EFFECTIVE LAGRANGIANS IN THE RESONANCE REGION

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- The Large N_C Limit
- Chiral Perturbation Theory (χPT)
- Resonance Chiral Theory (RχT)
- ε'/ε
- Quantum Loops in RχT

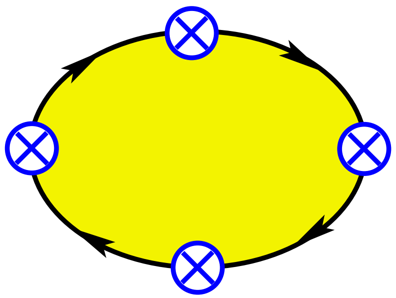
Benasque, 3 August 2004

Energy Scale	Fields	Effective Theory
M_W	W, Z, γ, g τ, μ, e, ν_i t, b, c, s, d, u	Standard Model
	$\downarrow$ OPE	
$\lesssim m_c$	$\gamma, g\ ;\ \mu, e, \nu_i$ s, d, u	$\mathcal{L}_{\mathrm{QCD}}^{(n_f=3)}$, $\mathcal{L}_{\mathrm{eff}}^{\Delta S=1,2}$
	$\downarrow$ $N_C \to \infty$	
M_K	$\gamma\ ;\ \mu, e, \nu_i$ π, K, η	χPT

Energy Scale	Fields	Effective Theory
M_W	t, b, c $s, d, u\ ;\ G^a$	$\mathrm{QCD}^{N_f=6}$
$\lesssim m_c$	$s, d, u\ ;\ G^a$	$\mathrm{QCD}^{N_f=3}$
Λ_χ	V, A, S, P π, K, η	RχPT
$\lesssim M_K$	π, K, η	$\chi\mathrm{PT}^{N_f=3}$
$\lesssim M_\pi$	π	$\chi\mathrm{PT}^{N_f=2}$

$\mathrm{N_C}$ COUNTING RULES

$g_s \sim 1/\sqrt{N_C}$; $\alpha_s \sim 1/N_C$; $\langle T\,(J_1 \cdots J_n)\rangle \sim N_C$

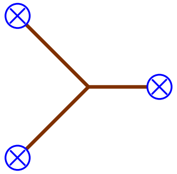
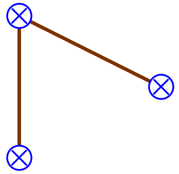


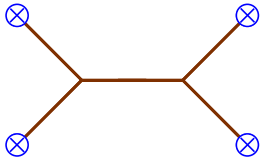
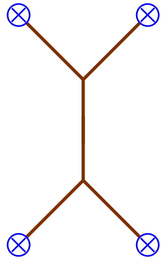
- Dominance of planar gluonic exchanges
- Non-planar diagrams suppressed by $1/N_C^2$
- Internal quark loops suppressed by $1/N_C$

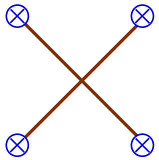
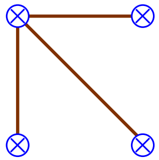
Colour Confinement ➡ $J\,|0\rangle \sim |1\ \text{Meson}\rangle$

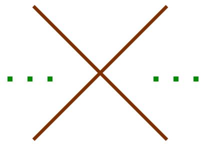
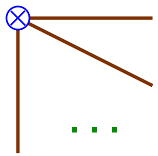
$$\langle J(k)\, J(-k)\rangle = \sum_n \frac{f_n^2}{k^2 - M_n^2}$$

- **Infinite** number of mesons $(\sim \ln k^2)$
- $f_n = \langle 0|J|n\rangle \sim \sqrt{N_C}$; $M_n \sim O(1)$
- Mesons are **free**, **stable** and **non-interacting**

<J J J> $=$ Σ  $+$ Σ 

<J J J J> $=$ Σ  $+$ Σ 

$+$ Σ  $+$ Σ 

 $\sim N_C^{1-\frac{n}{2}}$  $\sim N_C^{1-\frac{n}{2}}$

Crossing + Unitarity → **Tree Approximation to some Local Effective Meson Lagrangian**

χPT

Low–Energy Expansion (p^{2n}, m_q^n) **:** $\mathcal{L} = \sum_n \mathcal{L}_{2n}$

$$\mathcal{L}_2 = \frac{f^2}{4} \langle D_\mu U^\dagger D^\mu U + 2\, B_0 \left(U^\dagger \mathcal{M} + \mathcal{M}^\dagger U \right) \rangle$$

$$= D_\mu \pi^+ D_\mu \pi^- - M_\pi^2 \pi^+ \pi^- + \cdots + \frac{1}{6 f^2} \left(\pi^+ \overleftrightarrow{D}_\mu \pi^- \right) \left(\pi^+ \overleftrightarrow{D}^\mu \pi^- \right) + \cdots$$

$$\frac{M_\pi^2}{m_u + m_d} = \frac{M_{K^0}^2}{m_s + m_d} = \frac{M_{K^+}^2}{m_s + m_u} = B_0 = -\frac{\langle \bar{q} q \rangle}{f^2}$$

$\mathrm{N_C}$ Counting:

- $f^2 \approx f_\pi^2 \sim N_C$; $B_0 \sim O(1) \Longrightarrow M_\varphi \sim O(1)$
- $U \longrightarrow \sum_n (\varphi/f)^n \sim O(1)$; $V_n \sim f^{2-n} \sim N_C^{1-n/2}$
- $\mathcal{L} \sim N_C$; n-Loop $\sim 1/(16\pi^2 f^2)^n \sim 1/N_C^n$

$O(p^4)\quad \chi$PT

i) $\mathcal{L}_4$ **at tree level** (Gasser–Leutwyler)

$$\begin{aligned}\mathcal{L}_4 \; = \; & L_1 \langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2 \langle D_\mu U^\dagger D_\nu U\rangle \langle D^\mu U^\dagger D^\nu U\rangle + L_3 \langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle \\ + \; & L_4 \langle D_\mu U^\dagger D^\mu U\rangle \langle U^\dagger \chi + \chi^\dagger U\rangle + L_5 \langle D_\mu U^\dagger D^\mu U \left(U^\dagger \chi + \chi^\dagger U\right)\rangle \\ + \; & L_6 \langle U^\dagger \chi + \chi^\dagger U\rangle^2 + L_7 \langle U^\dagger \chi - \chi^\dagger U\rangle^2 + L_8 \langle \chi^\dagger U \chi^\dagger U + U^\dagger \chi U^\dagger \chi\rangle \\ - \; & i\, L_9 \langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10} \langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle\end{aligned}$$

$$F_J^{\mu\nu} \equiv \partial^\mu J^\nu - \partial^\nu J^\mu - i[J^\mu, J^\nu] \quad ; \quad J^\mu = v^\mu \pm a^\mu \quad ; \quad \chi \equiv 2\, B_0\, \mathcal{M}$$

ii) $\mathcal{L}_2$ **at one loop** **(unitarity)** $\qquad T_4 \sim p^4 \left\{a \log(p^2/\mu^2) + b(\mu)\right\}$

- Chiral Logarithms unambiguously predicted
- L_i's fixed by QCD dynamics [1–loop divergences ➡ $L_i^r(\mu)$]

iii) **Wess–Zumino–Witten term** **(anomaly):** $\qquad \pi^0 \to \gamma\gamma\, ,\ \eta \to \gamma\gamma\, ,\ \cdots$

$O(p^4)$ χPT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	$O(N_C)$	Source	Γ_i
1	0.4 ± 0.3	N_C	K_{e4}, $\pi\pi \to \pi\pi$	$\frac{3}{32}$
2	1.4 ± 0.3	N_C	K_{e4}, $\pi\pi \to \pi\pi$	$\frac{3}{16}$
3	-3.5 ± 1.1	N_C	K_{e4}, $\pi\pi \to \pi\pi$	0
4	-0.3 ± 0.5	1	Zweig rule	$\frac{1}{8}$
5	1.4 ± 0.5	N_C	F_K/F_π	$\frac{3}{8}$
6	-0.2 ± 0.3	1	Zweig rule	$\frac{11}{144}$
7	-0.4 ± 0.2	1	GMO, $L_{5,8}$	0
8	0.9 ± 0.3	N_C	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m})/(m_d - m_u)$	$\frac{5}{48}$
9	6.9 ± 0.7	N_C	$\langle r^2 \rangle_V^\pi$	$\frac{1}{4}$
10	-5.5 ± 0.7	N_C	$\pi \to e\nu\gamma$	$-\frac{1}{4}$

$$2L_1 - L_2 = (-0.6 \pm 0.6) \times 10^{-3} \sim O(1)$$

$$c\, \langle D_\mu U D_\nu U^\dagger D^\mu U D^\nu U^\dagger \rangle \;\to\; 2\,\delta L_1 = \delta L_2 = -\tfrac{1}{2}\,\delta L_3 = c$$

RχT

Include Resonance Nonet Multiplets (Ecker, Gasser, Pich, de Rafael)

$\mathbf{V}(1^{--})\,,\ \mathbf{A}(1^{++})\,,\ \mathbf{S}(0^{++})\,,\ \mathbf{P}(0^{-+})$

$$\mathcal{L}_2^V = \frac{F_V}{2\sqrt{2}}\,\langle V_{\mu\nu}\, f_+^{\mu\nu}\rangle + \frac{i\,G_V}{\sqrt{2}}\,\langle V_{\mu\nu}\, u^\mu\, u^\nu\rangle$$

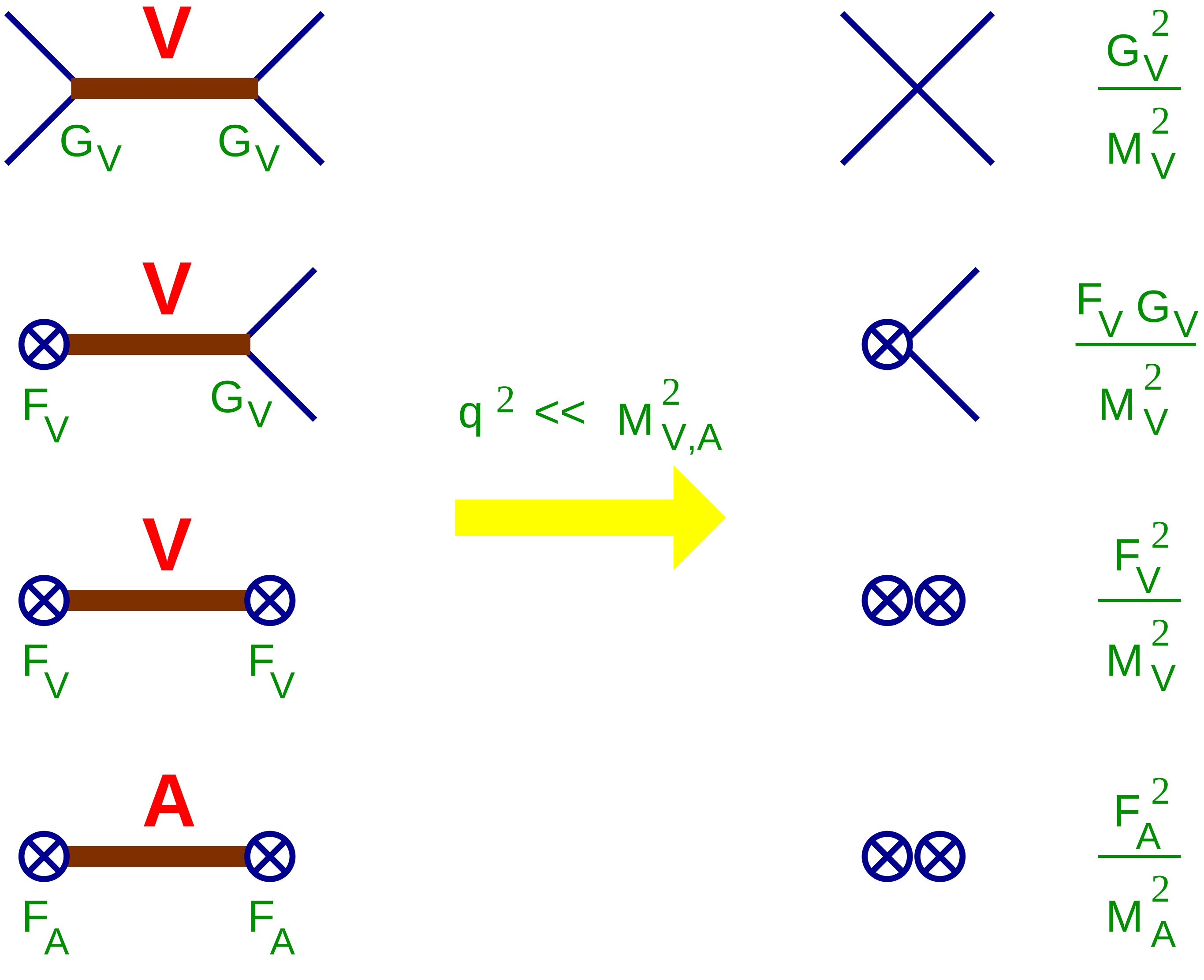
$$\mathcal{L}_2^A = \frac{F_A}{2\sqrt{2}}\,\langle A_{\mu\nu}\, f_-^{\mu\nu}\rangle$$

$$\mathcal{L}_2^S = c_d\,\langle S\, u^\mu\, u^\nu\rangle + c_m\,\langle S\,\chi_+\rangle$$

$$\mathcal{L}_2^P = i\, d_m\,\langle P\,\chi_-\rangle$$

$$u_\mu = i\, u^\dagger\, D_\mu U\, u^\dagger = u_\mu^\dagger \quad ; \quad U = u^2$$

$$f_\pm^{\mu\nu} = u\, F_L^{\mu\nu}\, u^\dagger \pm u^\dagger\, F_R^{\mu\nu}\, u \quad ; \quad \chi_\pm = u^\dagger\,\chi\, u^\dagger \pm u\,\chi^\dagger\, u$$



$$\left\{ \mathbf{V}\,[M_V, G_V, F_V]\ ,\ \mathbf{A}\,[M_A, F_A] \right\} \quad \longleftrightarrow \quad \left\{ \mathbf{S}\,[M_S, c_d, c_m]\ ,\ \mathbf{P}\,[M_P, d_m] \right\}$$

$\mathbf{O(N_C)}$:

$$2\,L_1 = L_2 = \sum_i \frac{G_{V_i}^2}{4\,M_{V_i}^2} \quad ; \quad L_3 = \sum_i \left\{ -\frac{3\,G_{V_i}^2}{4\,M_{V_i}^2} + \frac{c_{d_i}^2}{2\,M_{S_i}^2} \right\}$$

$$L_5 = \sum_i \frac{c_{d_i}\,c_{m_i}}{M_{S_i}^2} \quad ; \quad L_8 = \sum_i \left\{ \frac{c_{m_i}^2}{2\,M_{S_i}^2} - \frac{d_{m_i}^2}{2\,M_{P_i}^2} \right\}$$

$$L_9 = \sum_i \frac{F_{V_i}\,G_{V_i}}{2\,M_{V_i}^2} \quad ; \quad L_{10} = \frac{1}{4} \sum_i \left\{ \frac{F_{A_i}^2}{M_{A_i}^2} - \frac{F_{V_i}^2}{M_{V_i}^2} \right\}$$

$\mathbf{O(1)}$:

$$2\,L_1 - L_2 = L_4 = L_6 = 0 \quad ; \quad L_7 = -\frac{\tilde{d}_m^2}{2\,M_{\eta_1}^2}$$

BUT $\qquad M_{\eta_1}^2 \sim O\left(\frac{1}{N_C}, \mathcal{M}\right)$

SHORT–DISTANCE CONSTRAINTS

Vector Form Factor $\langle\pi|v_\mu|\pi\rangle$:

$$F_V(t) = 1 + \sum_i \frac{F_{V_i} G_{V_i}}{f^2} \frac{t}{M_{V_i}^2 - t}$$

$$\lim_{t\to\infty} F_V(t) = 0 \quad \longrightarrow \quad \boxed{\sum_i F_{V_i} G_{V_i} = f^2}$$

Axial Form Factor $\langle\gamma|a_\mu|\pi\rangle$:

$$G_A(t) = \sum_i \left\{ \frac{2 F_{V_i} G_{V_i} - F_{V_i}^2}{M_{V_i}^2} + \frac{F_{A_i}^2}{M_{A_i}^2 - t} \right\}$$

$$\lim_{t\to\infty} G_A(t) = 0 \quad \longrightarrow \quad \boxed{\sum_i \left(2 F_{V_i} G_{V_i} - F_{V_i}^2\right) / M_{V_i}^2 = 0}$$

Weinberg Sum Rules:

$$\Pi_{LR}(t) = -\frac{f^2}{t} + \sum_i \frac{F_{V_i}^2}{M_{V_i}^2 + t} - \sum_i \frac{F_{A_i}^2}{M_{A_i}^2 + t}$$

$$\lim_{t\to\infty} t\, \Pi_{LR}(t) = 0$$

$$\lim_{t\to\infty} t^2\, \Pi_{LR}(t) = 0$$

$\longrightarrow$

$$\sum_i \left(F_{V_i}^2 - F_{A_i}^2\right) = f^2$$

$$\sum_i \left(M_{V_i}^2 F_{V_i}^2 - M_{A_i}^2 F_{A_i}^2\right) = 0$$

Scalar Form Factor:

$$F^S_{K\pi}(s) \,=\, 1 \,+\, \sum_i \frac{4\, c_{m_i}}{f^2} \left[c_{d_i} + (c_{m_i} - c_{d_i}) \,\frac{M_K^2 + M_\pi^2}{M_{S_i}^2} \right] \frac{s}{M_{S_i}^2 - s}$$

$$\lim_{s\to\infty} F^S_{K\pi}(s) = 0 \quad \longrightarrow \quad \boxed{4 \sum_i c_{d_i}\, c_{m_i} \,=\, f^2 \quad ; \quad \sum_i \frac{c_{m_i}}{M_{S_i}^2} \left(c_{m_i} - c_{d_i} \right) \,=\, 0}$$

SS − PP Sum Rules:

$$\Pi_{SS-PP}(t) \,=\, 16\, B_0^2 \left\{ \sum_i \frac{c_{m_i}^2}{M_{S_i}^2 + t} \,-\, \sum_i \frac{d_{m_i}^2}{M_{P_i}^2 + t} \,-\, \frac{f^2}{8\, t} \right\}$$

$$\lim_{t\to\infty} t\ \Pi_{SS-PP}(t) = 0 \quad \longrightarrow \quad \boxed{8 \sum_i \left(c_{m_i}^2 - d_{m_i}^2 \right) = f^2}$$

Pseudoscalar Nonet:

$$\mathcal{L}_2 \,\doteq\, \frac{f^2}{4}\, \langle \chi\, U^\dagger + U\, \chi^\dagger \rangle \,\sim\, -i\, \frac{f}{\sqrt{24}}\, \eta_1\, \langle \chi_- \rangle \quad \longrightarrow \quad \boxed{\tilde{d}_m \,=\, -\frac{f}{\sqrt{24}}}$$

1–Resonance Approximation: (Ecker, Gasser, Leutwyler, Pich, de Rafael)

$$F_V = 2\,G_V = \sqrt{2}\,F_A = \sqrt{2}\,f \qquad ; \qquad M_A = \sqrt{2}\,M_V \qquad ; \qquad d_m = \frac{1}{2\sqrt{2}}\,f$$

$$c_m = c_d = \tfrac{1}{2}\,f$$ (Jamin, Oller, Pich)

$$\Rightarrow$$

$$2\,L_1 = L_2 = \tfrac{1}{4}\,L_9 = -\tfrac{1}{3}\,L_{10} = \frac{f^2}{8\,M_V^2}$$

$$L_3 = -\frac{3\,f^2}{8\,M_V^2} + \frac{f^2}{8\,M_S^2} \qquad ; \qquad L_5 = \frac{f^2}{4\,M_S^2}$$

$$L_8 = \frac{f^2}{8\,M_S^2} - \frac{f^2}{16\,M_P^2} \qquad ; \qquad L_7 = -\frac{f^2}{48\,M_{\eta_1}^2}$$

L_i'S FROM RESONANCE EXCHANGE

i	$L_i^r(M_\rho)$	V	A	S	η_1	Total	Total[b]
1	0.4 ± 0.3	0.6	0	0	0	0.6	0.9
2	1.4 ± 0.3	1.2	0	0	0	1.2	1.8
3	-3.5 ± 1.1	-3.6	0	0.6	0	-3.0	-4.3
4	-0.3 ± 0.5	0	0	0	0	0.0	0.0
5	1.4 ± 0.5	0	0	1.4[a]	0	1.4	2.1
6	-0.2 ± 0.3	0	0	0	0	0.0	0.0
7	-0.4 ± 0.2	0	0	0	-0.3	-0.3	-0.3
8	0.9 ± 0.3	0	0	0.9[a]	0	0.9	0.8
9	6.9 ± 0.7	6.9[a]	0	0	0	6.9	7.2
10	-5.5 ± 0.7	-10.0	4.0	0	0	-6.0	-5.4

[a] Input   [b] Short-Distance Constraints

Three–Point Functions:

- B. Moussallam
- M. Knecht, A. Nyffeler
- P.D. Ruiz-Femenía, A.Pich, J. Portolés
- V. Cirigliano, G. Ecker, M. Eidemüller, A.Pich, J. Portolés
- J. Bijnens, E. Gámiz, E. Lipartia, J. Prades

➡ **Constraints on $\mathcal{O}(p^6)$ χPT couplings**

Vertices with 2 or 3 Resonances. $\mathcal{O}(p^4)$ Couplings:

- V. Cirigliano, G. Ecker, M. Eidemüller, R. Kaiser, A.Pich, J. Portolés

More Resonance Multiplets:

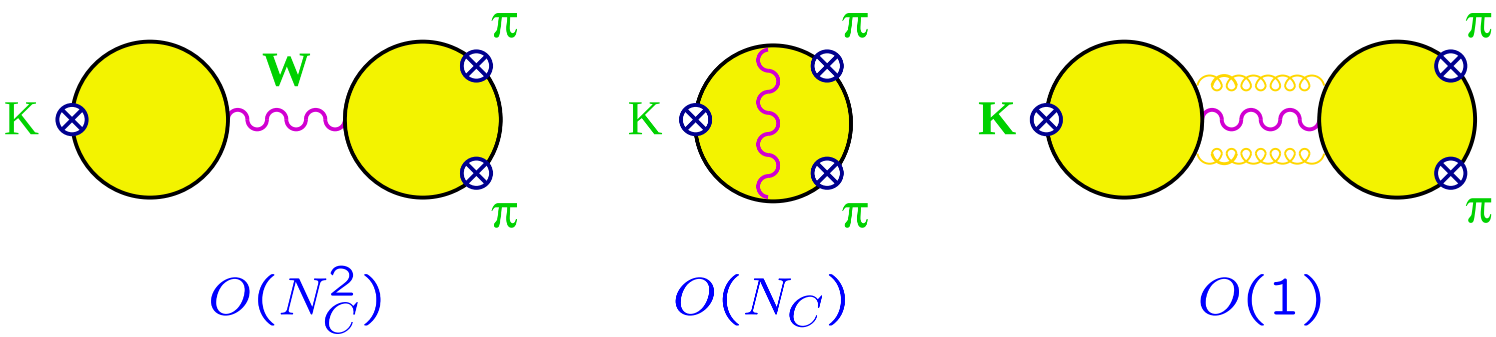
Minimal Hadronic Ansatz

- M. Knecht, S. Peris, M. Perrottet, B. Phily, E. de Rafael
- P.D. Ruiz-Femenía, J. Portolés

$1/N_C$ Corrections:

- **Resonance Widths**
  - F. Guerrero, A. Pich
  - D. Gómez–Dumm, A. Pich, J. Portolés
- **Unitarity Corrections** Final State Interactions
  - F. Guerrero, A. Pich
  - E. Pallante, A. Pich, I. Scimemi
  - J.J. Cillero, A. Pich, J. Portolés
  - M. Jamin, J.A. Oller, A. Pich
- **Quantum Loops in RχT**
  - J.J. Cillero, I. Rosell, A. Pich

Weak Currents Factorize at Large N_C



$$A[K^0 \to \pi^0\pi^0] = 0 \quad \Rightarrow \quad A_0 = \sqrt{2}\, A_2$$

No $\Delta I = \frac{1}{2}$ enhancement at leading order in $1/N_C$

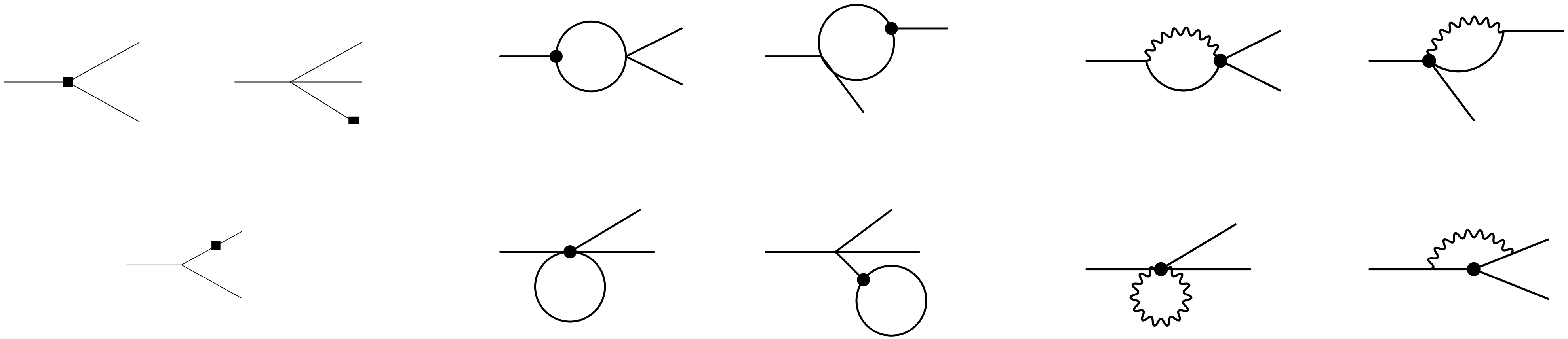
- **Multiscale problem:** **OPE** $\frac{1}{N_C} \log\left(\frac{M_W}{\mu}\right) \sim \frac{1}{3} \times 4$

Short-distance logarithms must be summed

- **Large χPT logarithms:** **FSI** $\frac{1}{N_C} \log\left(\frac{\mu}{M_\pi}\right) \sim \frac{1}{3} \times 2$

Infrared logarithms must also be included $[\delta_I \sim O(1/N_C)$, $\delta_0 - \delta_2 \approx 45^\circ]$

$O\left[p^4, (m_u - m_d)\, p^2, e^2 p^2\right]$ χ**PT**



Nonleptonic weak Lagrangian: $O(G_F\, p^4)$

$$\mathcal{L}_{\text{weak}}^{(4)} = \sum_i G_8\, N_i\, F^2\, O_i^8 + \sum_i G_{27}\, D_i\, F^2\, O_i^{27} + \text{h.c.}$$

Electroweak Lagrangian: $O(G_F\, e^2 p^2)$

$$\mathcal{L}_{\text{EW}} = e^2 \sum_i G_8\, Z_i\, F^4\, O_i^{EW} + \text{h.c.}$$

$O(e^2 p^2)$ **Electromagnetic** + $O(p^4)$ **Strong** $K_i\,,\, L_i$

$K \to \pi\pi, \pi\pi\gamma$ Inclusive, DAPHNE

$$\mathcal{A}_n^{(X)} = a_n^{(X)} \left[1 + \Delta_L \mathcal{A}_n^{(X)} + \Delta_C \mathcal{A}_n^{(X)}\right]$$

1) $O(p^4)$ χPT Loops: Large correction [NLO in $1/N_C$]

$$\Delta_L \mathcal{A}_{1/2}^{(8)} = 0.27 \pm 0.05 + 0.47\, i \quad ;$$

$$\Delta_L \mathcal{A}_{1/2}^{(27)} = 1.02 \pm 0.60 + 0.47\, i \quad ; \quad \Delta_L \mathcal{A}_{3/2}^{(27)} = -0.04 \pm 0.05 - 0.21\, i$$

$$\Delta_L \mathcal{A}_{1/2}^{(g)} = 0.27 \pm 0.05 + 0.47\, i \quad ; \quad \Delta_L \mathcal{A}_{3/2}^{(g)} = -0.50 \pm 0.20 - 0.21\, i$$

Pallante-Pich-Scimemi

2) All local $O(p^4)$ couplings fixed at $N_C \to \infty$ ➡ $\Delta_C \mathcal{A}_n^{(X)}$

Small correction to $O(p^2)$ results

3) Isospin Breaking: $O\left[(m_u - m_d)\, p^2, e^2 p^2\right]$ Sizeable corrections

Cirigliano-Ecker-Neufeld-Pich

4) Re(g_8), Re(g_{27}), $\chi_0 - \chi_2$ fitted to data

$$\frac{\varepsilon'_K}{\varepsilon_K} \sim \left[\frac{105\text{ MeV}}{m_s(2\text{ GeV})}\right]^2 \left\{B_6^{(1/2)}\,(1-\Omega_{\text{eff}}) - 0.4\, B_8^{(3/2)}\right\}$$

Delicate Cancellation. Strong Sensitivity to:

- m_s (quark condensate) $m_s(2\text{ GeV}) = 105 \pm 20\text{ MeV}$
- Isospin Breaking $(m_u \neq m_d\ ,\ \alpha)$ $\Omega_{\text{eff}} = 0.06 \pm 0.08$

  Cirigliano-Ecker-Neufeld-Pich
- Penguin Matrix Elements

χPT Loops (FSI): $B_{6,\infty}^{(1/2)} \times (1.35 \pm 0.05)$; $B_{8,\infty}^{(3/2)} \times (0.54 \pm 0.20)$

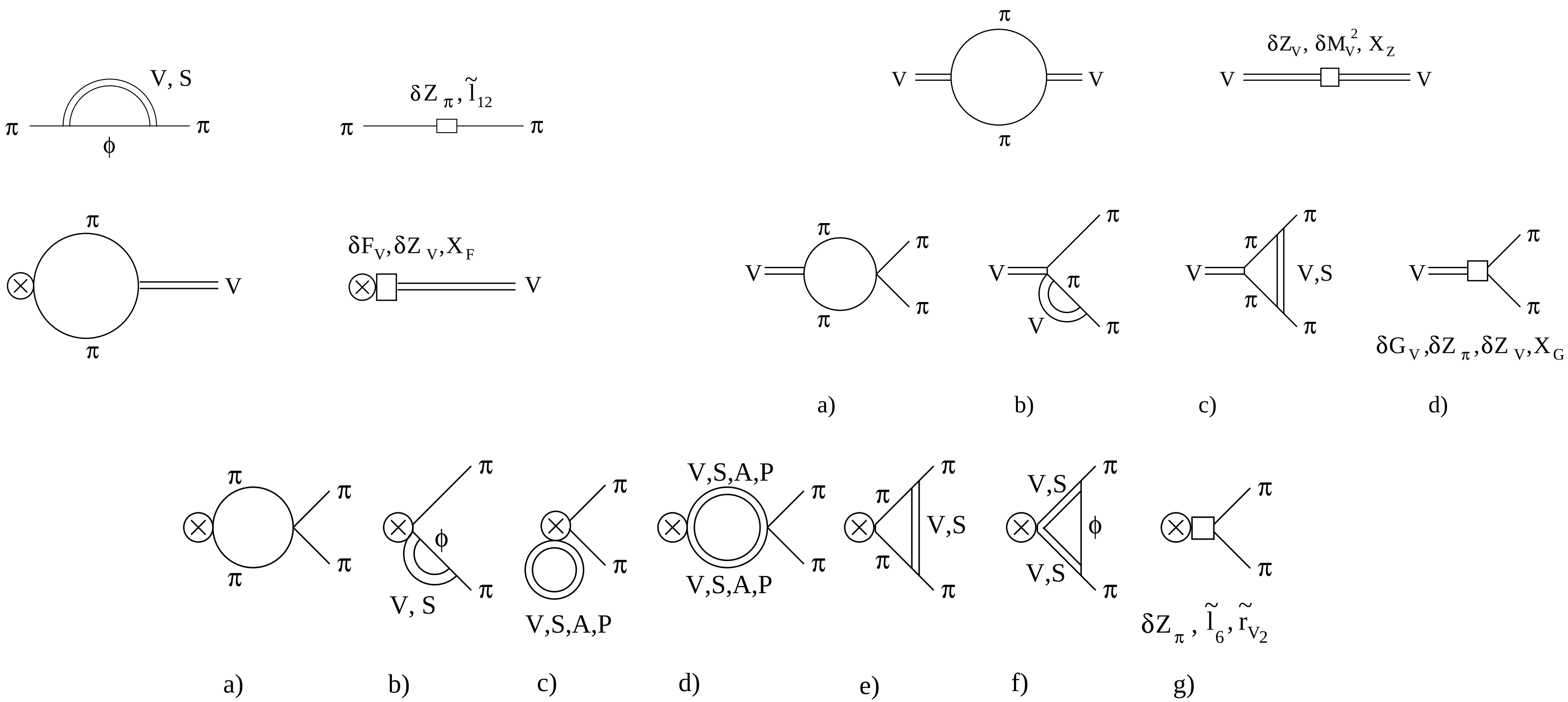
Pallante–Pich–Scimemi '01: $\text{Re}\left(\varepsilon'/\varepsilon\right) = \left(18 \pm 2_{\,\mu}\, {}^{+8}_{-5}{}_{\,m_s} \pm 5_{\,1/N_C}\right) \times 10^{-4}$

Experimental world average: $\text{Re}\,(\varepsilon'/\varepsilon) = (16.7 \pm 1.6) \times 10^{-4}$

Challenge: Control of subleading $1/\text{N}_\text{C}$ corrections to χPT couplings

QUANTUM LOOPS IN RχT: VFF

I. Rosell, J.J. Sanz-Cillero, A.P., arXiv:hep-ph/0407240



- $1/N_C$ Expansion
- Renormalization: Counterterms
- Ultraviolet Behaviour

COUNTERTERMS

$$\widetilde{\mathcal{L}}_{4\chi} = \frac{i\,\widetilde{\ell}_6}{4}\,\langle f_+^{\mu\nu}\,[u_\mu, u_\nu]\rangle \;-\; \widetilde{\ell}_{12}\,\langle \nabla^\mu u_\mu \nabla^\nu u_\nu\rangle \quad ; \quad \widetilde{\mathcal{L}}_{6\chi} = i\,\widetilde{c}_{51}\,\langle \nabla^\rho f_+^{\mu\nu}[h_{\mu\rho}, u_\nu]\rangle + i\,\widetilde{c}_{53}\,\langle \nabla_\mu f_+^{\mu\nu}[h_{\nu\rho}, u^\rho]\rangle$$

$$F_V(s) = 1 + \frac{F_V\,G_V}{f^2}\,\frac{s}{M_V^2 - s} - \widetilde{\ell}_6\,\frac{s}{F^2} + \widetilde{r}_{V2}\,\frac{s^2}{F^4}$$

$$\lim_{s\to\infty} F_V(s) = 0 \quad \xrightarrow{N_C\to\infty} \quad F_V\,G_V = f^2 \;,\; \widetilde{\ell}_6 = 0 \;,\; \widetilde{r}_{V2} \equiv 4f^2\,(\widetilde{c}_{53} - \widetilde{c}_{51}) = 0$$

$$\mathcal{L}_{4Z} = \frac{X_{Z_1}}{2}\,\langle\, \nabla^2 V^{\mu\nu}\,\{\nabla_\nu, \nabla^\sigma\}\,V_{\mu\sigma}\,\rangle + \cdots$$

$$\mathcal{L}_{4F} = X_{F_1}\,\langle\, V_{\mu\nu}\nabla^2 f_+^{\mu\nu}\,\rangle + \cdots$$

$$\mathcal{L}_{4G} = i\,X_{G_1}\,\langle\,\{\nabla^\alpha, \nabla_\mu\}\,V^{\mu\nu}\,[u_\nu, u_\alpha]\,\rangle + \cdots$$

EOM $\Longrightarrow$

$$X_Z\left\{\frac{M_V^4}{2}\,\langle V^{\mu\nu}V_{\mu\nu}\rangle + \frac{M_V^2 F_V}{\sqrt{2}}\,\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \cdots\right\}$$

$$-X_F\,\left\{M_V^2\,\langle V_{\mu\nu} f_+^{\mu\nu}\rangle + \cdots\right\}$$

$$-X_G\,\left\{2i\,M_V^2\,\langle V^{\alpha\nu}\,[u_\alpha, u_\nu]\rangle + \cdots\right\}$$

$$\widetilde{\ell}_6^{\,\mathrm{eff}} = \widetilde{\ell}_6 + 2\,X_Z F_V G_V - 2\sqrt{2}\,X_F G_V - 4\sqrt{2}\,X_G F_V \quad , \quad F_V^{\mathrm{eff}} = F_V + 2\,X_Z M_V^2 F_V - 2\sqrt{2}\,X_F M_V^2$$

$$G_V^{\mathrm{eff}} = G_V + 2\,X_Z M_V^2 G_V - 4\sqrt{2}\,X_G M_V^2 \quad , \quad (M_V^2)^{\mathrm{eff}} = M_V^2 + 2\,X_Z M_V^4 \quad , \quad \widetilde{r}_{V2}^{\,\mathrm{eff}} = \widetilde{r}_{V2}$$

Low-Energy Limit: χPT

$$F_V(s) = 1 - \frac{s}{f^2}\left\{\ell_6^r(\mu) + \frac{1}{96\pi^2}\left[\ln\left(-\frac{q^2}{\mu^2}\right) - \frac{5}{3}\right]\right\} + \frac{s^2}{F^4}\left\{r_{V2}^r(\mu) + \cdots\right\} + \mathcal{O}\left(\frac{s^3}{F^6}\right)$$

$$-\frac{1}{96\pi^2}\left[\bar{\ell}_6 + \log\frac{m_\pi^2}{M_V^2}\right] \equiv \ell_6^r(\mu) + \frac{1}{96\pi^2}\ln\frac{M_V^2}{\mu^2} = -\frac{f^2}{M_V^2} + \hat{\ell}_6 + \frac{1}{96\pi^2}\left[\ln\frac{M_S^2}{4M_V^2} + \frac{13}{6}\right]$$

$$r_{V2}^r(\mu) - \frac{f^2}{192\pi^2 M_V^2}\left(5 - \frac{M_V^2}{M_S^2}\right)\ln\frac{M_V^2}{\mu^2} =$$

$$\frac{f^4}{M_V^4} + \hat{r}_{V2} + \frac{f^2}{192\pi^2 M_V^2}\left\{\left(1 - \frac{M_V^2}{M_S^2}\right)\ln\frac{M_S^2}{M_V^2} - \ln 2 - \frac{64}{15} + \frac{9}{5}\frac{M_V^2}{M_S^2}\right\}$$

$\bar{\ell}_6 = 16.0 \pm 0.5 \pm 0.7 \quad , \quad r_{V2}^r(M_\rho) = (1.6 \pm 0.5)\cdot 10^{-4}$ Bijnens-Colangelo-Talavera

$\longrightarrow \quad \hat{\ell}_6 = (-0.2 \pm 0.9)\cdot 10^{-3} \quad , \quad \hat{r}_{V2} = (-0.2 \pm 0.5)\cdot 10^{-4}$

$\ell_6^{N_C\to\infty} = -f^2/M_V^2 = -0.014 \quad , \quad r_{V2}^{N_C\to\infty} = f^4/M_V^4 = 2.1\cdot 10^{-4}$

$$F_V(s) \;=\; A(s)\,\frac{M_V^2}{M_V^2 - s - \Sigma_V^r(s)} \;+\; B(s)$$

High-Energy Limit: $s \longrightarrow \infty$

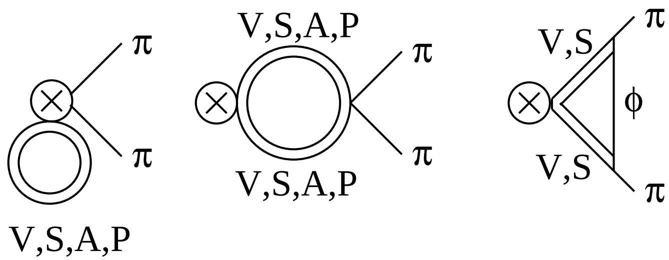
$$A(s) = \frac{M_V^2}{16\pi^2 F^2}\left\{ -\ln\frac{-s}{M_V^2}\left[\ln\frac{s}{M_V^2} - 2\right] + \frac{1}{2}\ln^2\frac{s}{M_V^2}\right\} + \mathcal{O}(1)$$

$$\Sigma_V^r(s) = \frac{-s^2}{96\pi^2 F^2}\left\{\ln\frac{-s}{\mu^2} - \frac{5}{3} + 192\pi^2 F^2\, X_Z^r(\mu)\right\}$$

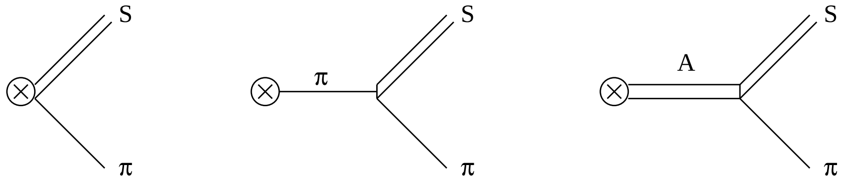
$$B(s) = \frac{1}{16\pi^2 F^2}\left\{-\,s^2\left[\frac{1}{6}\left(\frac{1}{M_V^2} + \frac{1}{M_A^2}\right)\left(\ln\frac{-s}{\mu^2} - \frac{2}{3}\right) - \frac{16\pi^2}{F^2}\,\widetilde{r}_{V2}^{\,r}(\mu)\right] + \frac{s}{3}\ln\frac{-s}{\mu^2} + \mathcal{O}(s)\right\}$$

BAD behaviour generated

by the 2-Resonance cuts



- $\langle\, S^0_{I=0}\, \pi^- | \bar{d}\gamma_\mu\gamma_5 u | 0\,\rangle \;=\; -\,2\,i\,\mathcal{F}_{S\pi}(q^2)\,\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right)\, p_\pi^\nu \qquad , \qquad q^\mu = (p_\pi + p_S)^\mu$

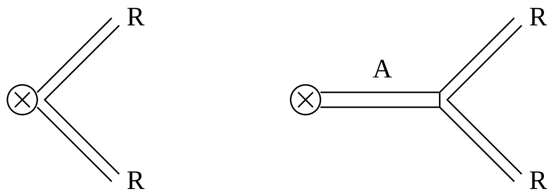


$$\mathcal{L}^2_{SA} \;=\; \lambda_1^{SA}\,\langle\,\{\nabla^\mu S, A_{\mu\nu}\}\,u^\nu\}\,\rangle$$

$$\lambda_1^{SA} \;=\; -\,\frac{\sqrt{2}\,c_d}{F_A} \;=\; -\,\frac{1}{\sqrt{2}}$$

$$\mathcal{F}_{S\pi}(q^2) \;=\; \frac{2c_d}{F} \;-\; \sqrt{2}\,\lambda_1^{SA}\,\frac{F_A}{F}\,\frac{q^2}{M_A^2 - q^2} \;=\; \frac{M_A^2}{M_A^2 - q^2}$$

- $\langle\, R^0_{I=1}(p_1)\, R^-(p_2) |\, \bar{d}\gamma^\mu u\, | 0\,\rangle \;=\; \sqrt{2}\,(p_2 - p_1)^\mu\, \mathcal{F}_{RR}(q^2) \qquad , \qquad R = S,\ P$



$$\mathcal{L}^2_{VRR} \;=\; i\,\lambda^{VRR}\,\langle\, V^{\mu\nu}\,\nabla_\mu R\,\nabla_\nu R\,\rangle$$

$$\lambda^{VRR} \;=\; \frac{\sqrt{2}}{F_V} \;=\; \frac{1}{F}$$

$$\mathcal{F}_{RR}(q^2) \;=\; 1 \;+\; \frac{F_V}{\sqrt{2}}\,\lambda^{VRR}\,\frac{q^2}{M_V^2 - q^2} \;=\; \frac{M_V^2}{M_V^2 - q^2}$$

SUMMARY

- $\mathbf{N_C} \to \infty$ provides a sensible approximation to the mesonic world
- Useful tool for quantitative non-perturbative analyses at low energies
- Some observables/physics only appear at subleading topologies
  - $U(1)_A$ Anomaly
  - Anomalous dimensions of four–quark (non-penguin) operators

    Short-distance logarithms
  - χPT loops (Final State Interactions)

    Long-distance logarithms
- Successful leading–order phenomenology

Challenge: **Rigorous control of subleading corrections**