0 < t < T, \\ y(x,0) = y^0(x), & 0 < x < L. \end{cases}$$

This is the so-called boundary control problem. It is by now well known that (8.1) is null controllable in any time $T > 0$ (see, for instance, Russell [96, 97]). To be more precise, the following holds: *For any $T > 0$ and $y^0 \in L^2(0,L)$ there exists a control $v \in L^2(0,T)$ such that the solution y of* (8.1) *satisfies $y(x,T) \equiv 0$ in $(0,L)$.*

This null controllability result is equivalent to a suitable observability inequality for the adjoint system

$$\text{(8.2)} \qquad \begin{cases} u_t + u_{xx} = 0, & 0 < x < L,\ 0 < t < T, \\ u(0,t) = u(L,t) = 0, & 0 < t < T, \\ u(x,T) = u^0(x), & 0 < x < L. \end{cases}$$

Note that, in this case, due to the time irreversibility of the state equation and its adjoint, in order to guarantee that the latter is well-posed, we take the initial

[22] This is true for low-frequency solutions. However, the decay rate may be lower for low-frequency ones when k is large enough. This can easily be seen by means of Fourier decomposition. This is the so-called *overdamping* phenomenon.

conditions at the final time $t = T$. The corresponding observability inequality is as follows[23]: *For any $T > 0$ there exists $C(T) > 0$ such that*

$$\int_0^L u^2(x,0)dx \leq C \int_0^T |u_x(L,t)|^2 \, dt \tag{8.3}$$

holds for every solution of (8.2).[24]

Let us now consider semidiscrete versions of these problems:

$$\begin{cases} y_j' - \frac{1}{h^2}\left[y_{j+1} + y_{j-1} - 2y_j\right] = 0, & 0 < t < T, \quad j = 1, \dots, N, \\ y_0 = 0, \, y_{N+1} = v, & 0 < t < T, \\ y_j(0) = y_j^0, & j = 1, \dots, N; \end{cases} \tag{8.4}$$

$$\begin{cases} u_j' + \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] = 0, & 0 < t < T, \, j = 1, \dots, N, \\ u_0 = u_{N+1} = 0, & 0 < t < T, \\ u_j(T) = u_j^0, & j = 1, \dots, N. \end{cases} \tag{8.5}$$

According to the Kalman criterion for controllability in section 2, for any $h > 0$ and for all time $T > 0$ system (8.4) is controllable and (8.5) observable. In fact, in this case, in contrast with the results we have described for the wave equation, these properties hold uniformly as $h \to 0$. More precisely, the following results hold.

THEOREM 8.1 (see [71]). *For any $T > 0$ there exists a positive constant $C(T) > 0$ such that*

$$h \sum_{j=1}^N |u_j(0)|^2 \leq C(T) \int_0^T \left|\frac{u_N(t)}{h}\right|^2 dt \tag{8.6}$$

holds for any solution of (8.5) *and any $h > 0$.*

THEOREM 8.2 (see [71]). *For any $T > 0$ and $\left\{y_1^0, \dots, y_N^0\right\}$ there exists a control $v \in L^2(0,T)$ such that the solution of* (8.4) *satisfies*

$$y_j(T) = 0, \quad j = 1, \dots, N. \tag{8.7}$$

Moreover, there exists a constant $C(T) > 0$, independent of $h > 0$, such that

$$\| v \|^2_{L^2(0,T)} \leq C(T) h \sum_{j=1}^N \left|y_j^0\right|^2. \tag{8.8}$$

These results were proved in [71] using Fourier series and a classical result on the sums of real exponentials (see, for instance, Fattorini and Russell [30]) that plays the role of Ingham's inequality in the context of parabolic equations.

[23] This inequality has been greatly generalized to heat equations with potentials in several space dimensions, with explicit observability constants depending on the potentials, etc. (see, for instance, [33, 31])

[24] Note that the observability estimate (8.3) does not provide information on $u(T)$. In fact, due to the regularizing effect of the heat equation it would be impossible to get estimates on u^0 on any Sobolev norm (see [31]). At the control level, this corresponds to the fact that, again, due to the regularizing effect, it would be impossible to drive the state y in the final time T to an arbitrary L^2-function, for instance. The strong regularizing effect of the heat equation introduces time-irreversibility both in the problems of control and observation.

The explicit form of the spectrum and its properties play a key role in the proof. One uses in particular that there exists $c > 0$ such that $\lambda_j^h \geq cj^2$ for all $h > 0$ and $j = 1, \ldots, N$ and that the uniform gap condition is also satisfied. Recall that, in the context of the wave equation, the lack of gap for the square roots of these eigenvalues was observed for the high frequencies and that this was the main reason for the lack of uniform bound on the controls. In particular it was found that $\sqrt{\lambda_N^h} - \sqrt{\lambda_{N-1}^h} \sim h$. But, in the present case, it follows that $\lambda_N^h - \lambda_{N-1}^h \sim (\sqrt{\lambda_N^h} - \sqrt{\lambda_{N-1}^h})(\sqrt{\lambda_N^h} + \sqrt{\lambda_{N-1}^h}) \sim 1$, since $\sqrt{\lambda_N^h} + \sqrt{\lambda_{N-1}^h} \sim 1/h$. This fact describes clearly why the gap condition is fulfilled in this case.

Once the uniform observability inequality of Theorem 8.1 is proved, the controls for the semidiscrete heat equation (8.4) can be easily constructed by means of the minimization method described in section 4.5. The fact that the observability inequality is uniform implies the uniform bound (8.8) on the controls. The null controls for the semidiscrete equation (8.5) that one obtains in this way are such that, as $h \to 0$, they tend to the null control for the continuous heat equation (8.1) (see [71]).

8.2. The Beam Equation. In a recent work by León and the author [65] the problem of boundary controllability of finite difference space semidiscretizations of the beam equation $y_{tt} + y_{xxxx} = 0$ was addressed. This model has important differences with the wave equation even in the continuous case. First of all, at the continuous level, it turns out that the gap between consecutive eigenfrequencies tends to infinity. For instance, with the boundary conditions $y = y_{xx} = 0$, $x = 0, \pi$, the solution admits the Fourier representation formula $y(x,t) = \sum_{k \in \mathbf{Z}} a_k e^{i\lambda_k t} \sin(kx)$, where $\lambda_k = \mathrm{sgn}(k)k^2$. Obviously, the gap between consecutive eigenvalues is uniformly bounded from below. More precisely, $\lambda_{k+1} - \lambda_k = 2k + 1 \to \infty$ as $k \to \infty$. This allows us to apply a variant of Ingham's inequality for an arbitrarily small control time $T > 0$ (see [76]).[25] As a consequence, boundary exact controllability holds for any $T > 0$ too.

When considering finite difference space semidiscretizations, things are better than they are for the wave equation too. Indeed, as was proved in [65], roughly speaking, the asymptotic gap[26] also tends to infinity as $k \to \infty$, uniformly on the parameter h. This allows proving the uniform observability and controllability (as $h \to 0$) of the finite difference semidiscretizations. However, as we mentioned in section 4, due to the bad approximation that finite differences provide at the level of observing the high-frequency eigenfunctions, the control has to be split into two parts: the main part that strongly converges to the control of the continuous equation in the sharp $L^2(0,T)$ space and the oscillatory one that converges to zero in a weaker space $H^{-1}(0,T)$. Thus, in the context of the beam equation, with the most classical finite difference semidiscretization, we obtain what we had gotten for the wave equation with mixed finite elements. This fact was further explained by means of tools related with discrete Wigner measures in [72, 73].

The same results apply for the Schrödinger equation. Indeed, the beam equation under consideration is simply the composition of the Schrödinger operator with its

[25] Although in the classical Ingham inequality the gap between consecutive eigenfrequencies is assumed to be uniformly bounded from below for all indices k, in fact, in order for Ingham inequality to be true, it is sufficient to assume that all eigenfrequencies are distinct and that there is an asymptotic gap as $k \to \infty$. We refer to [76] for a precise statement where explicit estimates of the constants arising in the inequalities are given.

[26] In fact one needs to be more careful since, for $h > 0$ fixed, the gap between consecutive eigenfrequencies is not increasing. Indeed, in order to guarantee that the gap is asymptotically larger than any constant $L > 0$ one has to filter not only a finite number of low frequencies but also the highest ones. However, the methods and results in [76] apply in this context too (see [65]).

conjugate: $y_{tt} + y_{xxxx} = (-i\partial_t + \partial_x^2)(i\partial_t + \partial_x^2)y$. The same occurs for the numerical scheme we consider. According to this, the solutions of the continuous Schrödinger equation and of its numerical version are also solutions of the beam equation and its numerical discretization, respectively. Therefore, all of our analysis applies in the context of the Schrödinger equation, too.

Note, however, that, as we shall see in open problem 3 below, the situation is more complex in several space dimensions in which the dissipative and dispersive effects added by the heat and Schrödinger equations do not suffice. That is the case since, in several space dimensions, in addition to the numerical dispersive effects, new geometric issues arise. Indeed, for instance, there are discrete eigenvectors that violate the unique continuation properties of the eigenfunctions of the continuous Laplacian.[27] Adding dissipativity or dispersivity does not rule out this spurious eigenmodes and therefore does not suffice to reestablish the uniform observability and controllability properties in the same geometric setting of the corresponding continuous models.

9. Further Comments and Open Problems.

9.1. Further Comments.

1. *Stabilization.* The problem of controllability has been addressed. Nevertheless, similar developments could be carried out, with the same conclusions, in the context of stabilization. The connections between controllability and stabilization are well known (see, for instance, [96, 107]).

For the wave equation, it is well known that the GCC suffices for stabilization and more precisely to guarantee the uniform exponential decay of solutions when a damping term, supported in the control region, is added to the system. More precisely, when the subdomain ω satisfies the GCC the solutions of the damped wave equation $y_{tt} - \Delta y + 1_\omega y_t = 0$ with homogeneous Dirichlet boundary conditions are known to decay exponentially in the energy space: there exist constants $C > 0$ and $\gamma > 0$ such that $E(t) \leq Ce^{-\gamma t}E(0)$ holds for every finite energy solution of the Dirichlet problem for this damped wave equation.

It is then natural to analyze whether the decay rate is uniform with respect to the mesh size for numerical discretizations. The answer is in general negative due to spurious high-frequency oscillations. This negative result also has important consequences in many other issues related with control theory like infinite horizon control problems, Riccati equations for the optimal stabilizing feedback (see [89]), etc. The uniformity of the exponential decay rate can be reestablished if we add an internal viscous damping term to the equation (see [100, 80]). This is closely related to the enhanced observability inequality (7.2) in which the extra internal viscous term added in the observed quantity guarantees the observability constant to be uniform. We shall return to this issue in open problem 5 below.

2. *Space-time discretizations.* The analysis we have developed in this article applies as well to fully space-time finite difference discretizations. Except for the very particular case of the centered discretization of the 1D wave equation with equal space and time steps ($\Delta x = \Delta t$) addressed in [83], filtering of high frequencies is also needed. The discrete version of the Ingham inequality developed in [84] allows proving

[27] For the eigenfunctions of the Laplacian it is well known that if an eigenfunction vanishes in an open nonempty subset, then it has to vanish everywhere. As we shall see in open problem 3 below, the discrete analogue of this is not true.

boundedness and convergence of controls once the filtering parameter and the control time have been chosen in an appropriate way.

3. *Homogenization.* Most of the analysis presented here has also been developed in the context of a more difficult problem, related to the behavior of the observation/control properties of the wave equation in the context of homogenization. In that context the coefficients of the wave equation oscillate rapidly on a scale ε that tends to zero, so that the equation homogenizes to a constant coefficient one. The interaction of high-frequency waves with the microstructure produces localized waves at high frequency. These localized waves are an impediment for the uniform observation/control properties to hold. It has been proved in a number of situations that this filtering technique suffices to reestablish uniform observation and control properties (see [17] and [62]). The analogies between both problems (homogenization and numerical approximation) are clear: the mesh size h in numerical approximation schemes plays the same role as the ε parameter in homogenization (see [115] and [19] for a discussion of the connection between these problems). Although the analysis of the numerical problem is much easier from a technical point of view, it was developed only after the problem of homogenization was understood. This is due in part to the fact that, from a control theoretical point of view, there was a conceptual difficulty to match the existing finite-dimensional and infinite-dimensional theories. In this article we have shown how this may be done in the context of the wave equation, a model of purely conservative dynamics in infinite dimension.

4. *Optimal and approximate control.* We have shown that the property of exact controllability is badly behaved for the wave equation with respect to numerical approximations. However, this is no longer true for classical optimal control problems (LQR (linear quadratic regulator), finite-time-horizon optimal control, etc.) or even for approximate controllability problems in which the objective is to drive the solution to any state of size less than a given ε. *Approximate controllability* is a relaxed version of the exact controllability property, the goal being to drive the solution of the controlled wave equation (3.4) not exactly to the equilibrium as in (3.5) but rather to an ε-state such that

$$||y(T)||_{L^2(0,1)} + ||y_t(T)||_{H^{-1}(0,1)} \leq \varepsilon. \tag{9.1}$$

When for all initial data (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$ and for all ε there is a control v (obviously, the control v will normally depend on ε, too) such that (9.1) holds, we say that the system (3.5) is approximately controllable. Obviously, approximate controllability is a weaker notion than exact controllability, and whenever the wave equation is exactly controllable, it is approximately controllable too.

Of course, the approximate controllability property by itself, as stated, does not provide any information as to what the cost of controlling to an ε-state is, as in (9.1), i.e., what norm of the control v_ε is needed to achieve the approximate control condition (9.1). Roughly speaking, when exact controllability does not hold (for instance, in several space dimensions, when the GCC is not fulfilled), the cost of controlling blows up exponentially as ε tends to zero (see [92]).[28] But this issue will not be addressed here.

[28]This type of result has been also proved in the context of the heat equation in [31]. But there the difficulty does not come from the geometry but rather from the regularizing effect of the heat equation.

Once ε is fixed, we know that when $T \geq 2$, for all initial data (y^0, y^1) in $L^2(0,1) \times H^{-1}(0,1)$, there exists a control $v_\varepsilon \in L^2(0,T)$ such that (9.1) holds. This is a consequence of the exact controllability property of the wave equation in section 3.

We are interested in the behavior of this property under numerical discretization. Thus, let us consider the semidiscrete controlled version of the wave equation (4.21). We fix the initial data in (4.21) "independently of h" (roughly, by taking a projection over the discrete mesh of fixed initial data (y^0, y^1) or by truncating its Fourier series).

Of course, (4.21) is also approximately controllable.[29] The question we address is as follows: *Given initial data which are "independent of h," with ε fixed, and given also the control time $T \geq 2$, is the control v_h of the semidiscrete system* (4.21) *(such that the discrete version of* (9.1) *holds) uniformly bounded as $h \to 0$?*

In the previous sections we have shown that the answer to this question in the context of the exact controllability (which corresponds to taking $\varepsilon = 0$) is negative. However, *in the context of approximate controllability, the controls v_h do remain uniformly bounded as $h \to 0$. Moreover, they can be chosen such that they converge to a limit control v for which* (9.1) *is realized for the continuous wave equation.*

This positive result on the uniformity of the approximate controllability property under numerical approximation when $\varepsilon > 0$ does not contradict the fact that the controls blow up for exact controllability. These are in fact two complementary and compatible facts. For approximate controllability, one is allowed to concentrate an ε amount of energy on the solution at the final time $t = T$. For the semidiscrete problem this is done precisely in the high-frequency components that are badly controllable as $h \to 0$, and this makes it possible to keep the control fulfilling (9.1), bounded as $h \to 0$.

We refer to [119] for the details of the proof of this positive result.

The same can be said about finite horizon optimal control problems (see [119]).

In view of this discussion it becomes clear that the source of divergence in the limit process as $h \to 0$ in the exact controllability problem is the requirement of driving the high-frequency components of the numerical solution exactly to zero. As we mentioned in the introduction, taking into account that optimal and approximate controllability problems are relaxed versions of the exact controllability one, this negative result should be considered as a warning about the limit process as $h \to 0$ in general control problems.

9.2. Open Problems.

Problem 1. Semilinear Equations. The questions we have addressed in this article are completely open in the case of the semilinear heat and wave equations with globally Lipschitz nonlinearities. For continuous models there are a number of fixed point techniques allowing one to extend the results of controllability of linear waves and heat equations to semilinear equations with moderate nonlinearities (globally Lipschitz ones, for instance [117]). These techniques need to be combined with Carleman or multiplier inequalities (see [33, 108]) allowing one to estimate the dependence of the observability constants on the potential of the linearized equation. However, the analysis we have pursued in this article relies very much on the Fourier decomposition of solutions, which does not suffice to obtain explicit estimates on the observability constants in terms of the potential of the equation. Thus, extending the positive

[29] In fact, in finite dimensions, exact and approximate controllability are equivalent notions and, as we have seen, the Kalman condition is satisfied for system (4.21).

results of uniform controllability presented in this paper (by means of filtering, mixed finite elements, multigrid techniques, etc.) to the numerical approximation schemes of semilinear PDE is a completely open subject of research.

Problem 2. Wavelets and Spectral Methods. In the previous sections we have described how filtering of high frequencies can be used to get uniform observability and controllability results. It would be interesting to develop the same analysis in the context of numerical schemes based on wavelets and spectral methods. We refer to [82, 8, 6] for some preliminary results.

Problem 3. Discrete Unique-Continuation. In the context of the continuous wave equation we have seen that the observability inequality and, consequently, exact controllability hold if and only if the domain where the control is being applied satisfies the GCC. However, very often in practice, it is natural to consider controls that are supported in a small subdomain. In those cases, when the control time is large enough, one obtains approximate controllability results. Approximate controllability is equivalent to a uniqueness or unique-continuation property for the adjoint system (see [67, 113, 117]): *If the solution* u *of* (5.6) *vanishes in* $\omega \times (0, T)$*, then it vanishes everywhere.* We emphasize that this property holds whatever the open subset ω of Ω may be, provided T is large enough, by Holmgren's uniqueness theorem (see [67]).

One could expect the same result to hold also for semidiscrete and discrete equations. But the corresponding theory has not been developed. The following example due to Kavian [54] shows that, at the discrete level, new phenomena arise. It concerns the eigenvalue problem for the five-point finite difference scheme for the Laplacian in the square. A grid function taking alternating values ± 1 along a diagonal and vanishing everywhere else (see Figure 11) is an eigenvector with eigenvalue $\lambda = 4/h^2$. According to this example, even at the level of the elliptic equation, the domain ω where the solution vanishes has to be assumed to be large enough to guarantee the unique-continuation property. In [22] it was proved that when ω is a "neighborhood of one side of the boundary," then unique continuation holds for the discrete Dirichlet problem in any discrete domain. Here by a "neighborhood of one side of the boundary" we mean the nodes of the mesh that are located immediately to one side of the boundary nodal points (left, right, top, or bottom). Indeed, if one knows that the solution vanishes at the nodes immediately to one side of the boundary, taking into account that they vanish in the boundary too, the five-point numerical scheme allows propagating the information and showing that the solution vanishes at all nodal points of the whole domain.

Getting optimal geometric conditions on the set ω, depending on the domain Ω where the equation holds, on the discrete equation itself, on the boundary conditions and, possibly, on the frequency of oscillation of the solution for the unique continuation property to hold at the discrete level is an interesting and widely open subject of research.

One of the main tools for dealing with unique continuation properties of PDEs are the so-called *Carleman inequalities.* It would be interesting to develop the corresponding discrete theory.

Obviously, the lack of unique continuation for the discrete eigenvectors shows that unique continuation will also fail for the discrete versions of the heat and Schrödinger equations. Therefore, even if in one dimension the dispersive and diffusive properties of the systems under consideration enhance its controllability, they are not sufficient to guarantee the controllability of the numerical discretizations in several space dimensions. Understanding the need of filtering of high frequencies for these systems in

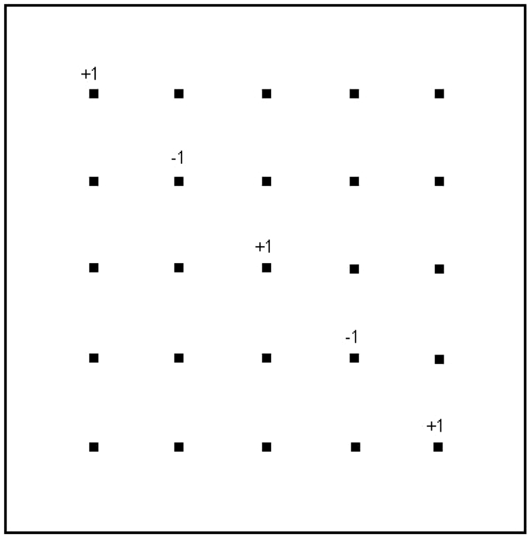

Fig. 11 *The eigenvector for the five-point finite difference scheme for the Laplacian in the square, with eigenvalue $\lambda = 4/h^2$, taking alternating values ± 1 along a diagonal and vanishing everywhere else in the domain.*

several space dimensions is also an interesting open problem.

Problem 4. Hybrid Hyperbolic-Parabolic Equations. We have discussed discretizations of the wave equation and have seen that, for most schemes, there are high-frequency spurious oscillations that need to be filtered to guarantee uniform observability and controllability. However, we have seen that the situation is much better for the 1D heat equation. It would also be interesting to analyze mixed models involving wave and heat components. There are two examples of such systems: (a) systems of thermoelasticity and (b) models for fluid-structure interaction (see [63] for the system of thermoelasticity and [109, 110, 114] for the analysis of a system coupling the wave and the heat equation along an interface). In particular, it would be interesting to analyze to what extent the presence of the parabolic component makes unnecessary the filtering of the high frequencies for the uniform observability property to hold for space or space-time discretizations.

Problem 5. Viscous Numerical Damping. In [100] we analyzed finite difference semidiscretizations of the damped wave equation $u_{tt} - u_{xx} + \chi_\omega u_t = 0$, where χ_ω denotes the characteristic function of the set ω where the damping term is effective. In particular we analyzed the following semidiscrete approximation in which an extra numerical viscous damping term is present:

$$u_j'' - \frac{1}{h^2}\left[u_{j+1} + u_{j-1} - 2u_j\right] - \left[u_{j+1}' + u_{j-1}' - 2u_j'\right] - u_j'\chi_\omega = 0. \tag{9.2}$$

It was proved that this type of scheme preserves the stabilization properties of the wave equation, uniformly as h tends to zero.

The extra numerical damping that this scheme introduces, namely, $[u_{j+1}' + u_{j-1}' - 2u_j']$, damps out the high frequency spurious oscillations that the classical finite difference discretization scheme produces and that are the cause of lack of uniform exponential decay in the presence of damping. We also refer to [79], where this type of scheme is used in order to develop an algorithm based on the level-set method for computing the optimal location of dampers for the wave equation, a topic that has been previously addressed from a theoretical point of view in [45].

The problem of whether this numerical scheme is uniformly observable or controllable as h tends to zero is an interesting open problem.

Note that the system above, in the absence of the damping term localized in ω, can be written in the vector form $\vec{u}'' + A_h\vec{u} + h^2 A_h \vec{u}' = 0$. Here $\vec{u}$ stands, as usual, for the vector unknown $(u_1, \ldots, u_N)^T$ and A_h for the tridiagonal matrix associated with the finite difference approximation of the Laplacian (4.3). In this form it is clear that the scheme above corresponds to a viscous approximation of the wave equation.

Whether this system has uniform (with respect to h) observability and controllability properties is an interesting open problem even in one space dimension.

Problem 6. Multigrid Methods. In section 7.2 we presented the two-grid algorithm introduced by Glowinski [38] and explained heuristically why it is a remedy for high-frequency spurious oscillations. In [38] the efficiency of the method was exhibited in several numerical examples and the convergence proved in [84] for one space dimension. The problem of convergence is open in several space dimensions.

Problem 7. Uniform Control of the Low Frequencies. We have seen that the most natural finite difference approximation scheme for the 1D wave equation fails to give convergent controls. However, Micu in [75] proved that the controls converge for some initial data, in particular for those that involve only a finite number of Fourier components. The 2D counterpart of the 1D positive result in [75] showing that the initial data involving a finite number of Fourier components are uniformly controllable as $h \to 0$ has not been proved in the literature. Such a result is very likely to hold for quite general approximation schemes and domains. But, up to now, it has been proved only in one dimension for finite difference semidiscretizations.

Problem 8. Extending the Wigner Measure Theory. As we mentioned above, Macià in [72, 73] developed a discrete Wigner measure theory to describe the propagation of semidiscrete and discrete waves at high frequency. However, this was done for regular grids and without taking into account boundary effects. The notion of polarization developed in [13] remains also to be analyzed in the discrete setting.

Problem 9. Theory of Inverse Problems and Optimal Design. This paper has been devoted mainly to the property of observability and its consequences for controllability. But, as we mentioned from the beginning, most of the results we have developed have consequences in other fields. This is the case, for instance, for the theory of inverse problems, where one of the most classical problems is the one of reconstructing the coefficients of a given PDE in terms of boundary measurements (see [52]). Assuming that one has a positive answer to this problem in an appropriate functional setting, it is natural to consider the problem of numerical approximation. Then, the following question arises: *Is solving the discrete version of the inverse problem for a discretized model an efficient way of getting a numerical approximation of the solution of the continuous inverse problem?*

According to the analysis above we can immediately say that, in general, the answer to this problem is negative. Consider, for instance, the wave equation

$$\rho u_{tt} - u_{xx} = 0, \quad 0 < x < 1,\, 0 < t < T, \qquad u(0,t) = u(1,t) = 0, \quad 0 < t < T, \tag{9.3}$$

with a constant but unknown density $\rho > 0$. Solutions of this equation are time-periodic of period $2\sqrt{\rho}$, and this can be immediately observed on the trace of normal derivatives of solutions at either of the two boundary points $x = 0$ or $x = 1$, by inspection of the Fourier series representation of solutions of (9.3). Thus, the value of ρ can be determined by means of boundary measurements.

Let us now consider the semidiscrete version of (9.3). In this case, according to the analysis above, the solutions do not have any well-defined time-periodicity property. On the contrary, for any given values of ρ and h, (9.3) admits a whole range of solutions that travel at different group velocities, ranging from $h/\sqrt{\rho}$ (for the high frequencies) to $1/\sqrt{\rho}$ (for the low frequencies). In particular, the high-frequency numerical solutions do behave more like a solution of the wave equation with an effective density ρ/h^2. This argument shows that the mapping that allows determining the value of the constant density from boundary measurements is unstable under numerical discretization.

Of course, most of the remedies that have been introduced in this paper to avoid the failure of uniform controllability and/or observability can also be used in this context of inverse problems. But developing these ideas in detail remains to be done.

The same can be said about optimal design problems. Indeed, in this context very little is known about the convergence of the optimal designs for the numerical discretized models towards the optimal design of the continuous models and, to a large extent, the difficulties one has to face in this context are similar to those we addressed all throughout this paper. Recently convergence of the discrete optimal shapes towards the continuous ones has been proved in [23] for the Dirichlet Laplacian in two dimensions.

Problem 10. Finite- versus Infinite-Dimensional Nonlinear Control. Most of this work has been devoted to analyzing linear problems. There is still a lot to be done to understand the connections between finite-dimensional and infinite-dimensional control theory, and, in particular, concerning numerical approximations and their behavior with respect to the control property. According to the analysis above, the problem is quite complex even in the linear case. Needless to say, one expects a much higher degree of complexity in the nonlinear frame.

There are a number of examples in which the finite-dimensional versions of important nonlinear PDEs have been solved from the point of view of controllability. Among them the following are worth mentioning: (a) The Galerkin approximations of the bilinear control problem for the Schrödinger equation arising in quantum chemistry (see [90, 104]); (b) the control of the Galerkin approximations of the Navier–Stokes equations [69].

In both cases nothing is known about the possible convergence of the controls of the finite-dimensional system to the control of a PDE as the dimension of the Galerkin subspace tends to infinity. This problem seems to be very complex. However, the degree of difficulty may be different in both cases. Indeed, in the case of the continuous Navier–Stokes and Euler equations for incompressible fluids there are a number of results in the literature indicating that they are indeed controllable (see [33, 25, 26]). However, for the bilinear control of the Schrödinger equations, it is known that the reachable set is very small in general, which indicates that one can only expect very weak controllability properties. This weakness of the controllability property at the continuous level makes it even harder to address the problem of passing to the limit on the finite-dimensional Galerkin approximations as the dimension tends to infinity.

Problem 11. Wave Equations with Irregular Coefficients. The methods we have developed do not suffice to deal with wave equations with nonsmooth coefficients. However, at the continuous level, in one space dimension, observability and exact controllability hold for the wave equation with BV coefficients. It would be interesting to see if the main results presented in this paper hold in this setting too. This seems to be a completely open problem. We refer to the book by Cohen [24] for the analysis of reflection and transmission indices for numerical schemes for wave equations with interfaces.

Problem 12. Convergence Rates. In this article we have described several numerical methods that do provide convergence of controls. The problem of the rate of convergence has not been addressed so far. Recently important progress has been made in this respect in the context of optimal control problems for semilinear elliptic equations (see [14, 15]).

Problem 13. Waves on Networks. The problems of wave propagation, observation, and control in planar networks of strings has been intensively analyzed. We refer to [27] for a survey on the state-of-the-art in the field. However, very little is known about the behavior of numerical methods. We refer to [9] for some preliminary results on the subject. The method of domain decomposition has also been studied [59]. But a lot is to be done in order to develop a complete theory for the observation, control, and numerical approximation of waves on networks.

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