Finite volume effects for masses and decay constants

Gilberto Colangelo

b
UNIVERSITÄT
BERN

Benasque, 10.8.04

Outline

- Introduction: CHPT in finite volume
- Lüscher's formula for masses
- Asymptotic formula for decay constants
- Numerics
- Summary

Work done in collaboration with S. Dürr, A. Fuhrer and C. Haefeli

Introduction

CHPT: expansion in m_{q_l}/Λ and p/Λ

In finite volume the momentum is quantized: $p = \frac{2\pi}{L} n$

Condition of applicability of CHPT:

$$m_{q_l} \ll \Lambda \quad \text{and} \quad \frac{2\pi}{L} \ll \Lambda$$

$$\Lambda \sim 4\pi F_\pi \quad \Rightarrow \quad 2LF_\pi \gg 1$$

Once this condition is respected we still have two different physical situations

$$LM_\pi \lesssim 1 \;\; \Rightarrow \;\; \epsilon\text{–regime} \quad M_\pi \sim \frac{1}{L^2} \sim O(\epsilon^2)$$

$$LM_\pi \gg 1 \;\; \Rightarrow \;\; p\text{–regime} \quad M_\pi \sim \frac{1}{L} \sim O(p)$$

p– or ϵ–regime?

Two alternatives:

- Chiral limit on the lattice $\Rightarrow$ ϵ–regime

  (unless one can simulate enormous volumes)

  $\Rightarrow$ Rely on CHPT to relate unphysical observables to physical quantities (cf. M. Laine's talk)

- $M_\pi > M_\pi^{\rm phys}$: choose $L \gg 1/M_\pi$, $\Rightarrow$ p–regime

  (e.g. $M_\pi = 300$ MeV, $L = 2$ fm, $M_\pi L \sim 3$)

  $\Rightarrow$ Rely on CHPT to make the chiral and the large volume extrapolation

p-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:

- the Lagrangian is the same as in infinite volume
- the propagators must be made periodic:

$$G_L(\vec{x}, t) = \sum_{\vec{n}} G_\infty(\vec{x} + \vec{n}L, t)$$

p-regime

Calculational rule in CHPT for isotropic finite box with periodic boundary conditions:
Examples: Gasser and Leutwyler (88)

$$M_\pi(L) = M_\pi \left[1 + \frac{1}{2N_f} \xi \, g_1(\lambda) + O(\xi^2) \right]$$

$$F_\pi(L) = F_\pi \left[1 - \frac{N_f}{2} \xi \, g_1(\lambda) + O(\xi^2) \right]$$

with

$$\lambda = M_\pi L, \quad \xi = (M_\pi / 4\pi F_\pi)^2$$

$$g_1(\lambda) = {\sum_{\vec{n}}}' \int_0^\infty dz \, e^{-\frac{1}{z} - \frac{z}{4} \vec{n}^2 \lambda^2} = \sum_{\vec{n} \neq \vec{0}} G_\infty(\vec{x} + \vec{n} L, t)|_{t = \vec{x} = 0}$$

Finite volume effects in the p-regime

Foundations: Gasser and Leutwyler (87)

quenched CHPT: Sharpe, Bernard and Golterman (90's)

Recent applications:

- two–pion states — Lin, Martinelli, Pallante, Sachrajda and Villadoro (03)
- F_K and B_K — Becirevic and Villadoro (03)
- m_p — QCDSF (03)
- m_N, μ_N and g_A — Beane and Savage (03-04)
- f_B and B_B — Arndt and Lin (04)
- m_p — Koma and Koma (04)

Finite volume effects in the p-regime

Foundations: Gasser and Leutwyler (87)

quenched CHPT: Sharpe, Bernard and Golterman (90's)

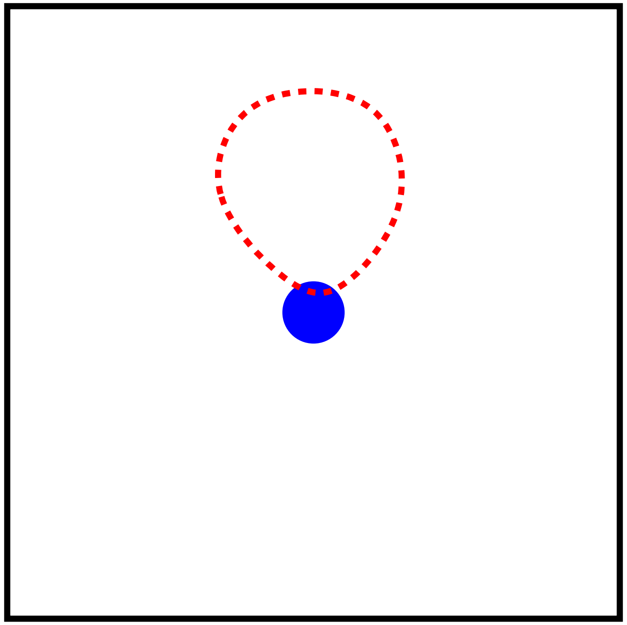
Talks at Lattice 2004:

- M_π, F_π and $\langle r^2 \rangle_V$ — R. Lewis
- Lüscher Formula for m_p — Y. Koma
- m_p and g_A — M. Goeckeler
- f_B and B_B — D. Lin
- Weak matrix elements in the ϵ-regime — H. Wittig
- Plenary — GC

Outline

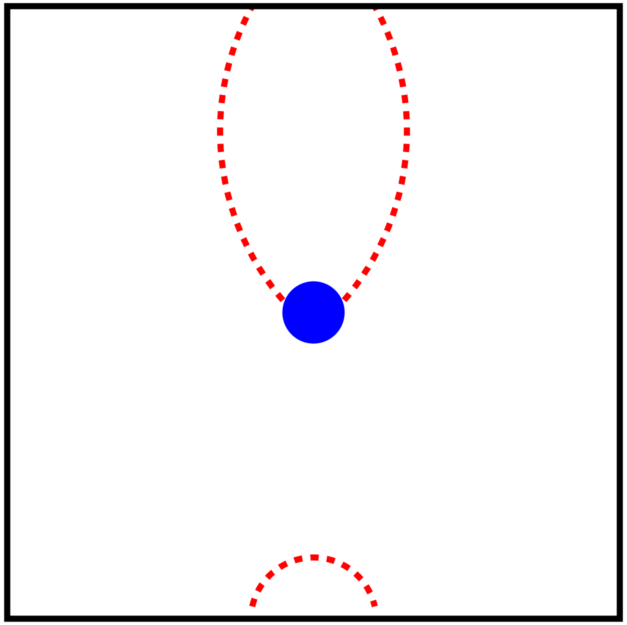
- Introduction: CHPT in finite volume
- Lüscher's formula for masses
- Asymptotic formula for decay constants
- Numerics
- Summary

Masses in finite volume



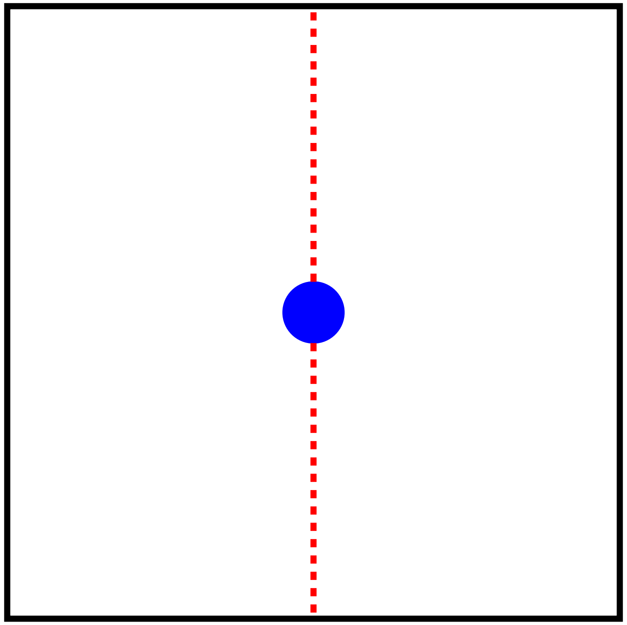
Loop-diagram

Masses in finite volume



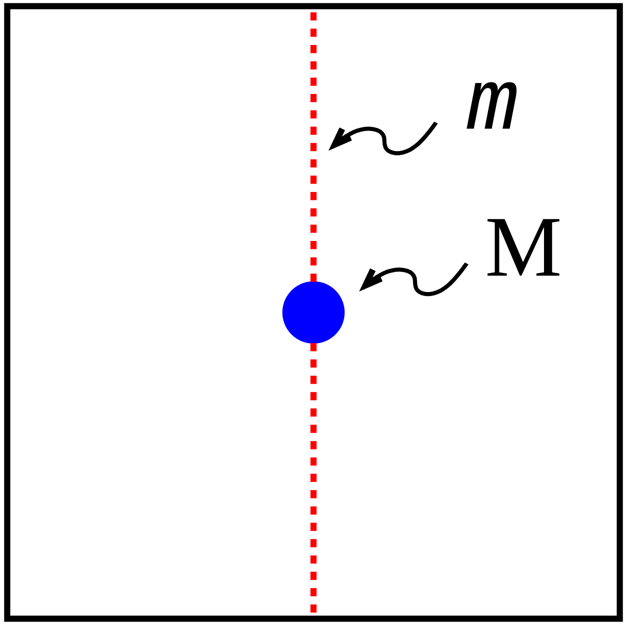
Loop diagram with
periodic
boundary conditions

Masses in finite volume



Loop diagram with periodic boundary conditions

Masses in finite volume

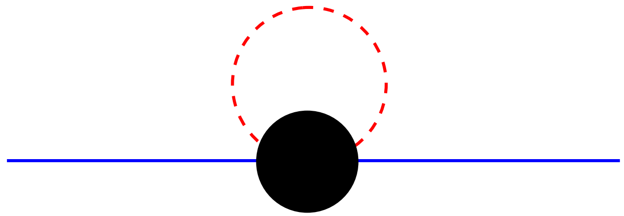


Loop diagram with periodic
boundary conditions

This diagram exists only for
$L \neq \infty$

Its effect is of the order
$\exp[-mL]$

Lüscher's Formula



$$= \int d\ell \; \Gamma(p,\ell,-\ell,-p) G_L(\ell)$$

$$G_L(\ell) = \sum_{\vec{n}} G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \qquad G_\infty(\ell) \sim \frac{1}{\ell^2 + m^2}$$

$$\begin{aligned} M_L - M_\infty &= \int d\ell \; \Gamma(p,\ell,-\ell,p) \left[G_L(\ell) - G_\infty(\ell) \right] \\ &= \sum_{\vec{n}\neq\vec{0}} \int d\ell \; \Gamma(p,\ell,-\ell,p) G_\infty(\ell) e^{i\vec{\ell}\cdot\vec{n}L} \end{aligned}$$

Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

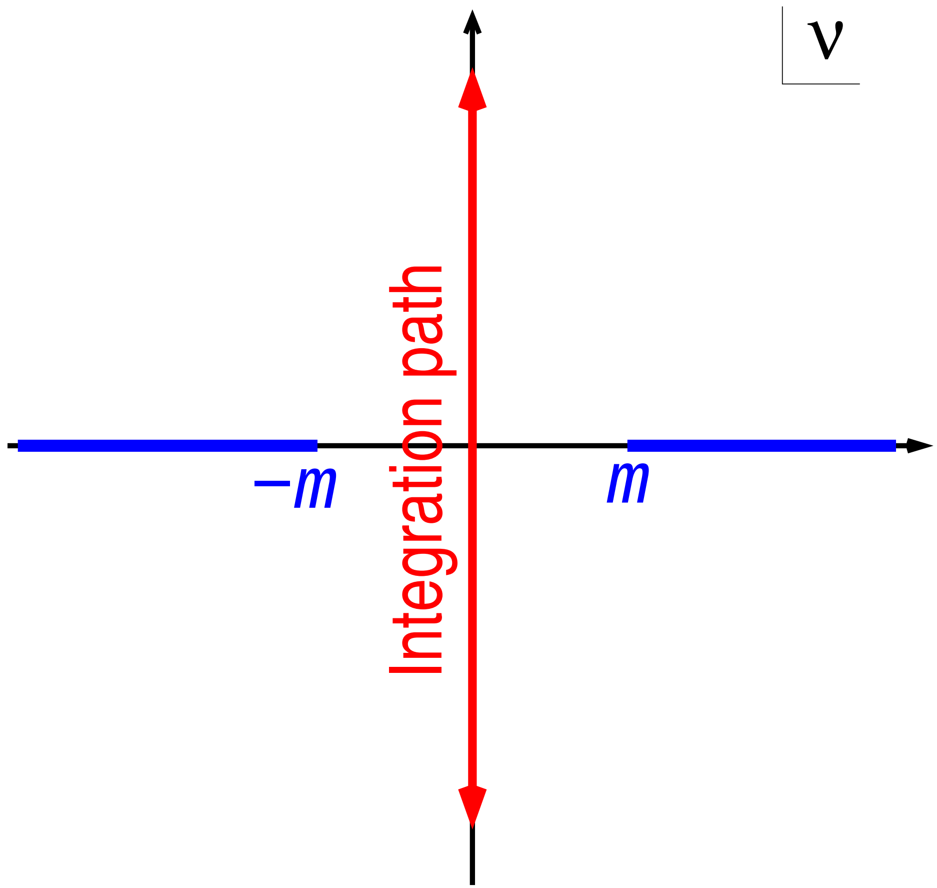
$$M_L - M_\infty = C \int_{-\infty}^{\infty} dy \, e^{-\sqrt{m^2+y^2}L} F(iy) + \ldots$$

where $F(\nu)$ is the scattering amplitude between the red (m) and blue (M) particle, and C a constant that depends from $L, \; m$ and M

Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

$$M_L - M_\infty = C \int_{-\infty}^{\infty} dy\, e^{-\sqrt{m^2+y^2}L} F(iy) + \ldots$$



Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

$$M_L - M_\infty = C \int_{-\infty}^{\infty} dy \, e^{-\sqrt{m^2+y^2}L} F(iy) + \ldots$$

The formula expresses the corrections as an integral over a (analytically continued) physical amplitude

Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

$$M_L - M_\infty = C \int_{-\infty}^{\infty} dy \, e^{-\sqrt{m^2+y^2}L} F(iy) + \ldots$$

The formula expresses the corrections as an integral over a (analytically continued) physical amplitude

What matters for the behaviour of the corrections is not the mass of the particle itself, but rather the mass of the lightest particle to which it is coupled

Lüscher's Formula

Leading correction for $mL \gg 1$: (Lüscher 86)

$$M_L - M_\infty = C \int_{-\infty}^{\infty} dy \, e^{-\sqrt{m^2+y^2}L} F(iy) + \ldots$$

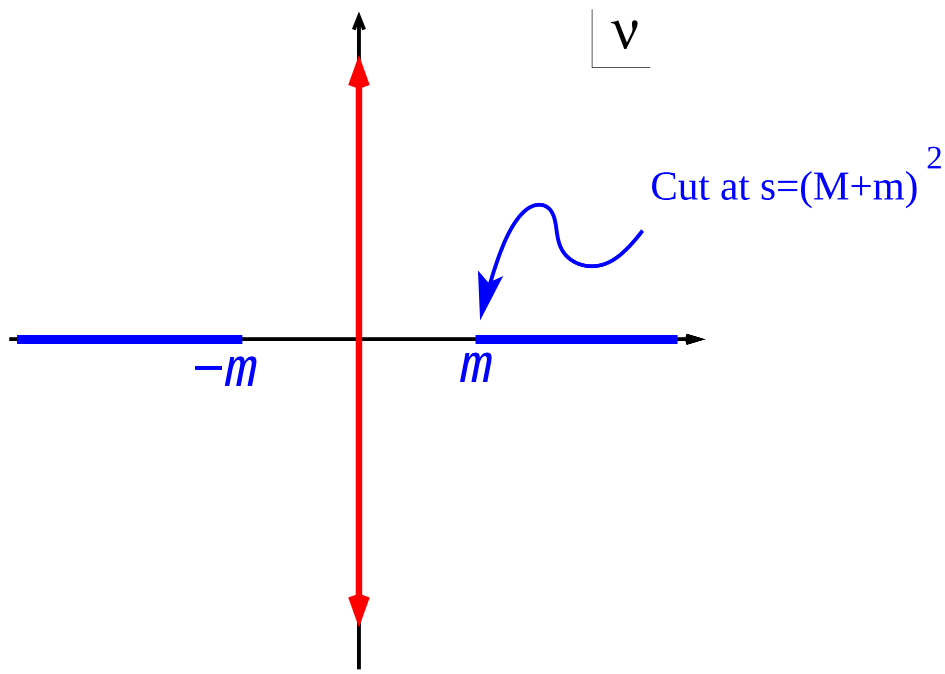
The formula expresses the corrections as an integral over a (analytically continued) physical amplitude

What matters for the behaviour of the corrections is not the mass of the particle itself, but rather the mass of the lightest particle to which it is coupled

e.g. both the corrections for the pion as well as those for the proton mass depend exponentially on $M_\pi L$

Cuts and poles in the scattering amplitude

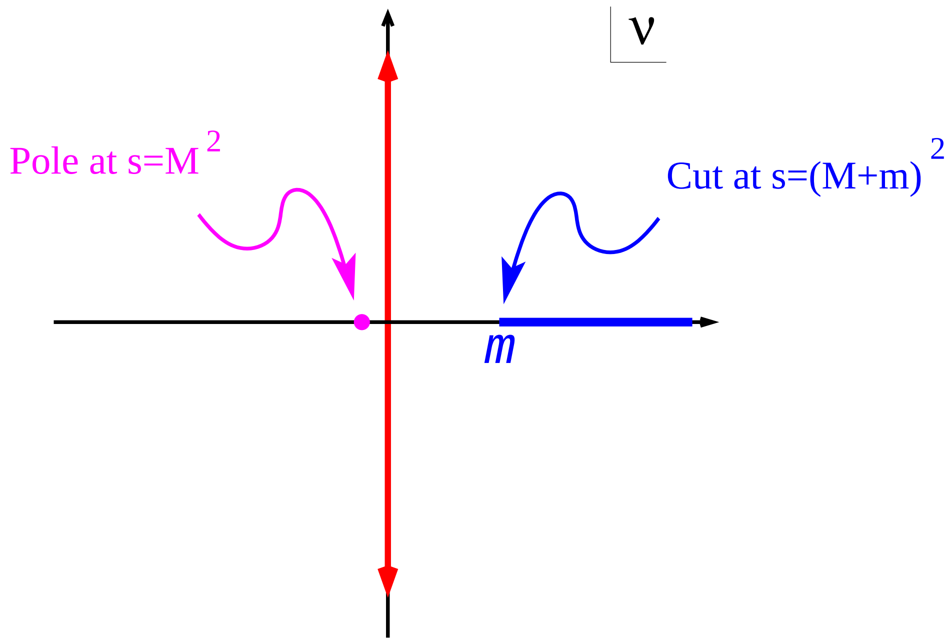
Any scattering amplitude must have a cut at
$s, u = (M+m)^2 \Rightarrow \nu = \frac{s-u}{4M} = \pm m$



Cuts and poles in the scattering amplitude

In addition it may have poles, e.g. at

$s, u = M^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp \frac{m^2}{2M}$

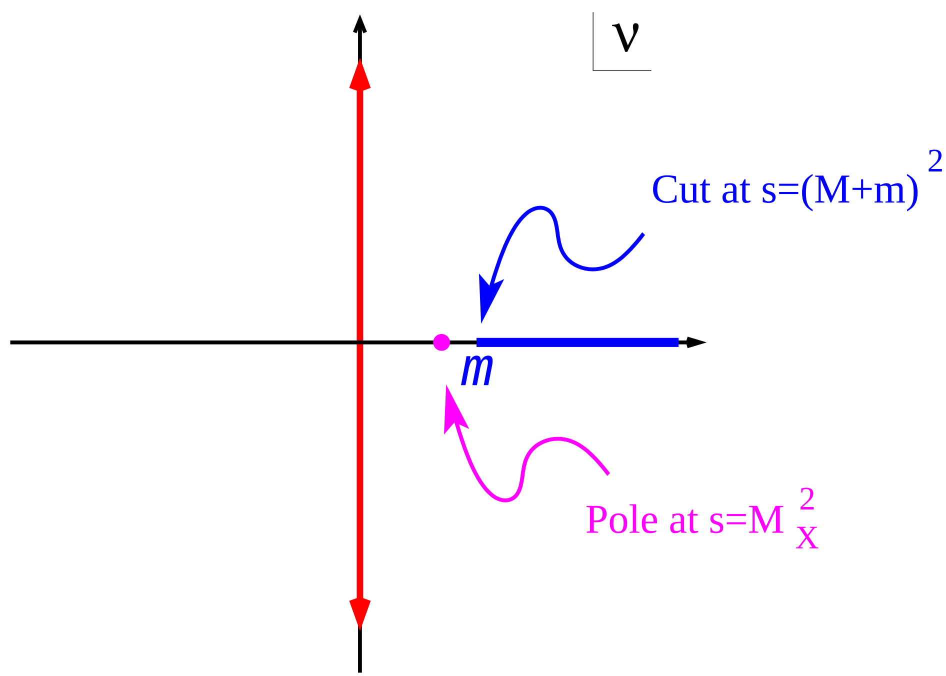


Poles on the lhs of the imaginary axis generate an extra term in the Lüscher's formula (cf. the formula for the nucleon mass)

Cuts and poles in the scattering amplitude

In addition it may have poles, or at

$$s, u = M_X^2 \Rightarrow \nu = \frac{s-u}{4M} = \mp\frac{m^2}{2M} + \Delta M \left(1 + \frac{\Delta M}{2M}\right) \qquad \Delta M = M_X - M$$

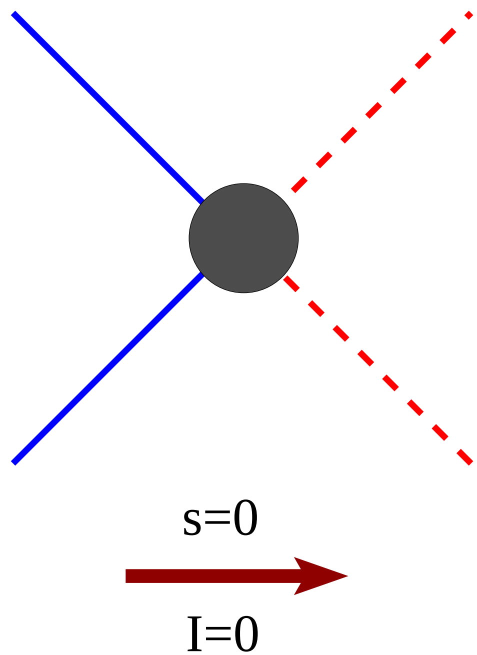


Poles on the rhs of the imaginary axis do not generate an extra term in the Lüscher's formula (cf. D. Lin's talk on heavy mesons)

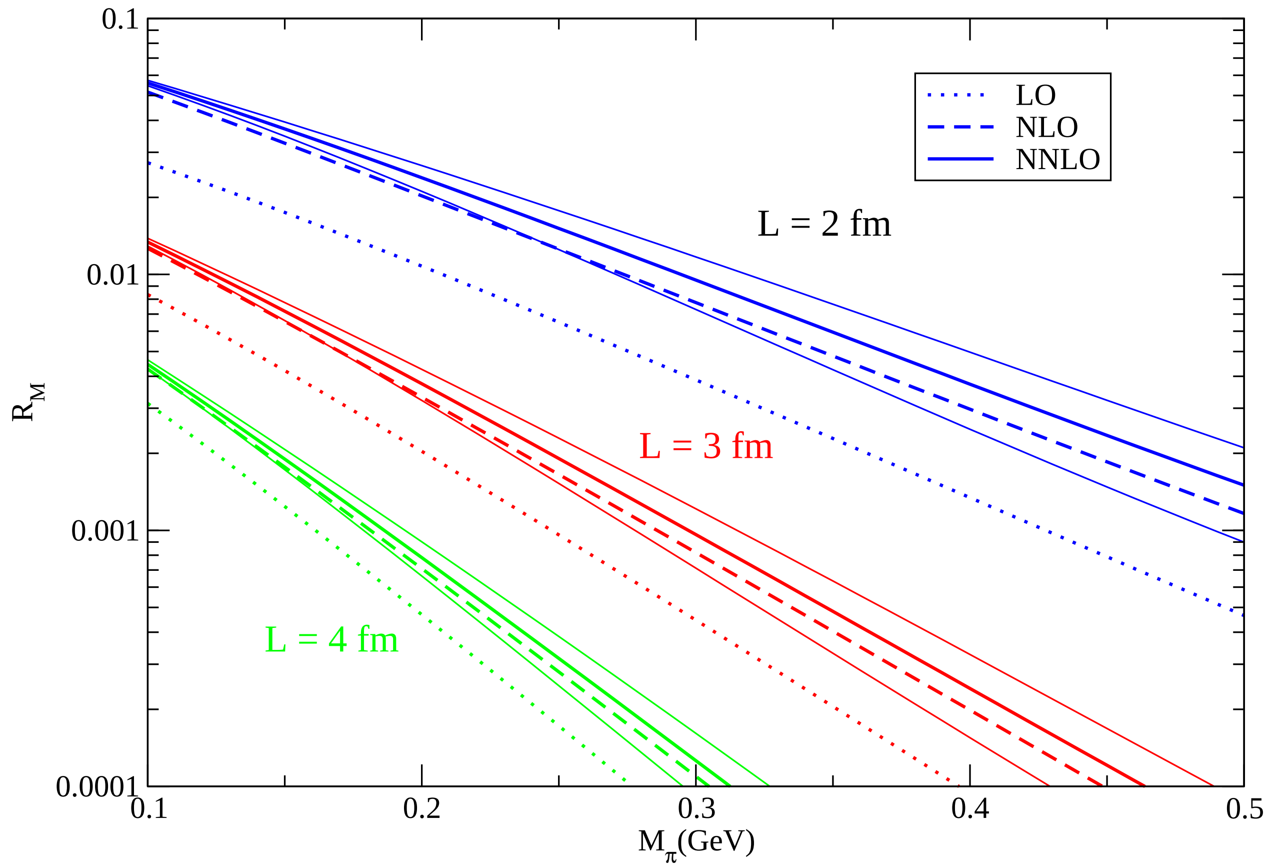
Corrections for M_π

$\pi\pi$ scattering amplitude with forward kinematics

$$F_{\pi\pi}(\nu) = T^{I=0}\left[0, 2M_\pi(M_\pi+\nu), 2M_\pi(M_\pi-\nu)\right] = -\frac{M_\pi^2}{F_\pi^2} + O(p^4)$$



Corrections for M_π



$$R_M = M_{\pi\, L}/M_\pi - 1$$

GC and S. Dürr 03

Lüscher's Formula or CHPT?

$$\Delta M_{\pi\,\text{Lüscher}}^{L} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\; F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M_{\pi\,\mathrm{CHPT}}^{L} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)_{|_{t=\vec{x}=0}} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda}\; K_1\left(|\vec{n}|\,\lambda\right)$$

where $m(|\vec{n}|)$ is the multiplicity of a vector of length $|\vec{n}|$ in 3-dimensional discretized space

Lüscher's Formula or CHPT?

$$\Delta M_{\pi}^{L}{}_{\text{Lüscher}} = \frac{-3}{16\pi^2\lambda} \int_{-\infty}^{\infty} dy\, F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

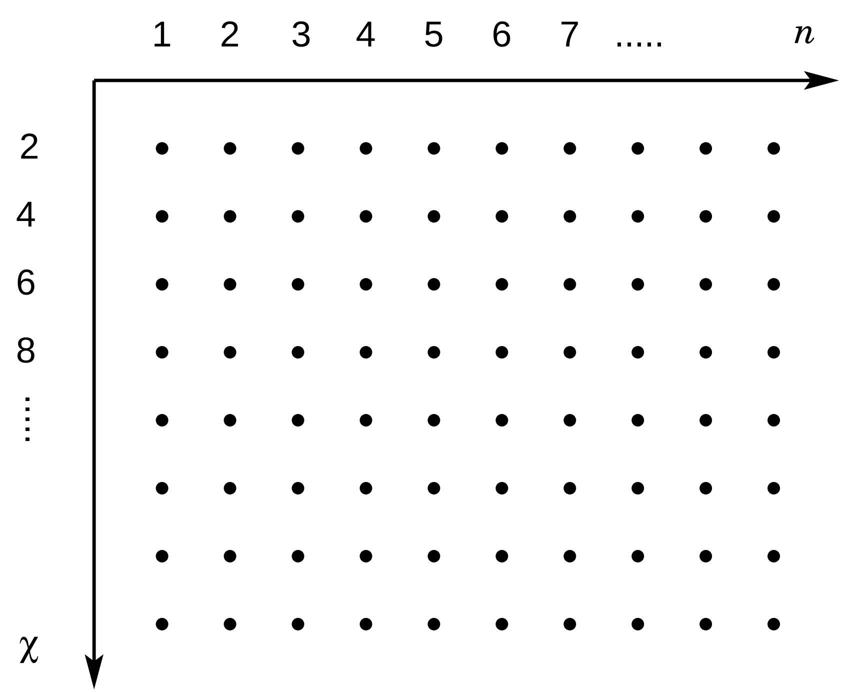
$$\Delta M_{\pi}^{L}{}_{\mathrm{CHPT}} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$

The two formulas give the leading term in two different expansions

Lüscher's Formula or CHPT?

$$\Delta M_{\pi\,\text{Lüscher}}^{L} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\; F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

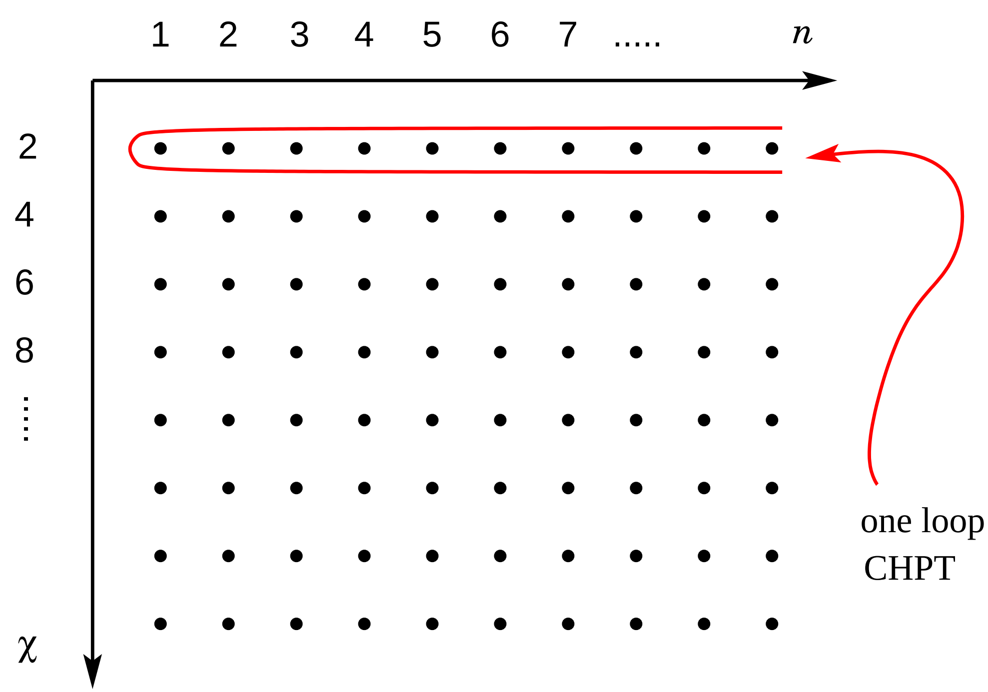
$$\Delta M_{\pi\,\mathrm{CHPT}}^{L} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$



Lüscher's Formula or CHPT?

$$\Delta M_\pi^L{}_{\text{Lüscher}} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\, F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

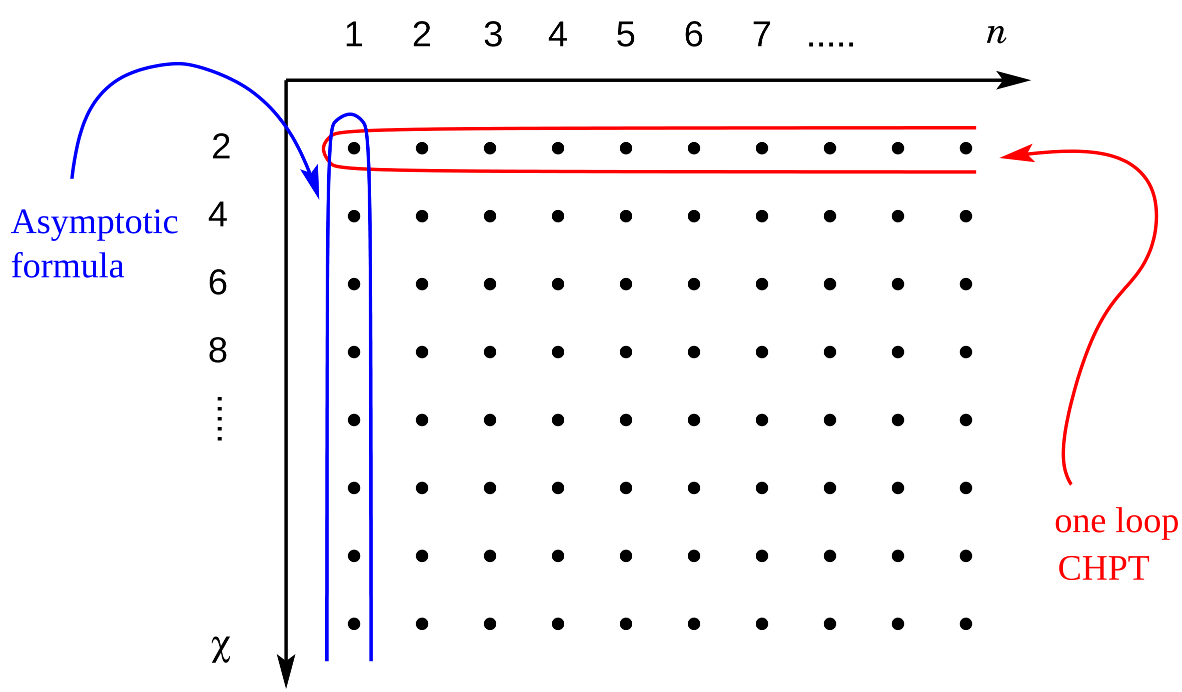
$$\Delta M_\pi^L{}_{\text{CHPT}} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$



Lüscher's Formula or CHPT?

$$\Delta M_{\pi\,\text{Lüscher}}^{L} = \frac{-3}{16\pi^2\lambda}\int_{-\infty}^{\infty} dy\, F(\mathrm{i}y)\, e^{-\sqrt{M_\pi^2+y^2}\,L} + O(e^{-\overline{M}L})$$

$$\Delta M_{\pi\,\text{CHPT}}^{L} = \frac{1}{4}\xi\, g_1(\lambda) + O(\xi^2)$$



Extension of the Lüscher’s Formula

One-loop CHPT corrections

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)\big|_{t=\vec{x}=0} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda}\; K_1\left(|\vec{n}|\,\lambda\right)$$

Extension of the Lüscher's Formula

One-loop CHPT corrections

$$g_1(\lambda) = \sum_{\vec{n}\neq\vec{0}} G_\infty(\vec{x}+\vec{n}L,t)_{|t=\vec{x}=0} = \sum_{|\vec{n}|=1}^{\infty} \frac{4m(|\vec{n}|)}{|\vec{n}|\,\lambda} K_1\left(|\vec{n}|\,\lambda\right)$$

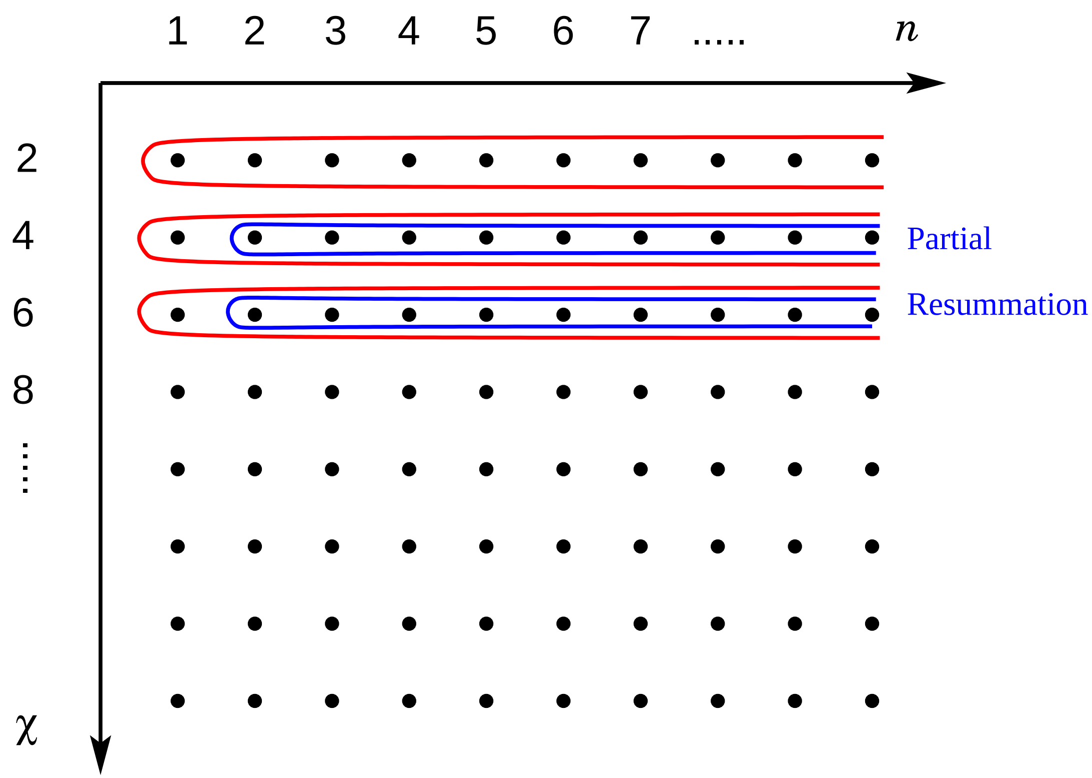
Analogously one can extend the Lüscher's Formula so that it contains contributions from all $|\vec{n}|$ of a single propagator:

$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda} \sum_{|\vec{n}|=1}^{\infty} \frac{m(|\vec{n}|)}{|\vec{n}|} \int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$

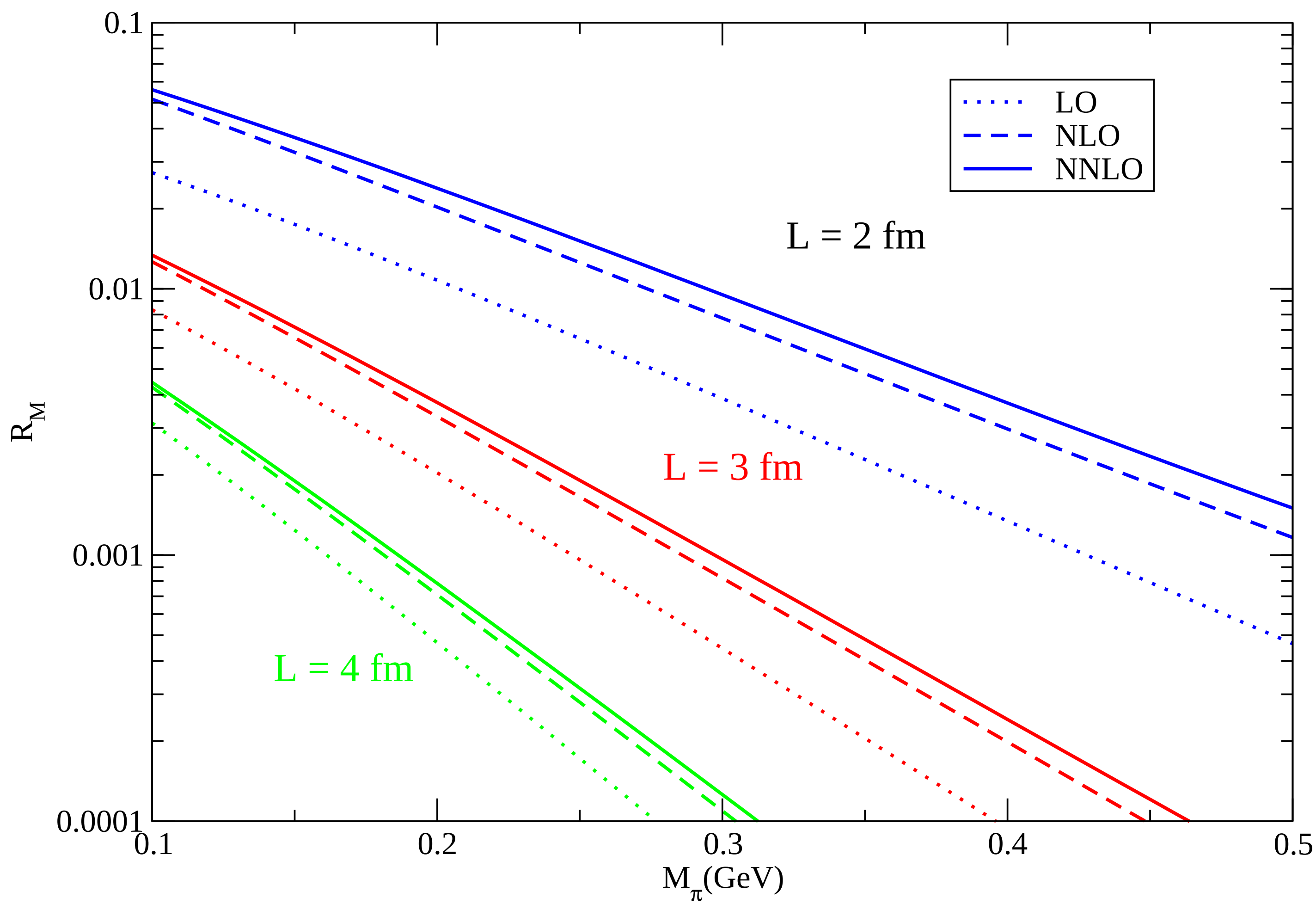
The extension does not provide all exponentially subleading terms!

Extension of the Lüscher's Formula

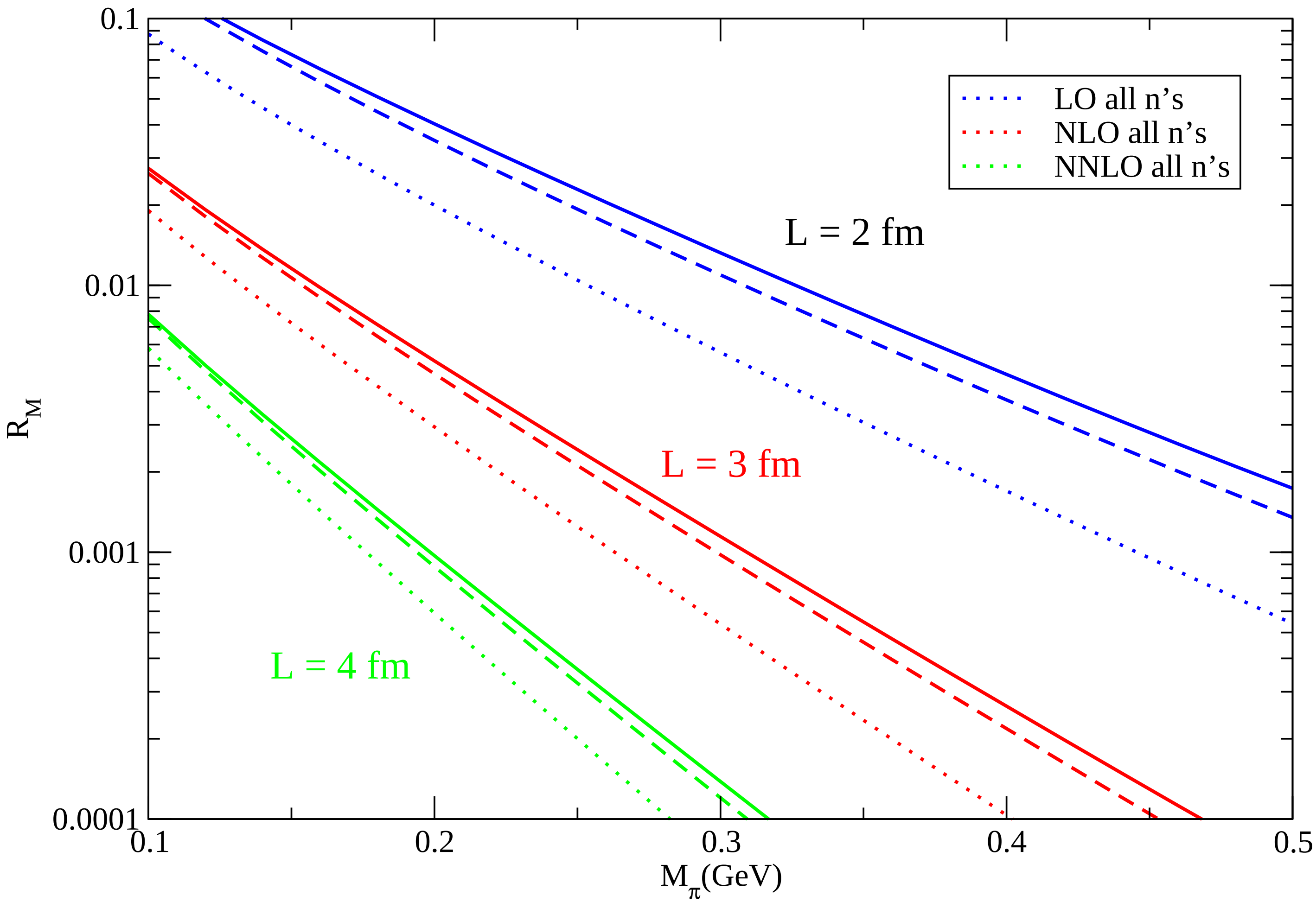
$$M_{\pi,L} - M_\pi = -\frac{1}{32\pi^2\lambda}\sum_{|\vec{n}|=1}^{\infty}\frac{m(|\vec{n}|)}{|\vec{n}|}\int_{-\infty}^{\infty} dy F(iy) e^{-\sqrt{\vec{n}^2(M_\pi^2+y^2)}L}$$



Nonleading exp. terms in $M_{\pi,L}$



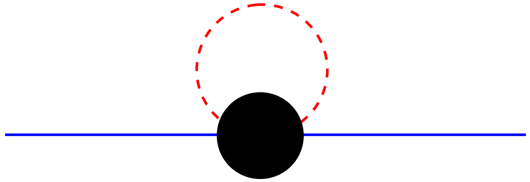
Nonleading exp. terms in $M_{\pi,L}$



Outline

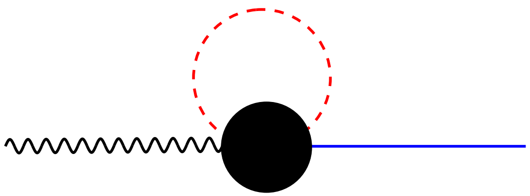
- Introduction: CHPT in finite volume
- Lüscher's formula for masses
- Asymptotic formula for decay constants
- Numerics
- Summary

Extension to decay constants



$$\Rightarrow \quad \Delta M \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \ldots$$

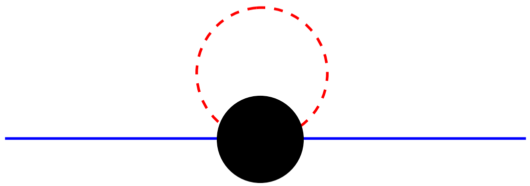
$$F(\nu) \Leftrightarrow \langle \pi\pi | T | \pi\pi \rangle$$



$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \ldots$$

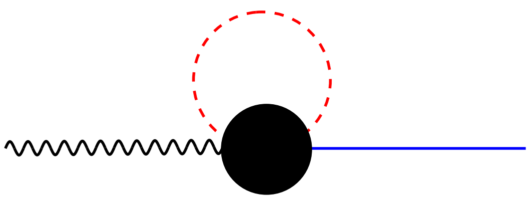
$$N(\nu) \Leftrightarrow \langle 0 | A_\mu | \pi\pi\pi \rangle \sim A(\tau \to 3\pi\nu_\tau)$$

Extension to decay constants



$$\Rightarrow \quad \Delta M \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} F(iy) + \ldots$$

$$F(\nu) \Leftrightarrow \langle \pi\pi | T | \pi\pi \rangle$$



$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \ldots$$

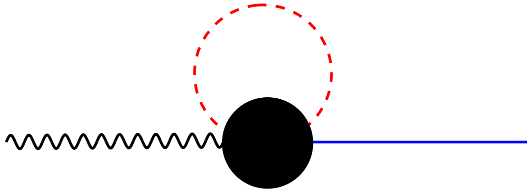
$$N(\nu) \Leftrightarrow \langle 0 | A_\mu | \pi\pi\pi \rangle \sim A(\tau \to 3\pi\nu_\tau)$$

The $\langle 0 | A_\mu | \pi\pi\pi \rangle$ amplitude must be subtracted:

$$N(\nu) = \langle (2\pi)_{I=0}\pi | A_\mu(0) | 0 \rangle - iQ_\mu \frac{F_\pi F(\nu)}{M_\pi^2 - Q^2}$$

GC and C. Haefeli 04

Extension to decay constants



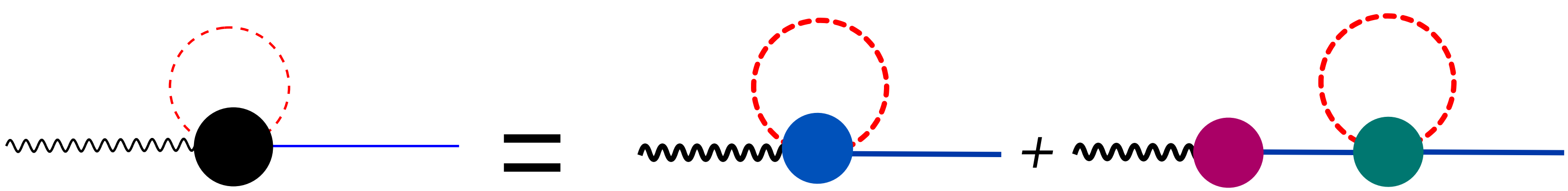
$$\Rightarrow \quad \Delta F \propto \int_{-\infty}^{\infty} dy\, e^{-\sqrt{M_\pi^2+y^2}L} N(iy) + \ldots$$

$$N(\nu) \Leftrightarrow \langle 0|A_\mu|\pi\pi\pi\rangle \sim A(\tau \to 3\pi\nu_\tau)$$

The $\langle 0|A_\mu|\pi\pi\pi\rangle$ amplitude must be subtracted:

$$N(\nu) = \langle (2\pi)_{I=0}\pi|A_\mu(0)|0\rangle - iQ_\mu \frac{F_\pi F(\nu)}{M_\pi^2 - Q^2}$$

GC and C. Haefeli 04



Ward identity in finite volume

$$\langle 0|A_\mu^i(0)|\pi^k(p)\rangle = i\delta^{ik}F_\pi p_\mu \qquad \langle 0|P^i(0)|\pi^k(p)\rangle = i\delta^{ik}G_\pi$$

Ward identity

$$F_\pi M_\pi^2 = \hat{m} G_\pi$$

The identity is valid also in finite Volume

$$R_G = R_F + 2R_M \qquad (R_X := \Delta X_\pi^L / X_\pi)$$

Ward identity in finite volume

$$\langle 0|A^i_\mu(0)|\pi^k(p)\rangle = i\delta^{ik}F_\pi p_\mu \qquad \langle 0|P^i(0)|\pi^k(p)\rangle = i\delta^{ik}G_\pi$$

Ward identity

$$F_\pi M_\pi^2 = \hat{m}G_\pi$$

The identity is valid also in finite Volume

$$R_G = R_F + 2R_M \qquad\qquad (R_X := \Delta X_\pi^L / X_\pi)$$

and must be satisfied by the asymptotic formulae :

$$C\int_{-\infty}^{\infty} dy e^{-\sqrt{M_\pi^2+y^2}L}\left[N_G(iy) - N_F(iy) + \frac{F_\pi}{M_\pi}F(iy)\right] = 0$$

Ward identity in finite volume

$$\langle 0|A^i_\mu(0)|\pi^k(p)\rangle = i\delta^{ik}F_\pi p_\mu \qquad \langle 0|P^i(0)|\pi^k(p)\rangle = i\delta^{ik}G_\pi$$

Ward identity

$$F_\pi M_\pi^2 = \hat{m}G_\pi$$

The identity is valid also in finite Volume

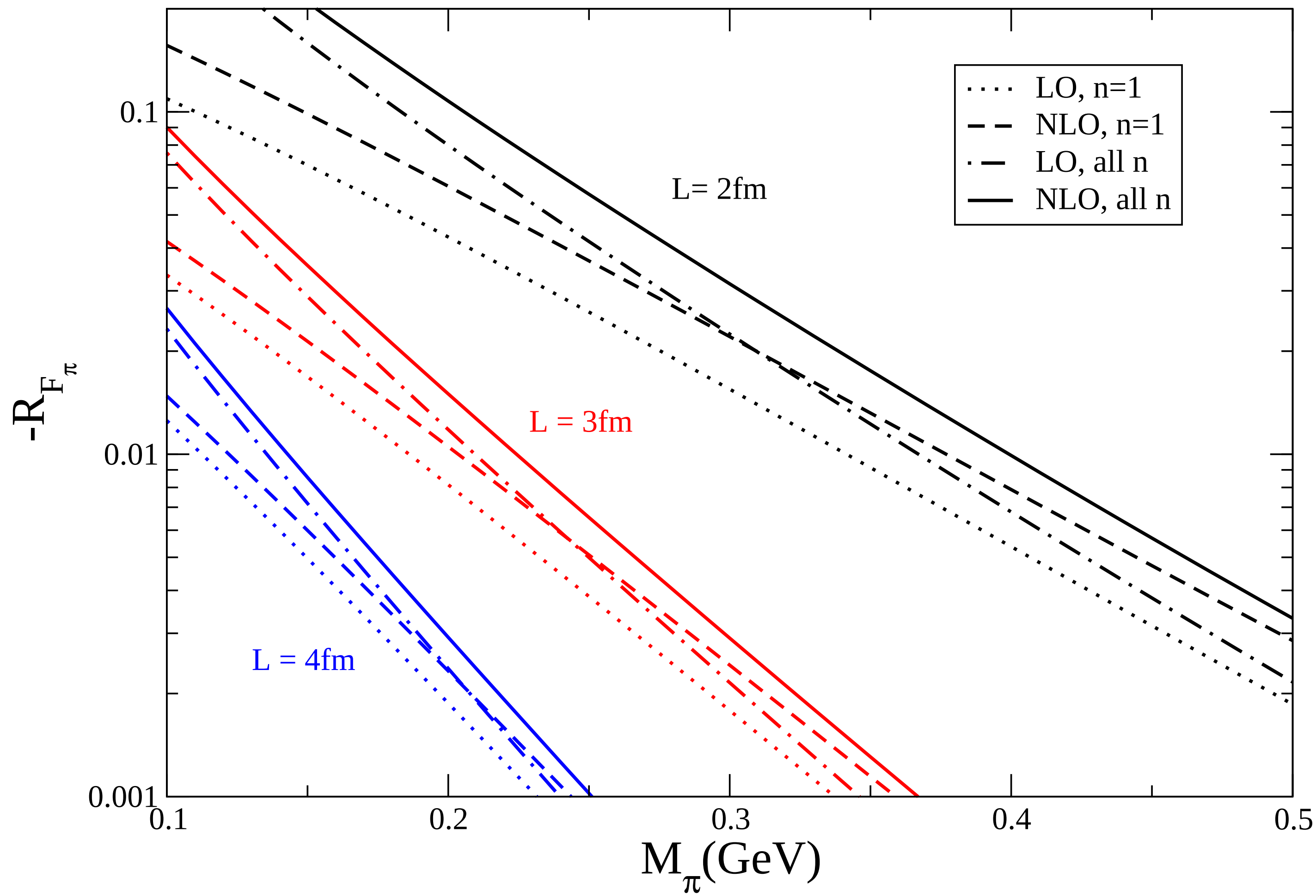
$$R_G = R_F + 2R_M \qquad (R_X := \Delta X_\pi^L / X_\pi)$$

and must be satisfied by the asymptotic formulae – in particular for the integrands:

$$N_G(\nu) - N_F(\nu) + \frac{F_\pi}{M_\pi}F(\nu) = 0$$

This is a Ward identity for 4-point functions!

Corrections for F_π



GC and C. Haefeli 04

Other applications

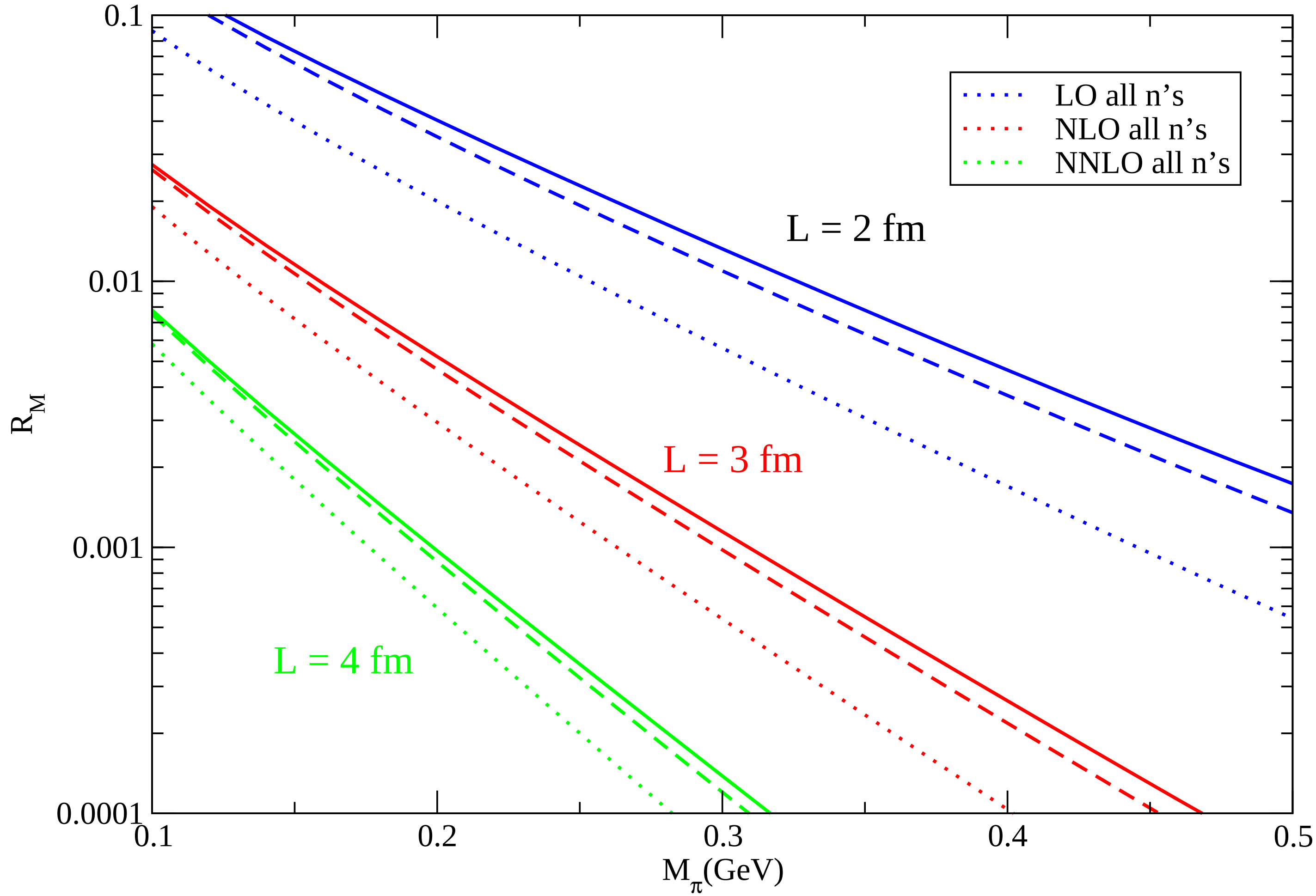
Quantity	Amplitude	Theory status
M_K	$A(\pi K \to \pi K)$	$O(p^6)$ (Bijnens et al.)
F_K	$A(K_{l4})$	$O(p^6)$ (Bijnens et al.)
M_η	$A(\pi\eta \to \pi\eta)$	$O(p^4)$ (Bernard et al.)
F_η	$A(\eta_{l4})$	?
M_N	$A(\pi N \to \pi N)$	$O(p^4)$ various Authors
M_B	$A(\pi B \to \pi B)$	?
F_B	$A(B_{l4})$	?

Work in progress: GC, S. Dürr, C. Haefeli, A. Fuhrer

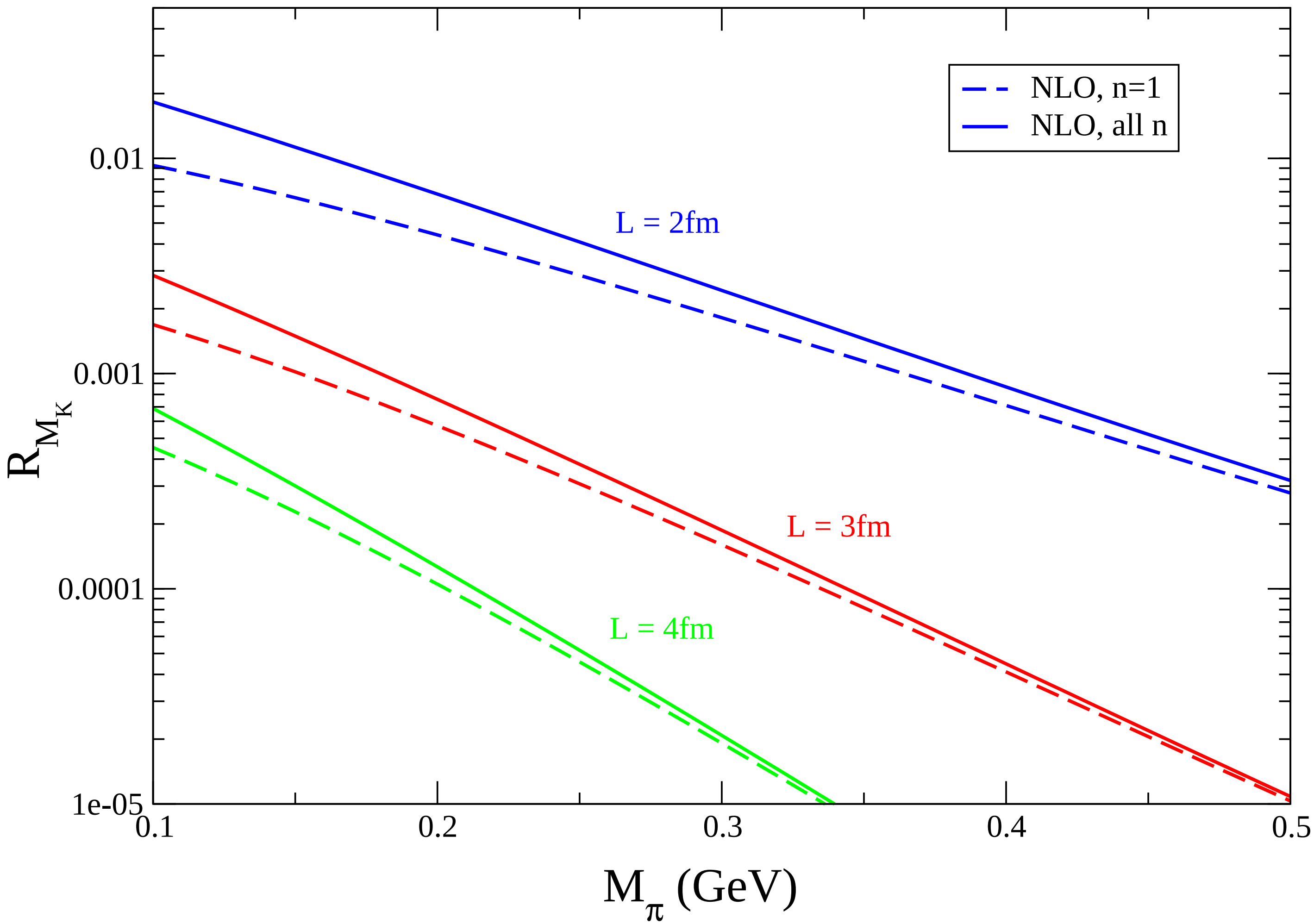
Outline

- Introduction: CHPT in finite volume
- Lüscher's formula for masses
- Asymptotic formula for decay constants
- Numerics
- Summary

Corrections for M_π

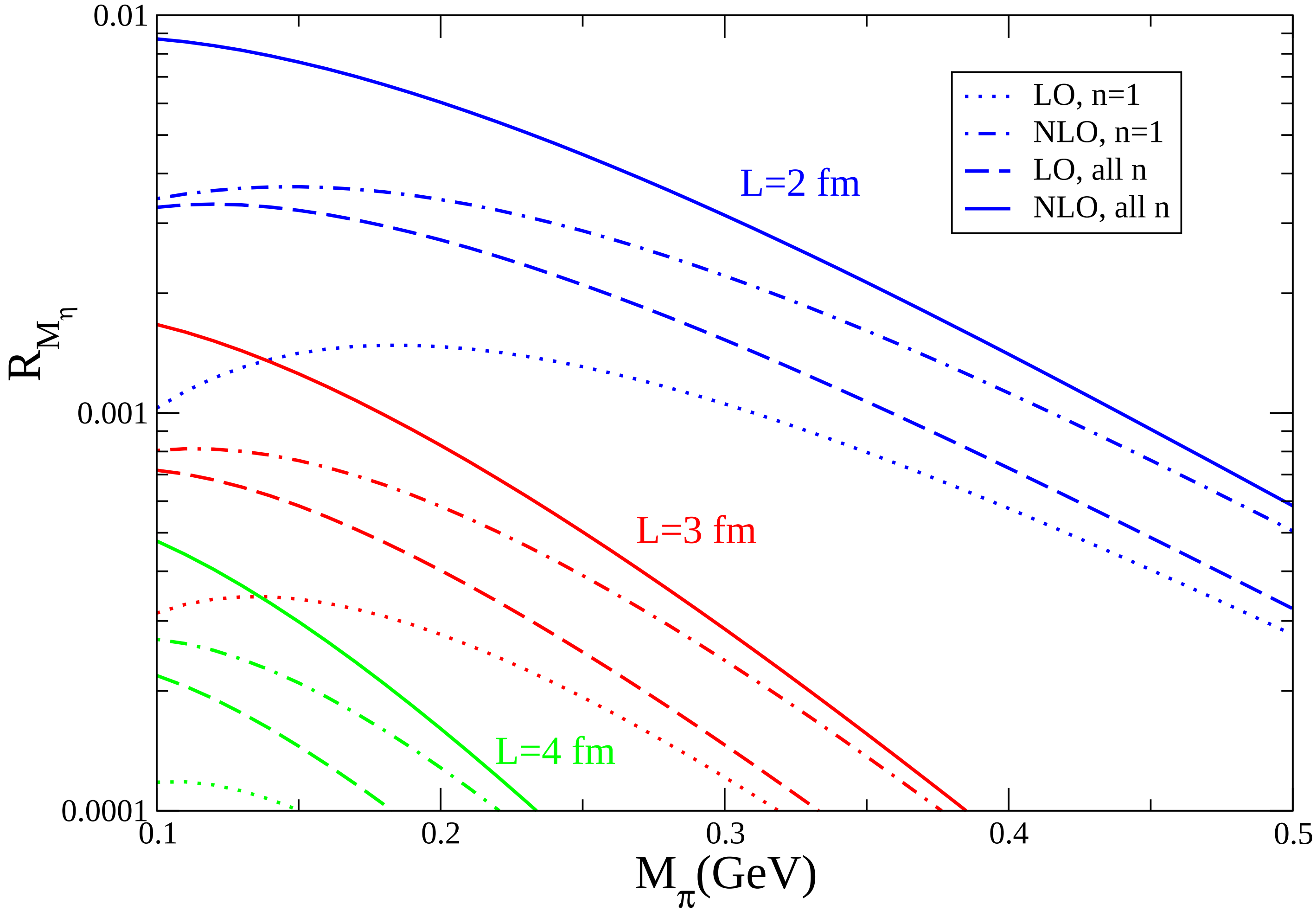


Corrections for M_K



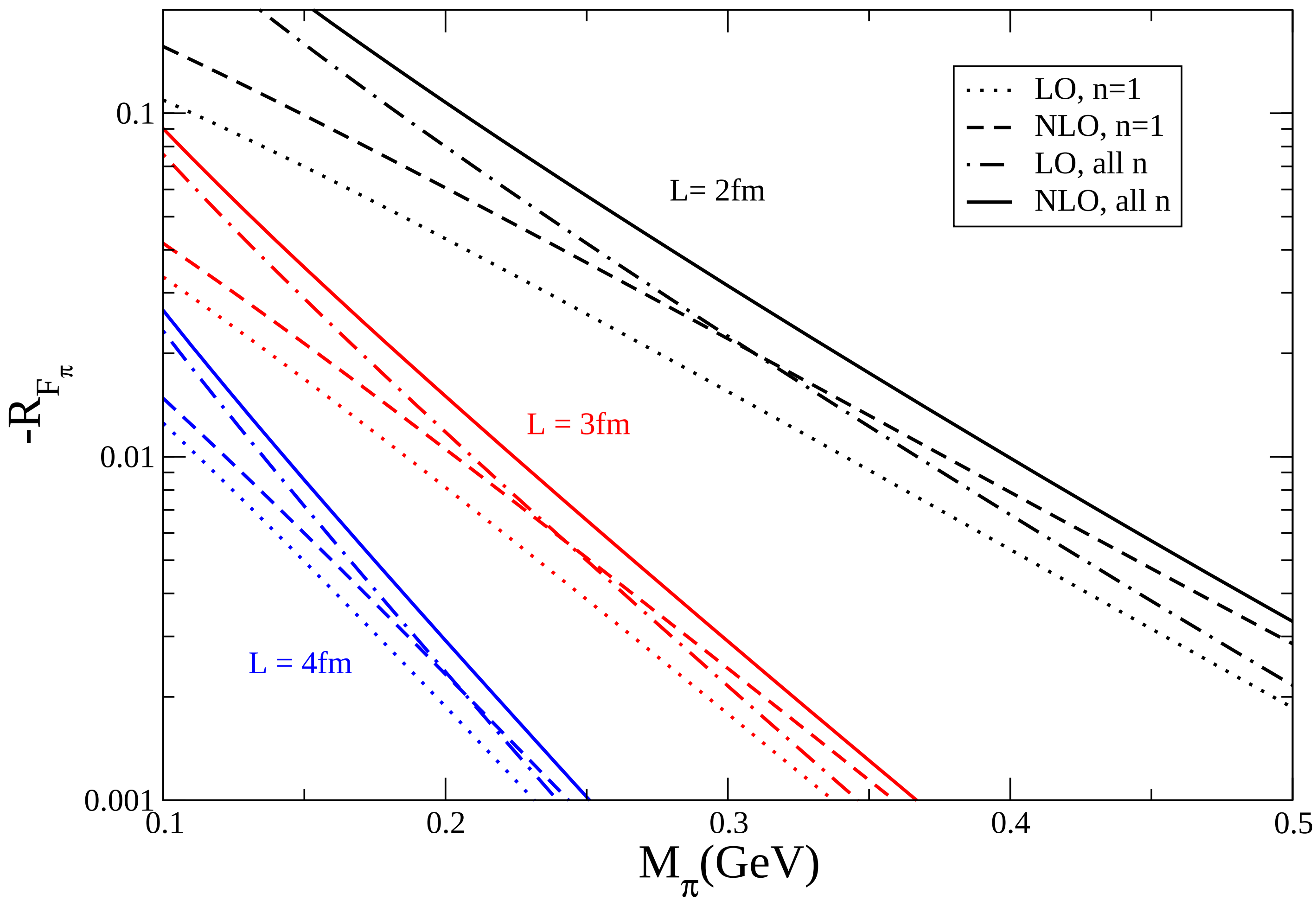
work in progress, G.C. and C. Haefeli

Corrections for M_η



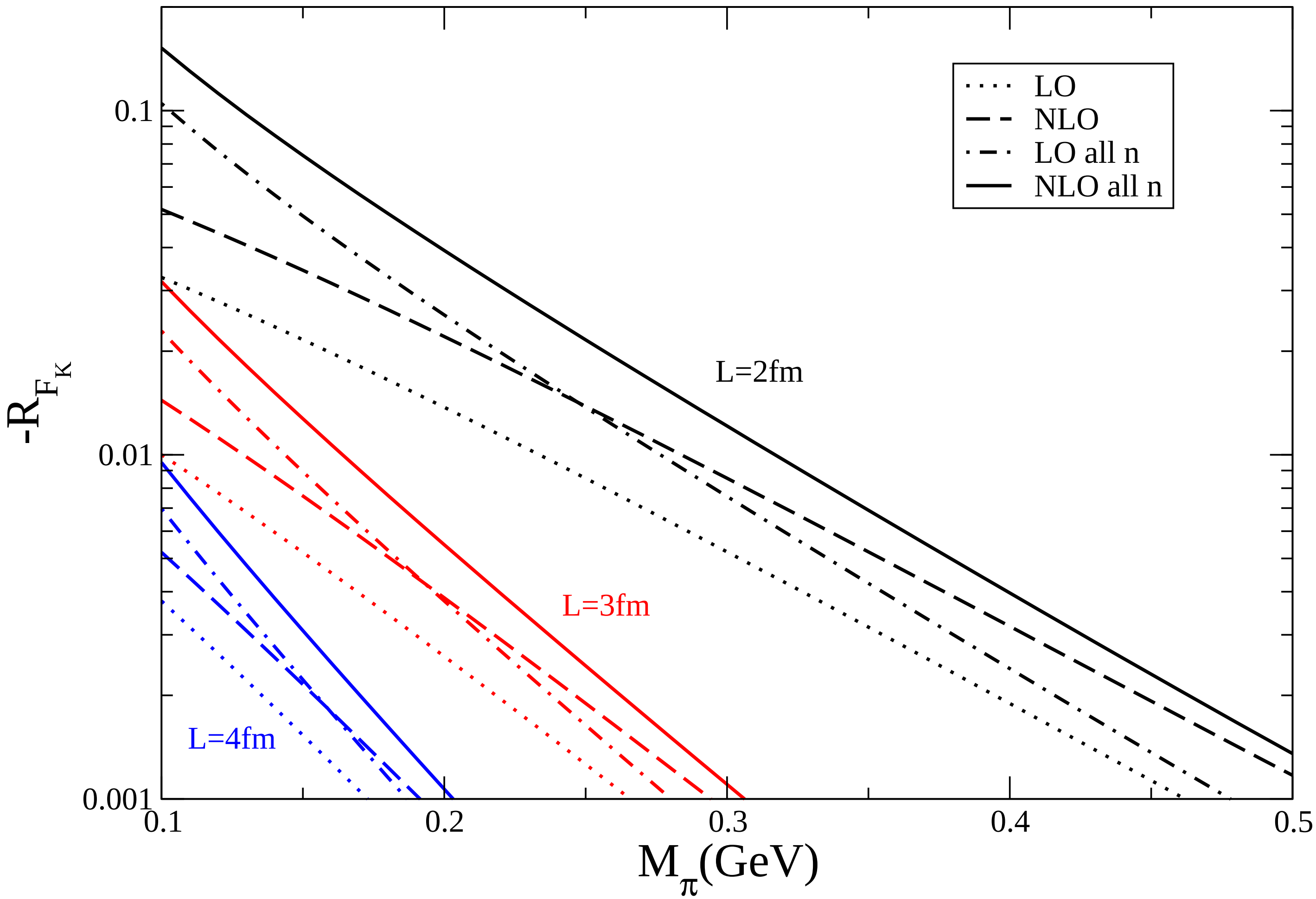
work in progress, G.C. and C. Haefeli

Corrections for F_π



G.C. and C. Haefeli 04

Corrections for F_K



LO agrees with Becirevic and Villadoro

work in progress, GC and C. Haefeli

Summary

- For large volumes ($2LF_\pi \gg 1$), finite–volume effects can be calculated analytically within CHPT
- several one-loop CHPT calculations in the p–regime ($M_\pi L \gg 1$) have appeared in the recent literature
- the combined use of CHPT and asymptotic formulae *à la* Lüscher offers the most efficient way to get to higher orders in CHPT
- I have presented numerical evaluations of these finite volume corrections for all pseudoscalar masses and decay constants, as well as for the nucleon (after the talk of T.Hemmert 10 days ago)

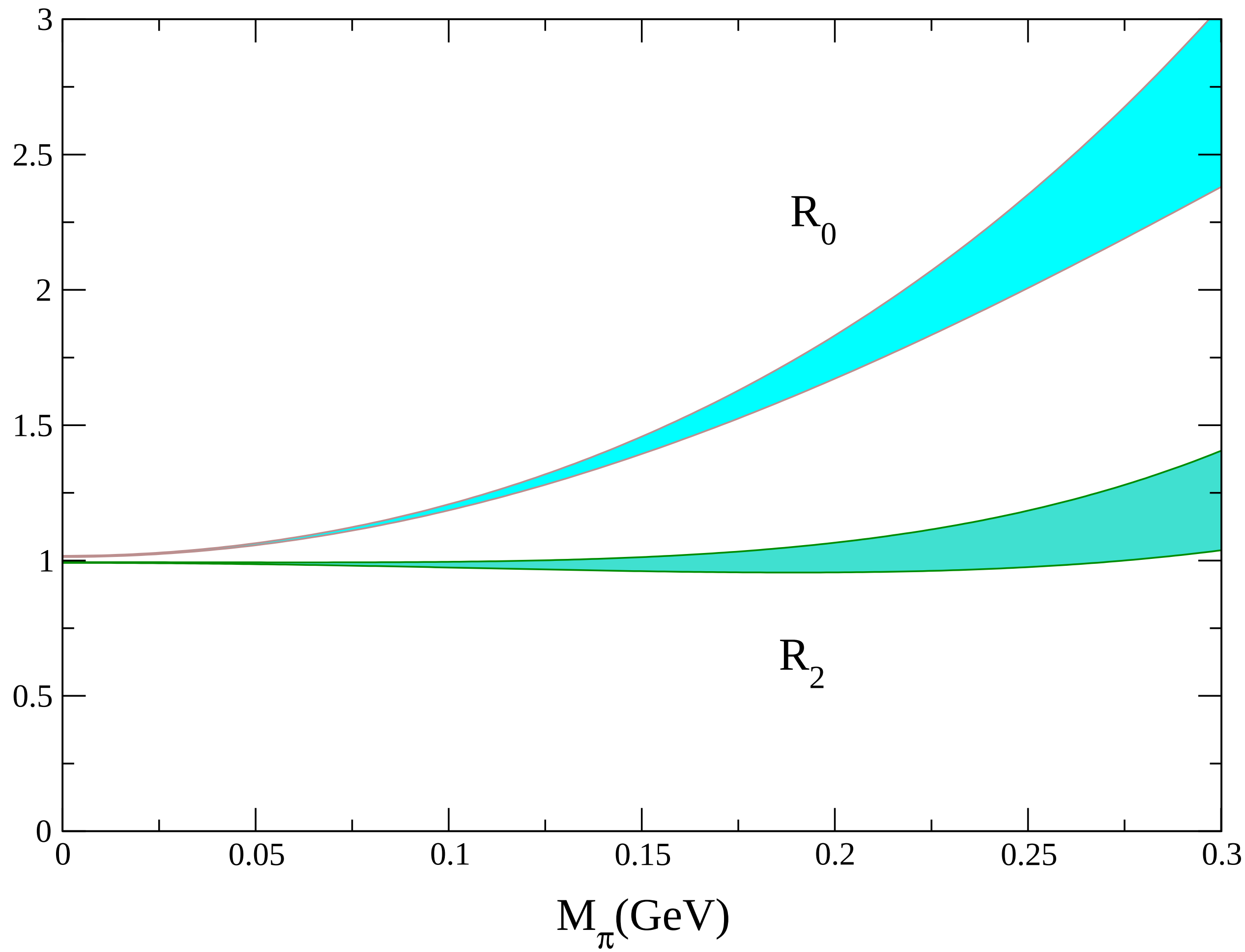
Summary

- For large volumes ($2LF_\pi \gg 1$), finite–volume effects can be calculated analytically within CHPT
- several one-loop CHPT calculations in the p–regime ($M_\pi L \gg 1$) have appeared in the recent literature
- the combined use of CHPT and asymptotic formulae *à la* Lüscher offers the most efficient way to get to higher orders in CHPT
- I have presented numerical evaluations of these finite volume corrections for all pseudoscalar masses and decay constants, as well as for the nucleon (after the talk of T.Hemmert 10 days ago)
- the extrapolation $L \to \infty$ can be made analytically!

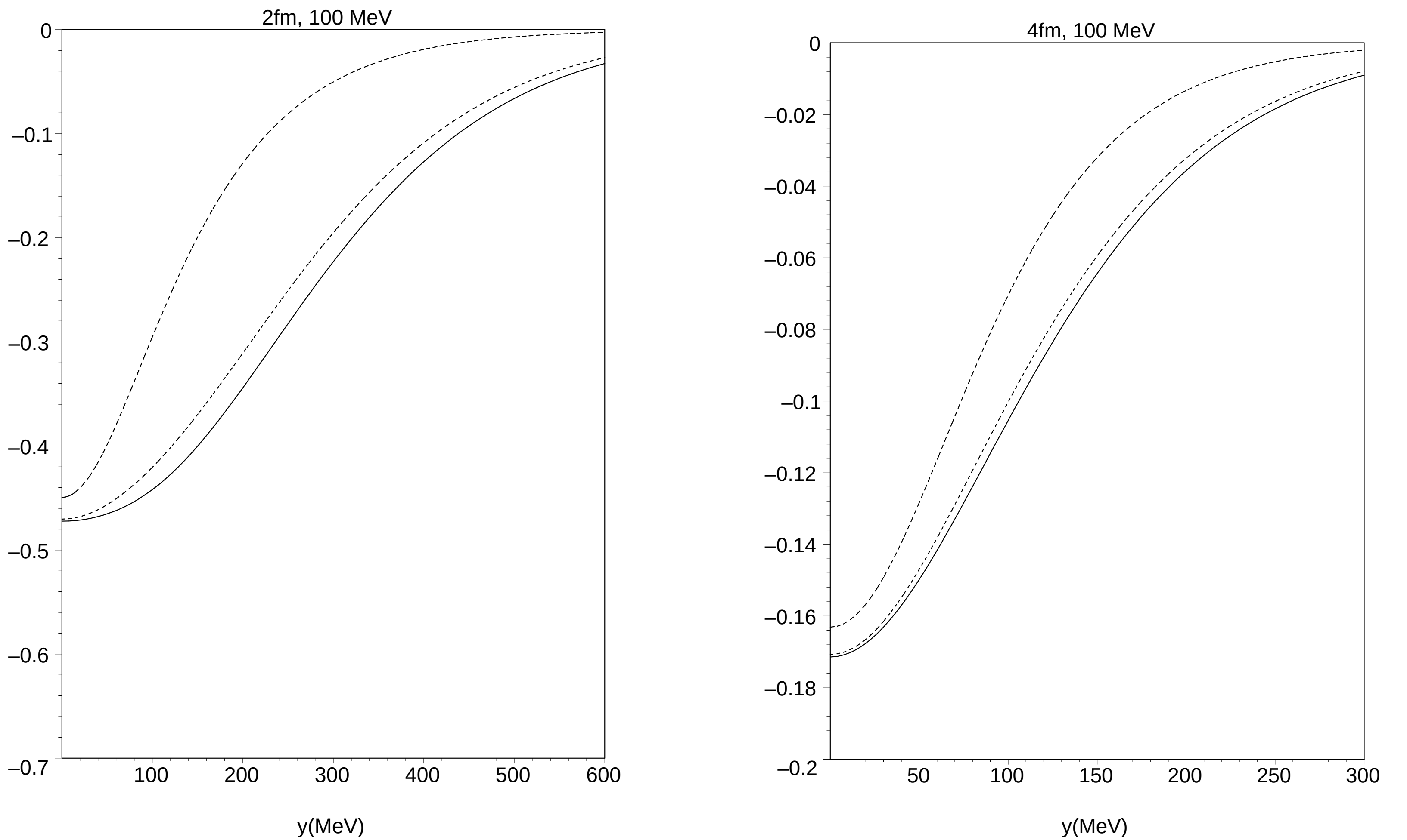
A remark/proposal

- When lattice calculation will become precise enough one will be able to use these effects to get information on the infinite-volume amplitudes
- From the corrections to the pion mass, e.g. one will be able to extract interesting information on the $I = 0$ amplitude in an unphysical region – or, in other words, on the relevant low-energy constants

The scattering lengths



Integrands for the pion mass



Integrands for the pion mass