Lattice Calculation of $K \to \pi\pi$ with On-shell Pions

Changhoan Kim
Norman H. Christ

Columbia University

RBC Collaboration

OUTLINE

- **Circumvent the Maini-Testa theorem by imposing anti-periodic boundary conditions on the final-state pions.**

- **Use Lellouch-Lüscher to relate the finite volume matrix elements with those at infinite volume.**

- **Preliminary $I = 2$ phase shifts.**

- **Status of $\Delta I = 3/2$ K-decay calculation.**

G-parity Boundary Conditions

- **G-parity operation on the pion:**

$$G|\pi^{\pm}> \; = \; -|\pi^{\pm}>$$
$$G|\pi^{0}> \; = \; -|\pi^{0}>$$

- **G-parity operation on the quark fields:**

$$G\begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} -\mathbf{d}^{C} \\ \mathbf{u}^{C} \end{pmatrix}$$

- **Must impose charge-conjugate boundary conditions on the gauge field to preserve gauge invariance.**
- **G-parity commutes with isospin but not with the chiral generators.**

H-parity Boundary Conditions

- **H-parity operation on the quark fields (definition):**

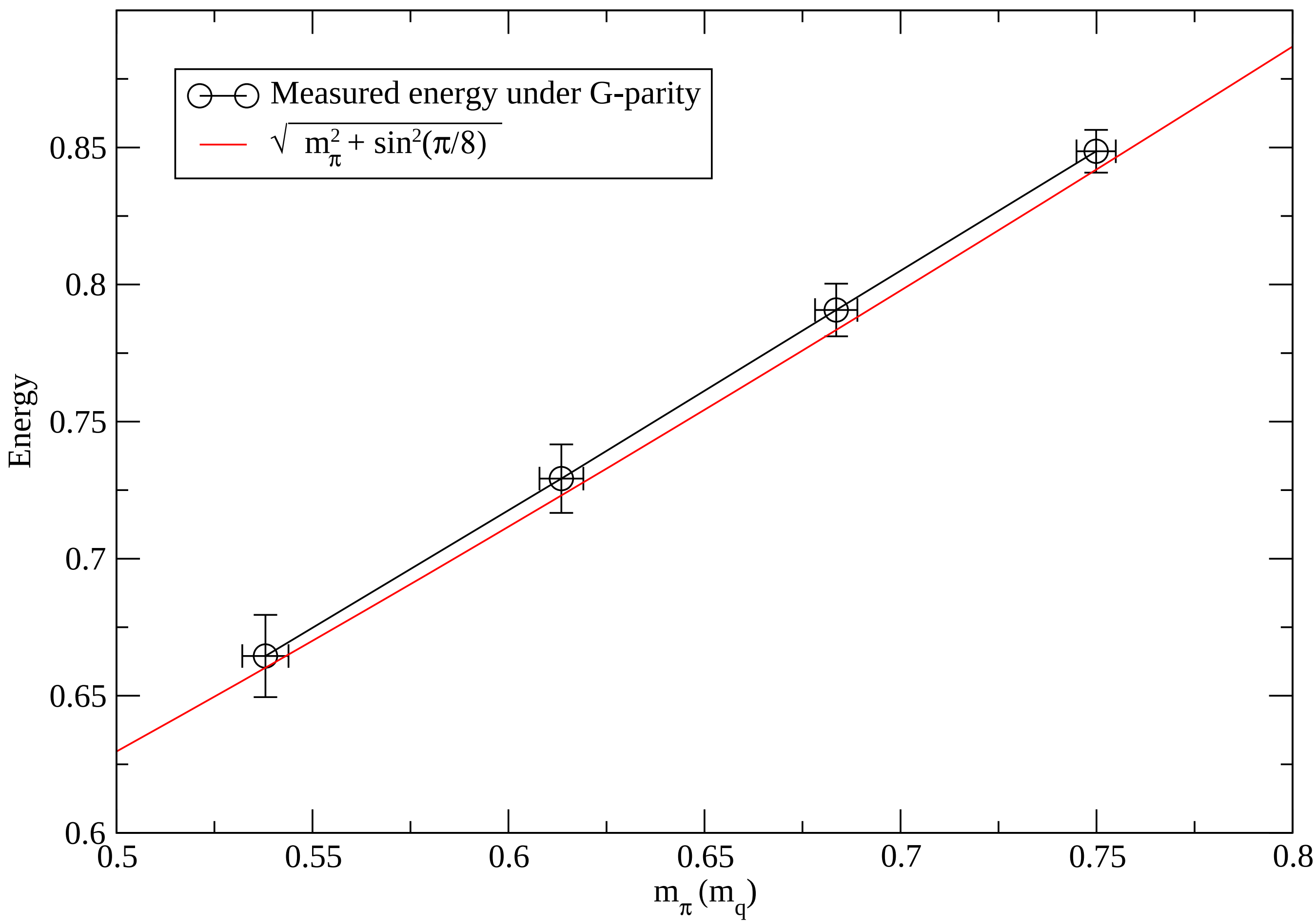
$$H \begin{pmatrix} \mathbf{u} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} -\mathbf{u} \\ \mathbf{d} \end{pmatrix}$$

- **H-parity operation on the pion:**

$$H|\pi^{\pm}> \, = \, -|\pi^{\pm}>$$
$$H|\pi^{0}> \, = \, +|\pi^{0}>$$

- **$\mathbf{I_z = 2}$, $\pi^+\pi^+$ state contains anti-symmetric pions with non-zero momenta.**
- **Not true for the $\mathbf{I = 0}$ $\pi\pi$ state.**
- **No modification of the usual gauge configurations is required— an advantage.**

Single Pion Result



$1/a$ = 0.978(14)GeV, β =5.7, $8^3 \times 32$, L_s=10

Wilson gauge action, domain wall fermions

H-parity boundary conditions

Two-pion states and the $I = 2$ phase shift

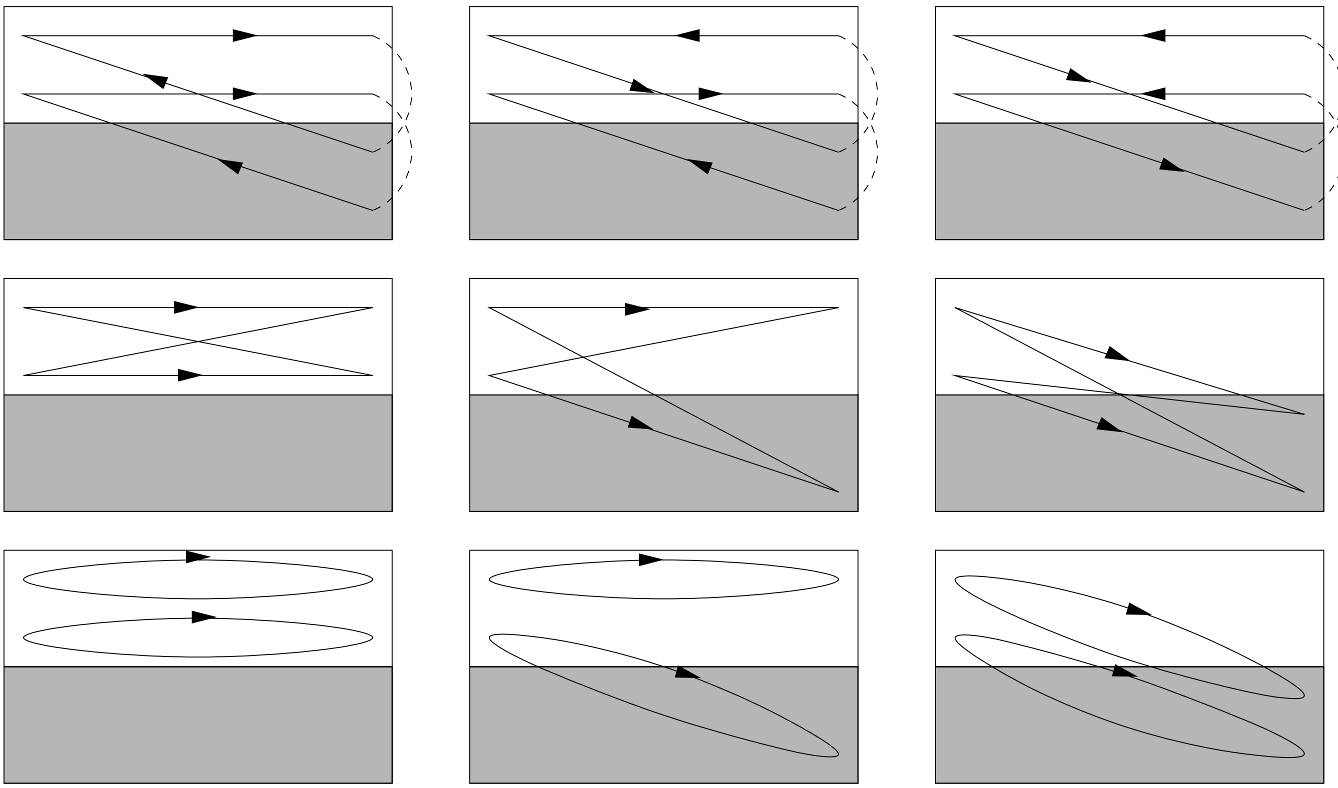
- **Before studying $K \to \pi\pi$ decays we examine the $I = 2$ phase shift.**

- **We use Lüscher's method to extract phase shift from the energy levels in a finite box.**

$$n\pi - \delta_0(k) \; = \; \phi(q) \qquad q \equiv \frac{kL}{2\pi}$$

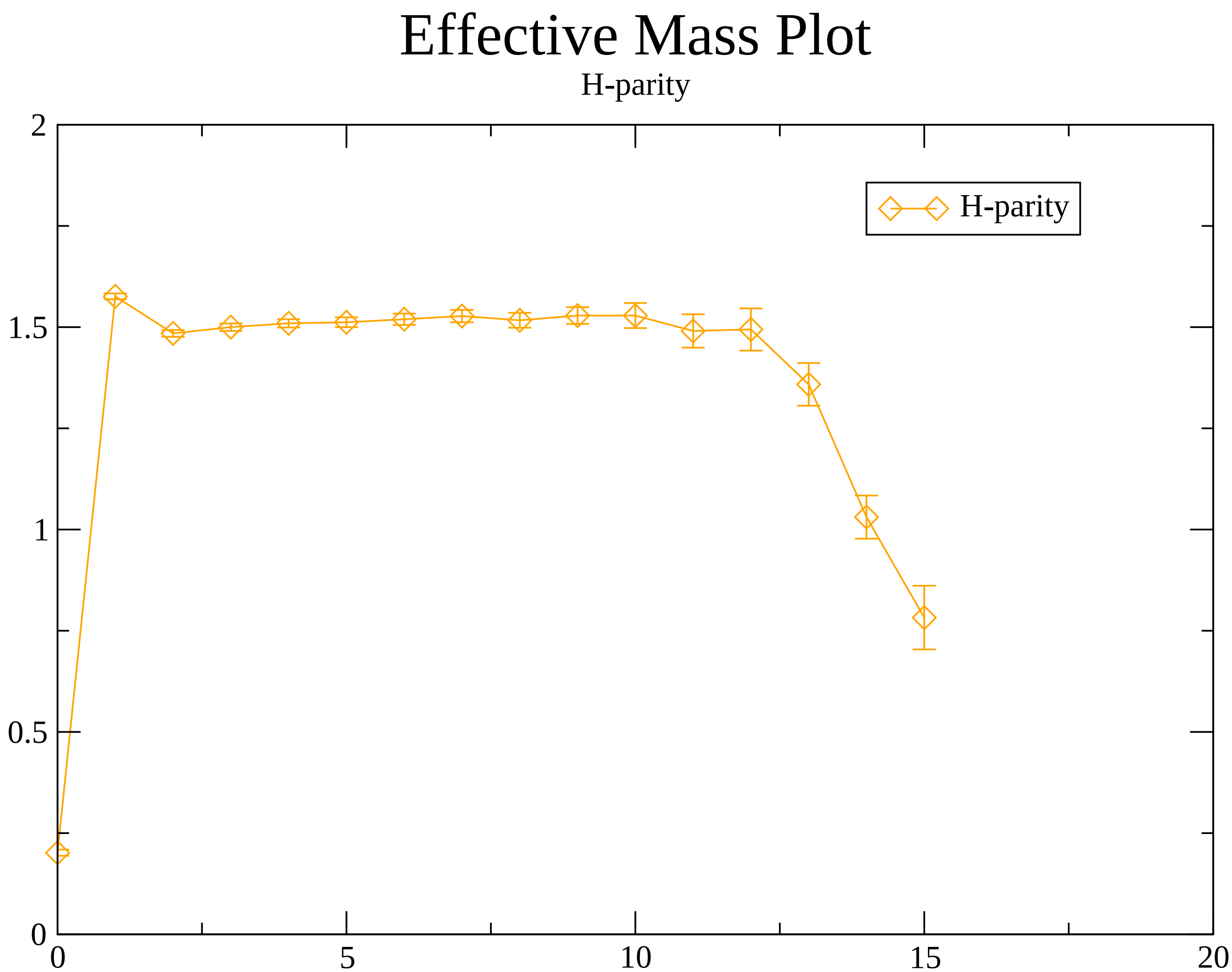
- **The new boundary conditions modify only the functional form of $\phi(q)$.**

Some of the diagrams entering the $\pi\pi - \pi\pi$ propagator



Open: **usual gauge links** $U_\mu(x)$

Shaded: **charge-conjugate gauge links** $U_\mu(x)^*$

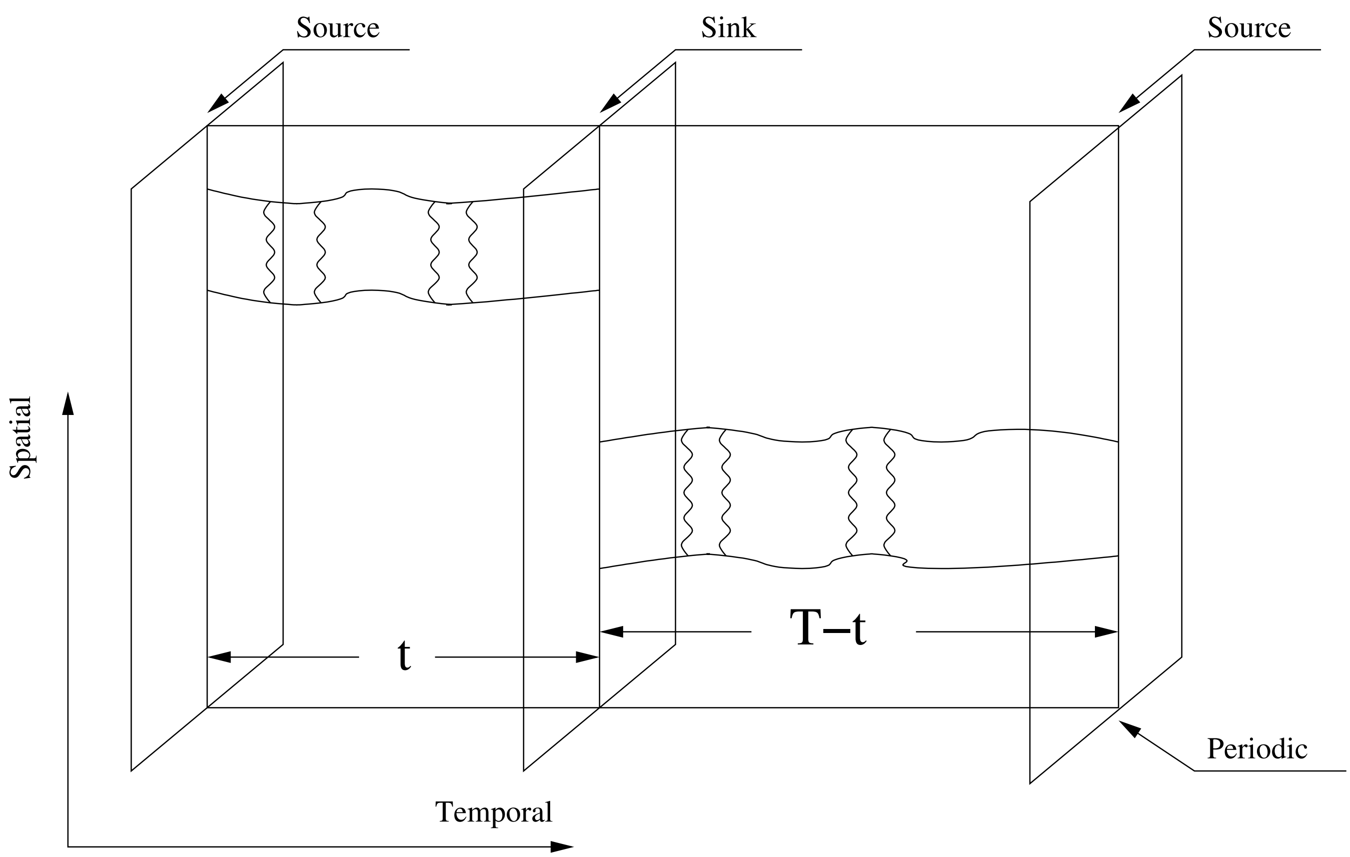


$1/a =$ **0.978(14)GeV,** β =**5.7,** $8^3 \times 32$, L_s=**10**

Wilson gauge action, domain wall fermions

H-parity boundary conditions

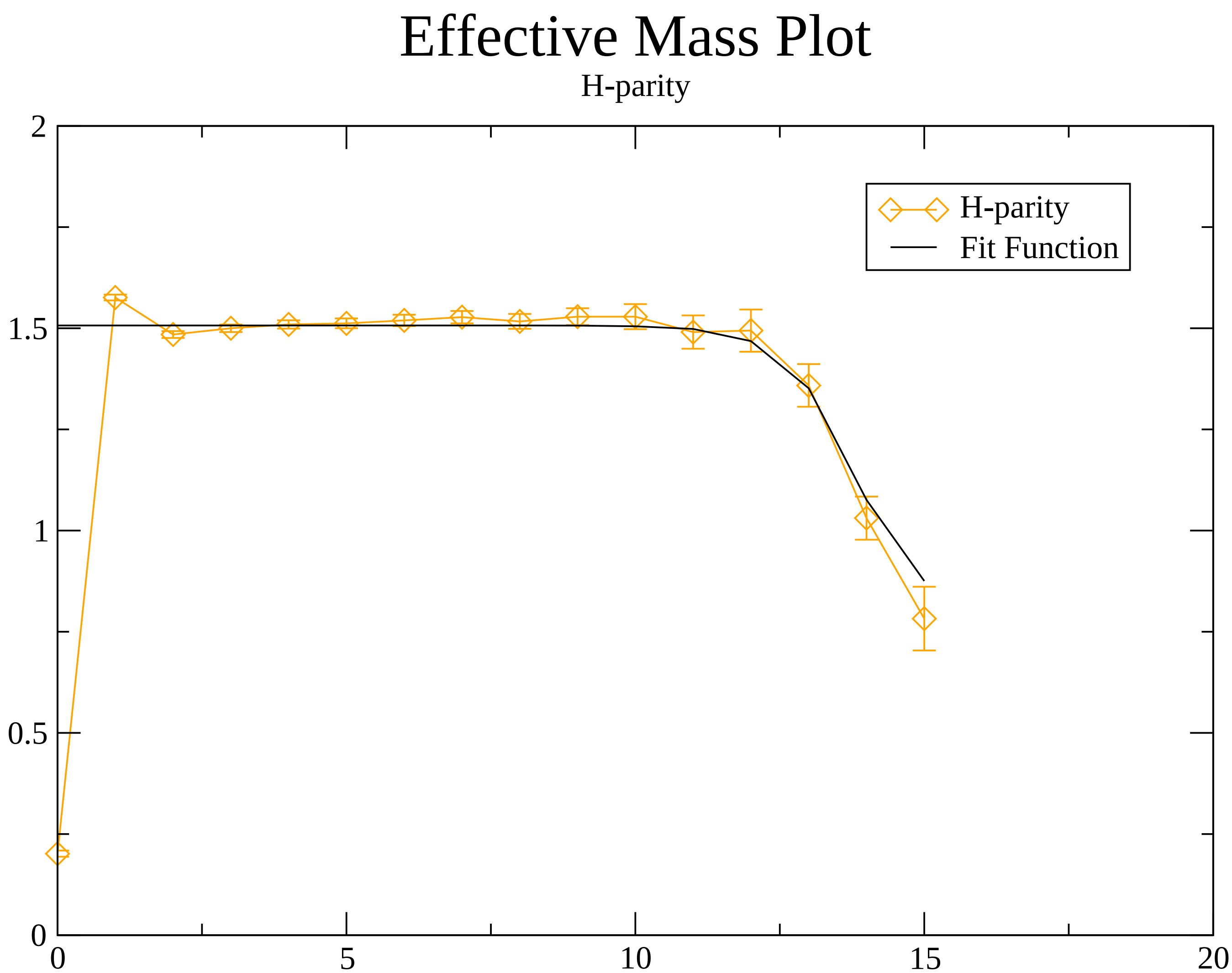
Explanation of constant term



A single pion propagates for all time:

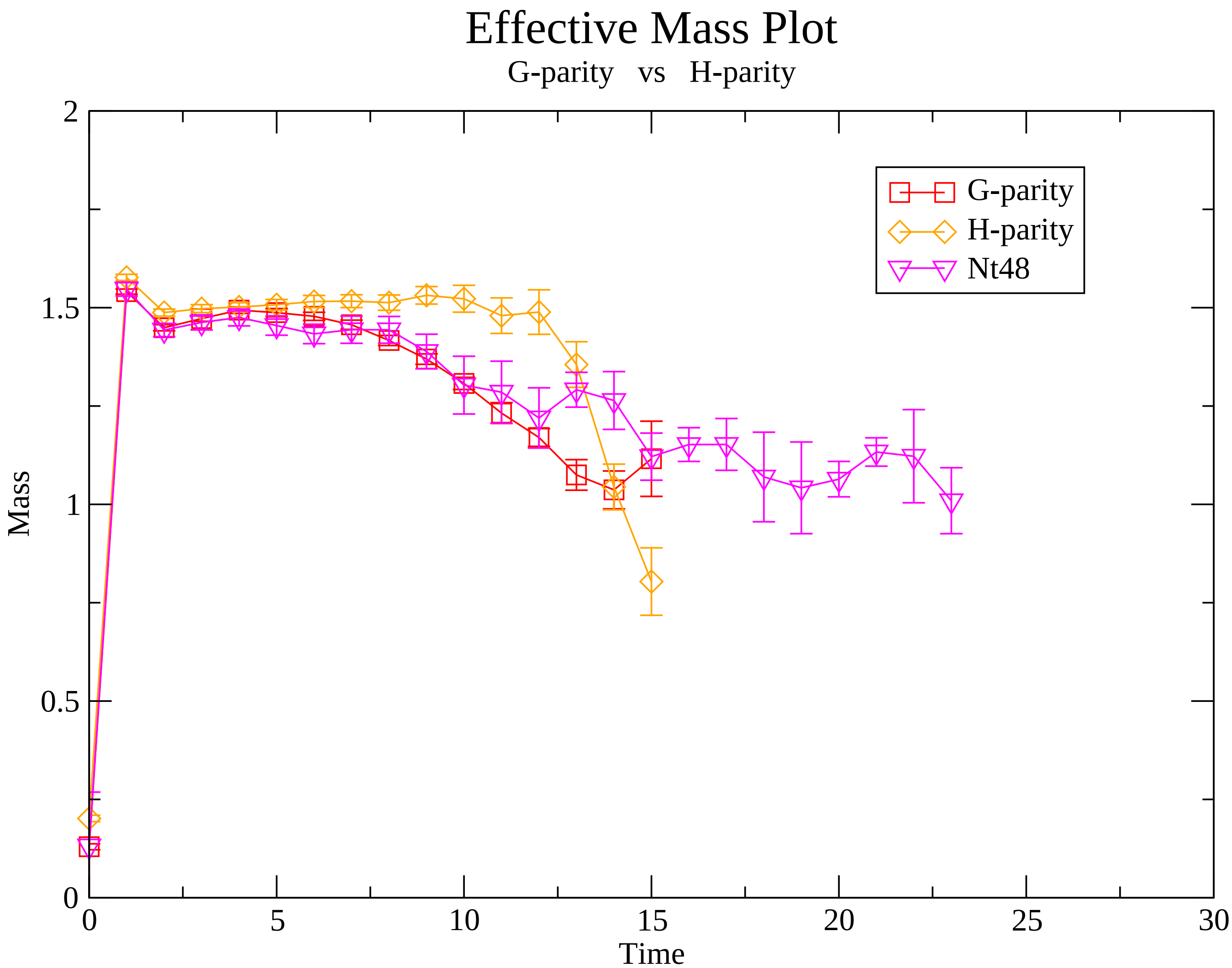
$$G(t) \approx e^{-m_\pi (T-t)} e^{-m_\pi t} = e^{-m_\pi T} = e^{-2m_\pi \frac{T}{2}}$$

Fit including a constant



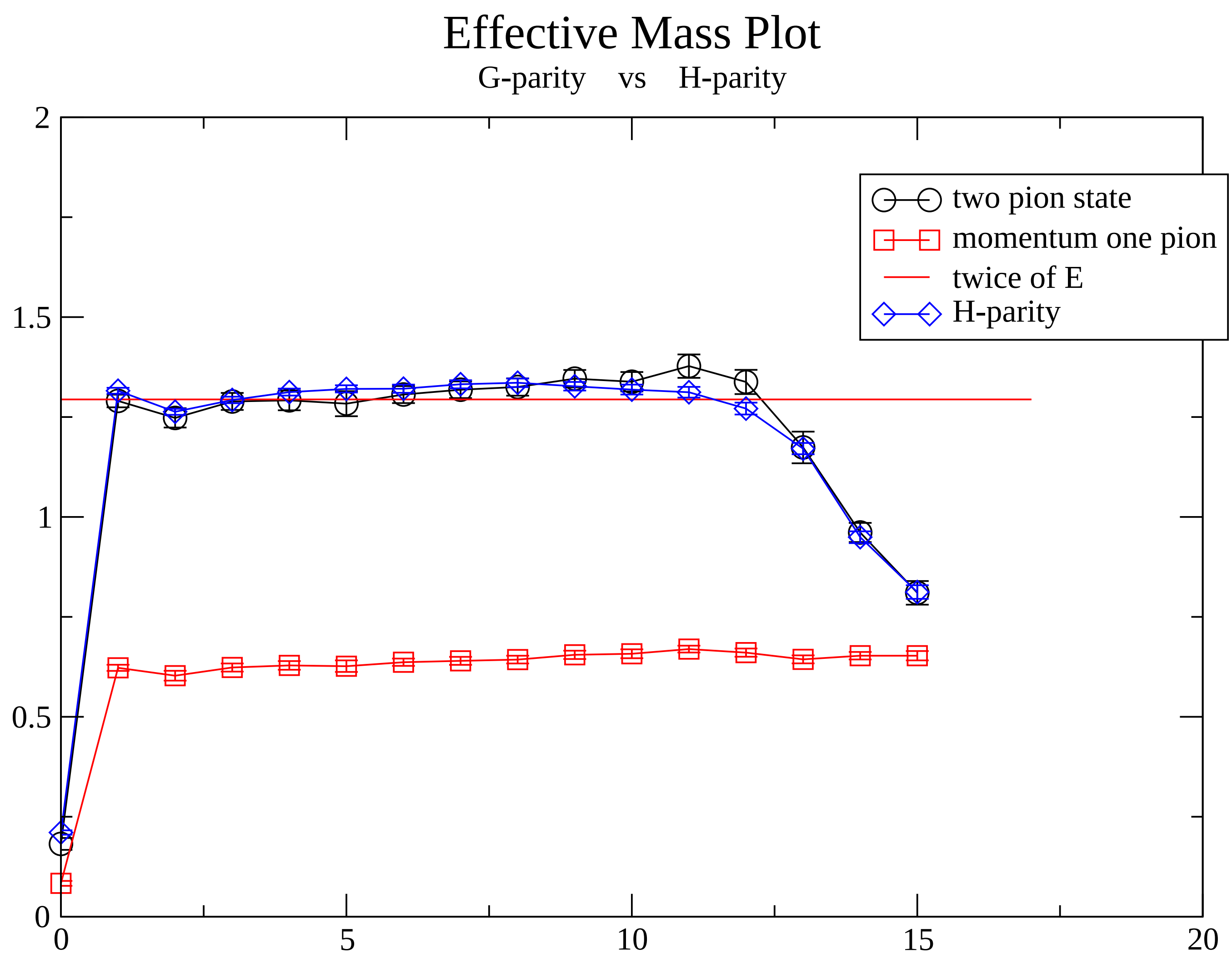
$1/a =$ **0.978(14)GeV,** β =**5.7,** $8^3 \times 32$, L_s=**10**
Wilson gauge action, domain wall fermions
H-parity boundary conditions

G-parity boundary conditions



$1/a =$ 0.978(14)GeV, β =5.7, $8^3 \times 32$ and $\times 48$
L_s=10, Wilson gauge action, domain wall fermions, H-parity boundary conditions

Examine a larger $8^2 \times 16 \times 32$ volume



$1/a =$ **0.978(14)GeV**, $\beta =$**5.7**

$8^2 \times 16 \times 32$ **and** $\times 48$, L_s**=10**

Wilson gauge action, domain wall fermions

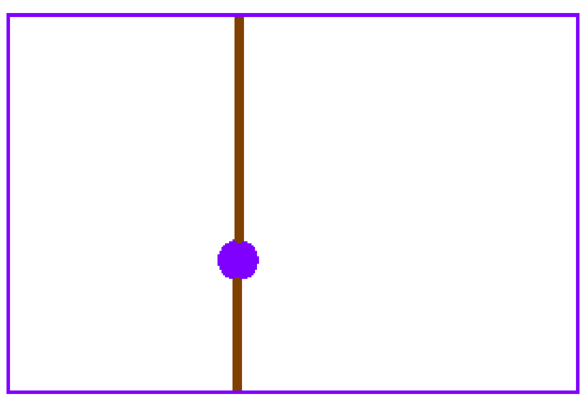
G- and H-parity boundary conditions

Finite-volume sensitivity of G-parity

- G-parity allows color flux tube going from q to q as well as q to $\overline{q}$ if it passes through the boundary.

- Additional interactions between quarks and their finite volume images.

- A single quark can propagate bound to its image with energy increasing linearly with the size of the z direction:

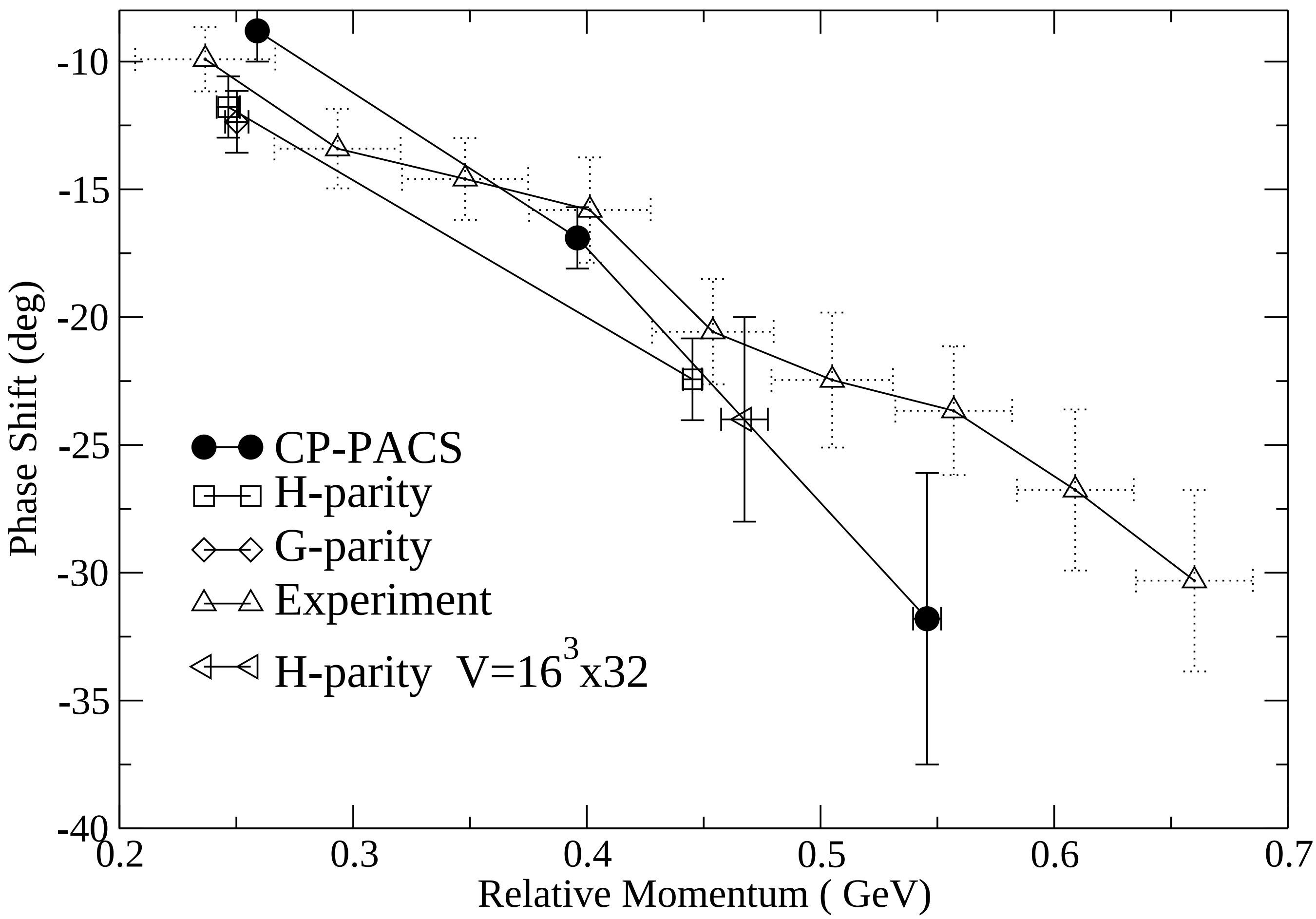


$I = 2$ $\pi\pi$ phase shift results (domain wall fermions)

	Vol	$1/a$(GeV)	#conf's.
p = 250MeV			
G-parity	$8^2 \times 16 \times 32$	**0.978(14)**	**91**
H-parity	$8^2 \times 16 \times 32$	**0.978(14)**	**172**
p = 450MeV			
H-parity	$8^3 \times 32$	**0.978(14)**	**270**
H-parity	$16^3 \times 32$	**1.98(3)**	**80**

- **Running parameters :** L_s =**10** , M_5 =**1.65**

Results for $\delta_{\pi\pi}^{I=2}$



$K \rightarrow (\pi\pi)_{I=2}$ Decay

- **Examine more recent results obtained over the past year.**

- **Choose more realistic parameters.**

- **Use the simpler H-parity boundary conditions.**

- **Examine matrix elements of the three $\Delta I = 3/2$ operators between $|K\rangle$ and physical $|\pi(p)\pi(p)\rangle$ states.**

- **Adjust m_K to achieve $m_K = E_{\pi\pi}$.**

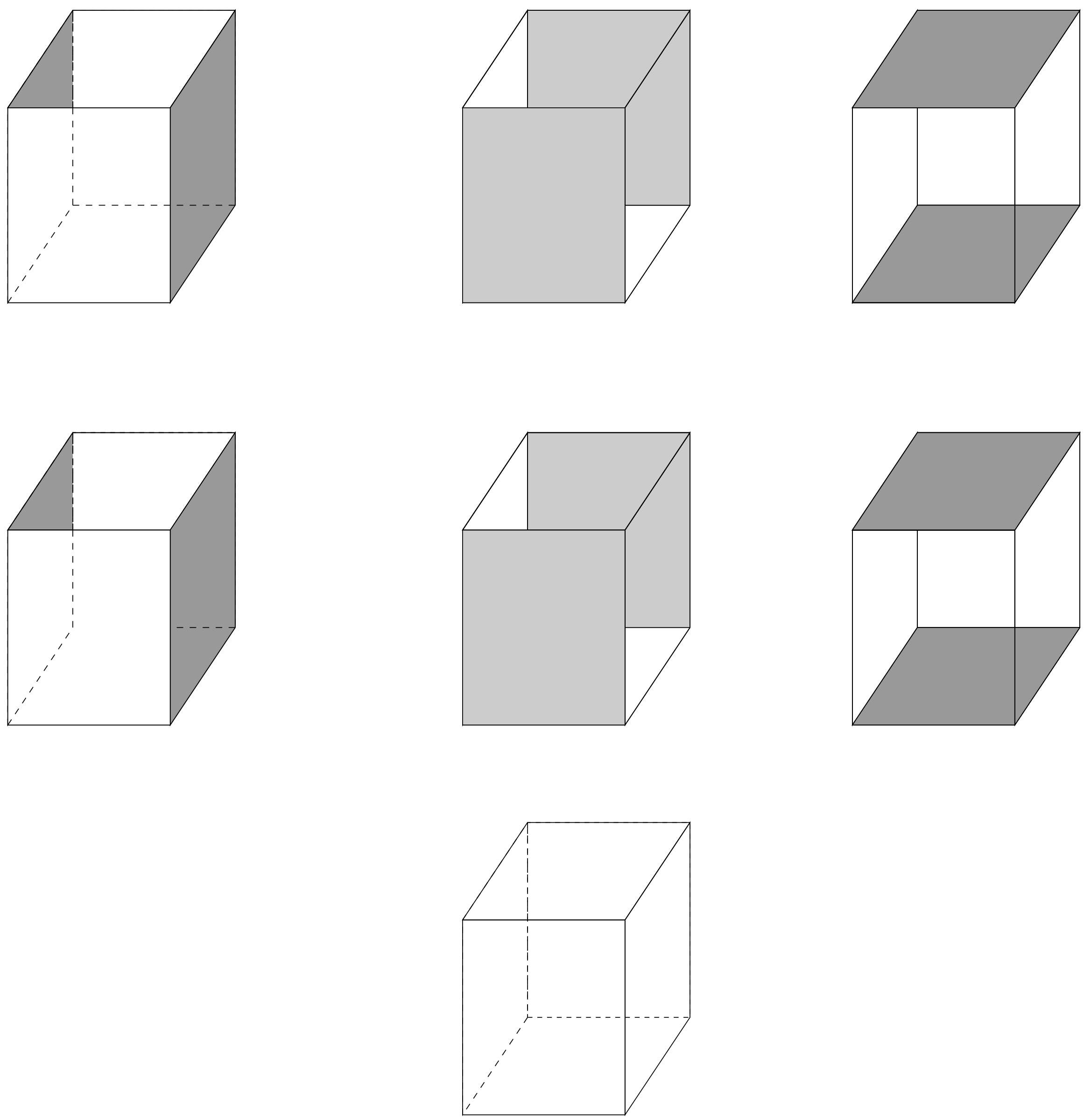
- **Show the character of data and errors — physically normalized results should be available in a few weeks.**

Simulation Parameters

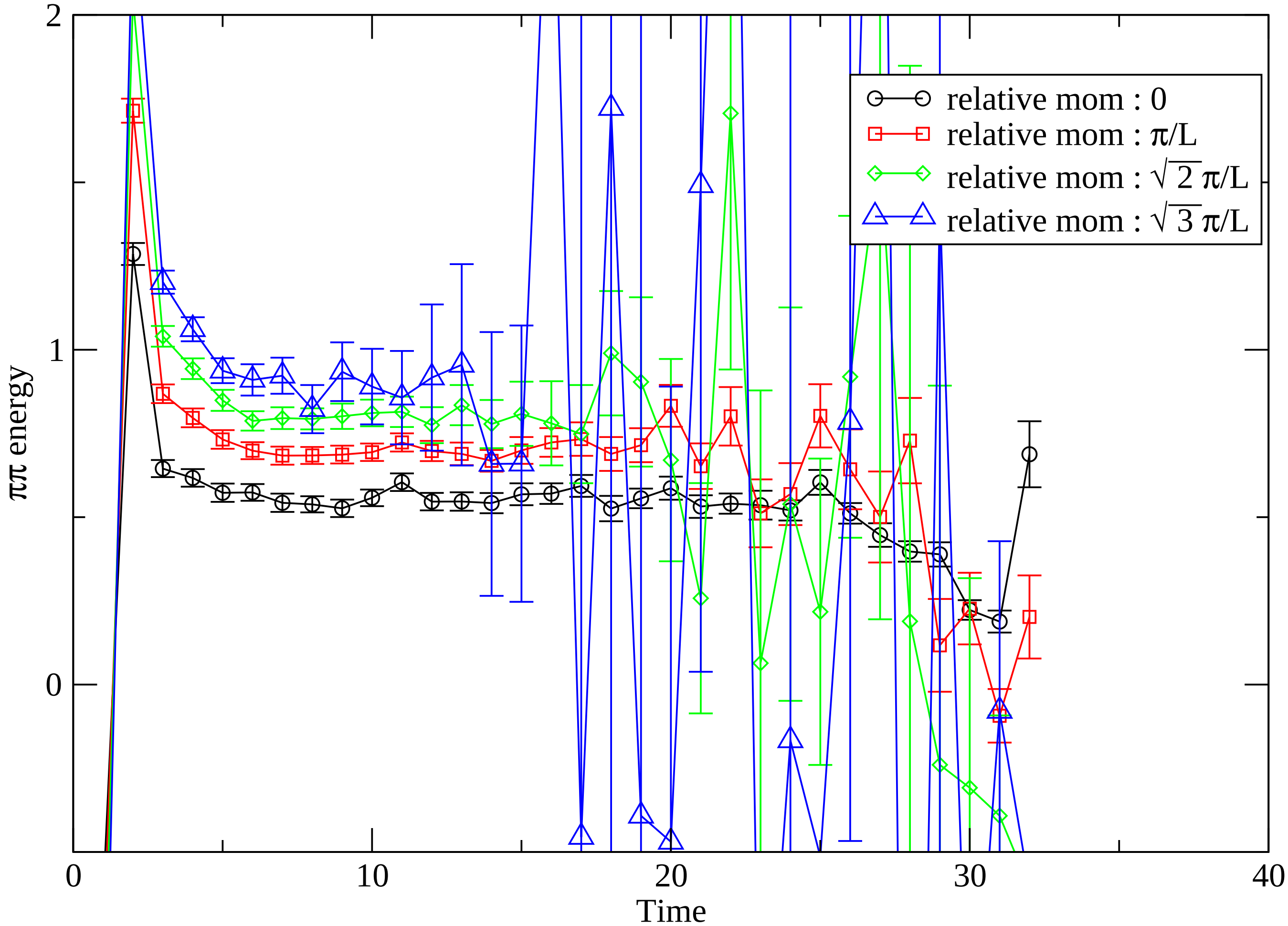
- **Lattice size:** $16^3 \times 32$
- **Pion mass: 352**MeV
- **Kaon mass: 712**MeV **- 1.29**GeV
- **Lattice spacing:** a^{-1}**=1.3**GeV
- **Action: DBW2**
- **Number of Configurations: 120**
- **Domain Wall Fermions:** M_5**=1.8,** L_s**=12**
- **Resulting kinematics:**

	m_K	m_π	p_π
Simulation	**910 MeV**	**352 MeV**	**290 MeV**
Nature	**496 MeV**	**138 MeV**	**206 MeV**

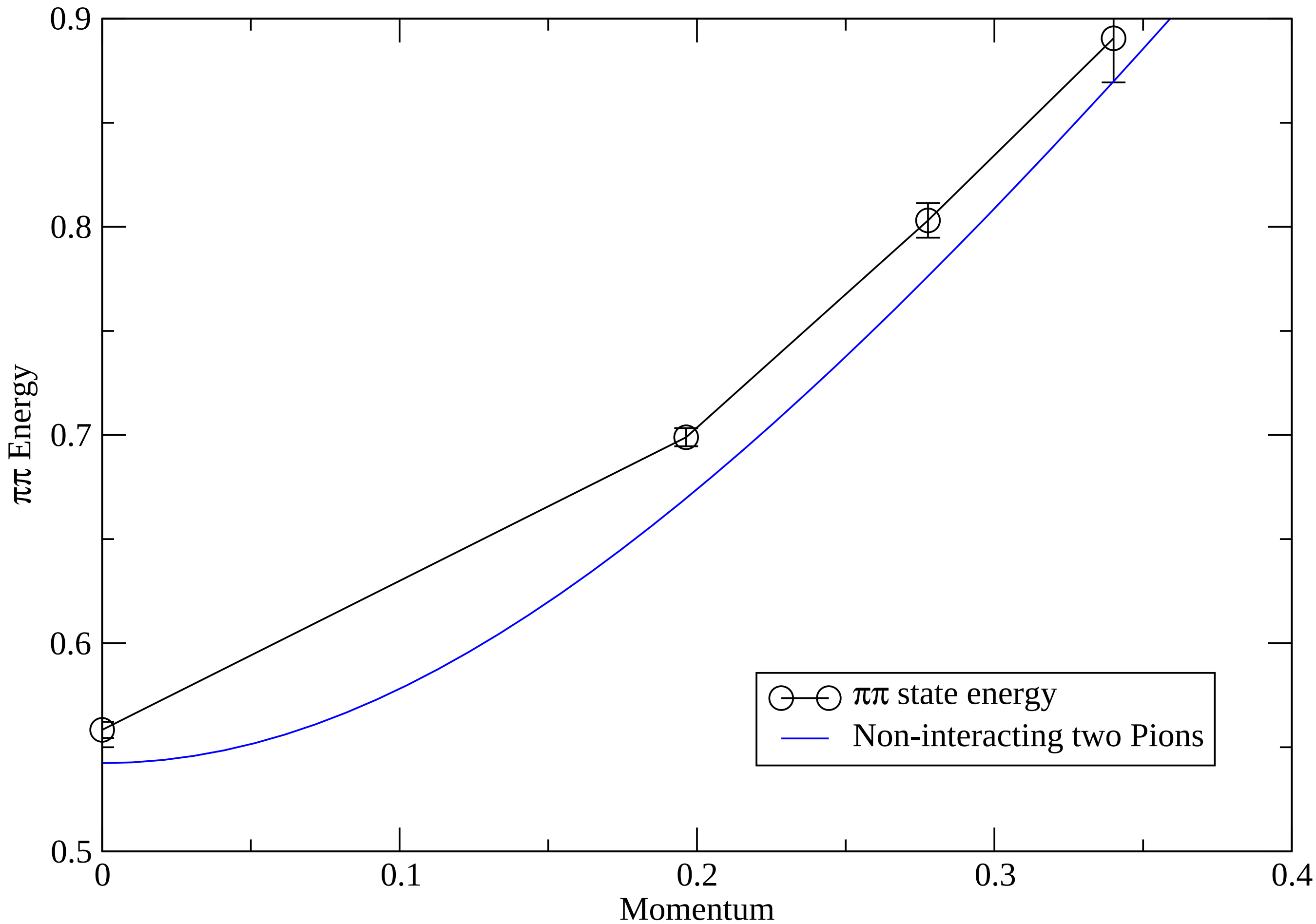
Obtaining three possible momenta



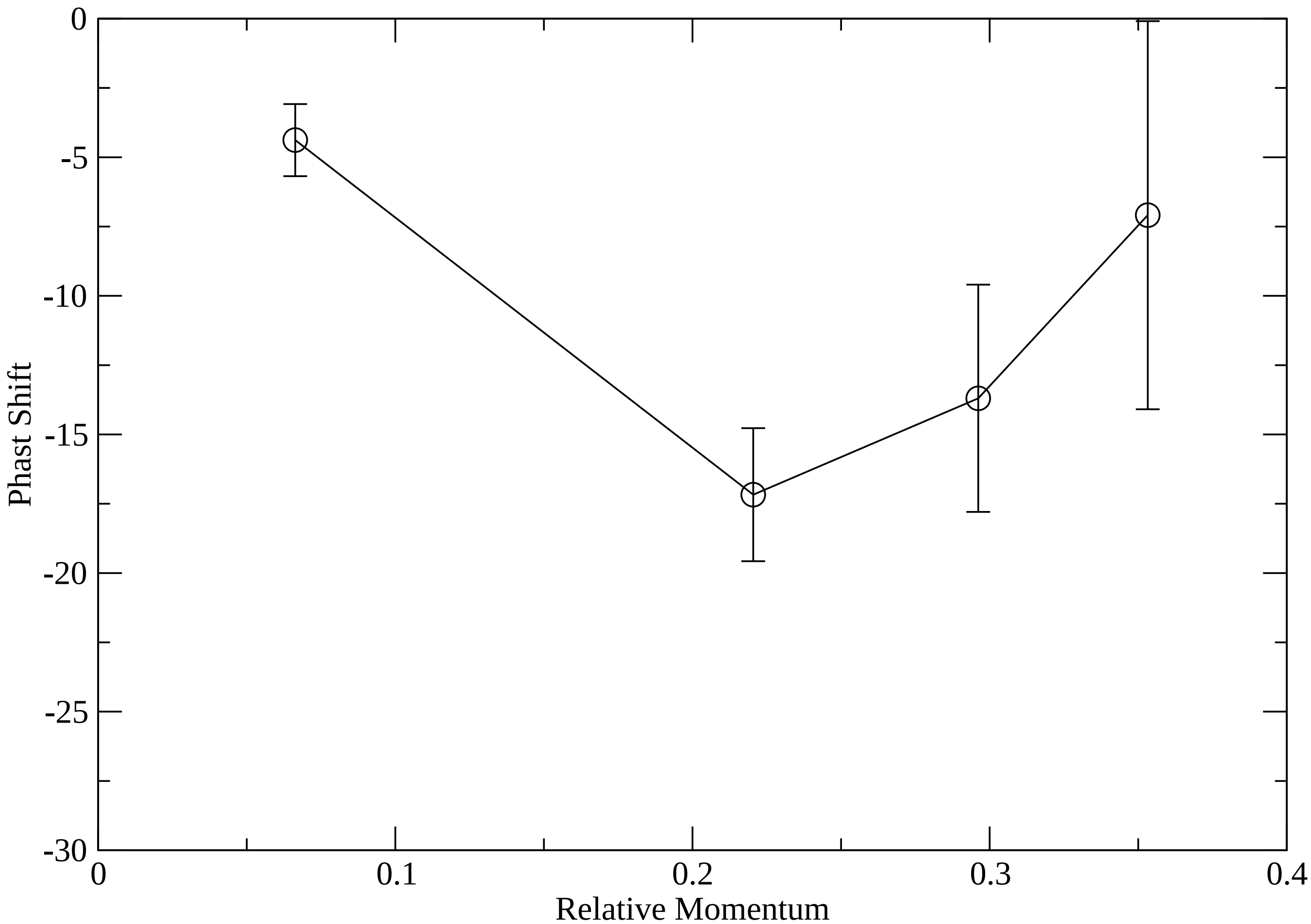
$\pi\pi$ effective mass



$\pi\pi$ energy


0.9
0.8
0.7
0.6
0.5
$\pi\pi$ Energy
0
0.1
0.2
0.3
0.4
Momentum
$\pi\pi$ state energy
Non-interacting two Pions

$\pi - \pi$ phase shifts



Operators with $\Delta I = 3/2$ and H-parity

- **Kaon isospin :** $I_z = 1/2$.

- $\pi\pi$ **state with relative momentum under** H-**parity :** $I_z = 2$.

- **No terms in the effective Hamiltonian have** $\Delta I_z = 3/2$.

- **Use Wigner-Eckart theorem:**

$$\langle K|O^{I=3/2}_{I_z=1/2}|\pi\pi\rangle = \frac{\langle 2, \frac{3}{2}; 1, \frac{1}{2}|2, \frac{3}{2}; \frac{1}{2}, \frac{1}{2}\rangle}{\langle 2, \frac{3}{2}; 2, \frac{3}{2}|2, \frac{3}{2}; \frac{1}{2}, \frac{1}{2}\rangle}\langle K|O^{I=3/2}_{I_z=3/2}|\pi\pi\rangle.$$

Normalized matrix elements of lattice operators

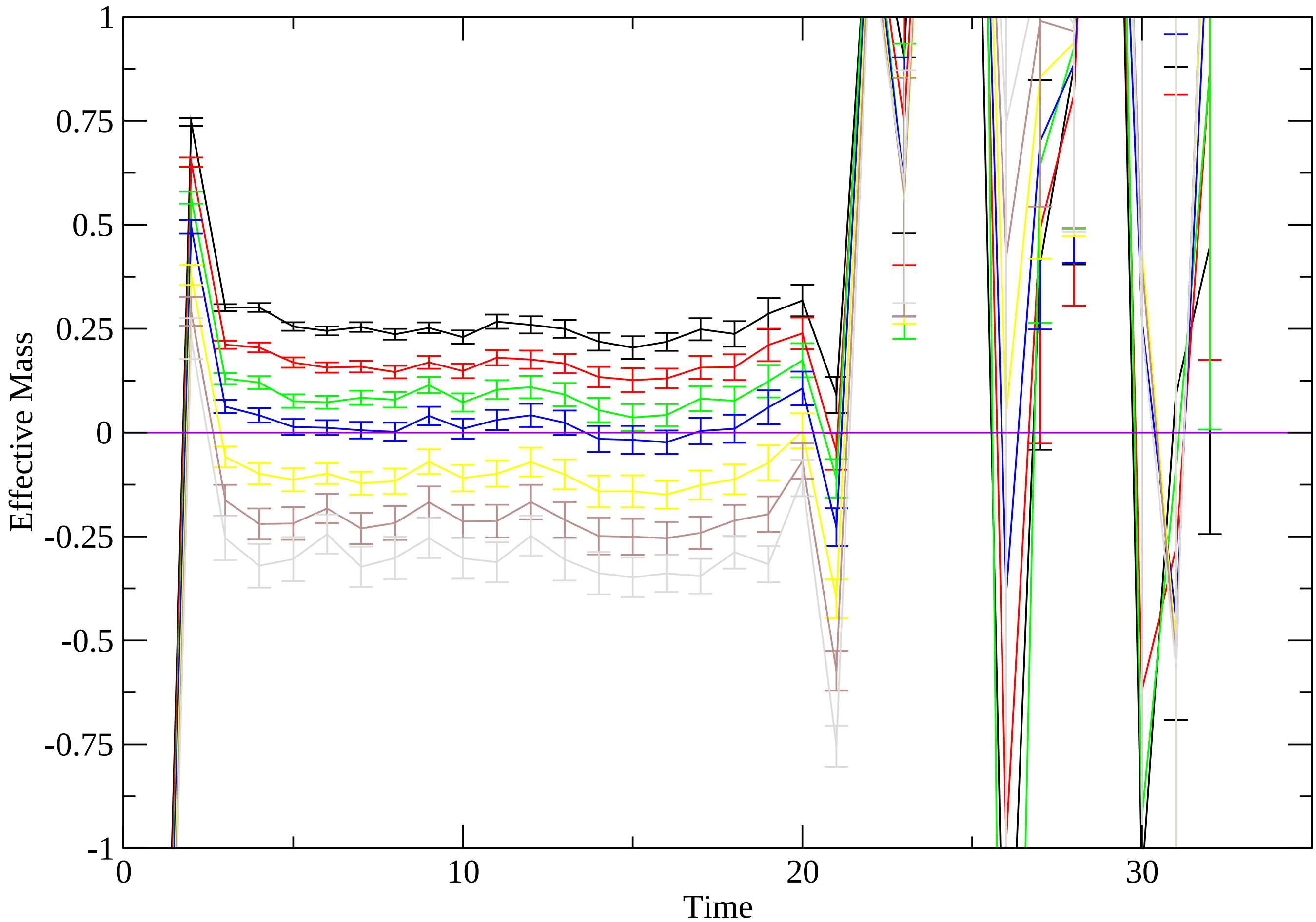
Evaluate three Greens functions in the usual way:

$$\lim_{t_{\pi\pi} \gg t \gg t_K} G_{(\pi\pi)\,O\,K}(t) \;\longrightarrow\; \langle 0|P_{\pi\pi}|\pi^+\pi^+\rangle\langle\pi^+\pi^+|O|K^+\rangle\langle K^+|P_K|0\rangle \, e^{-m_K(t-t_K)}e^{-E_{\pi\pi}(t_{\pi\pi}-t)}$$

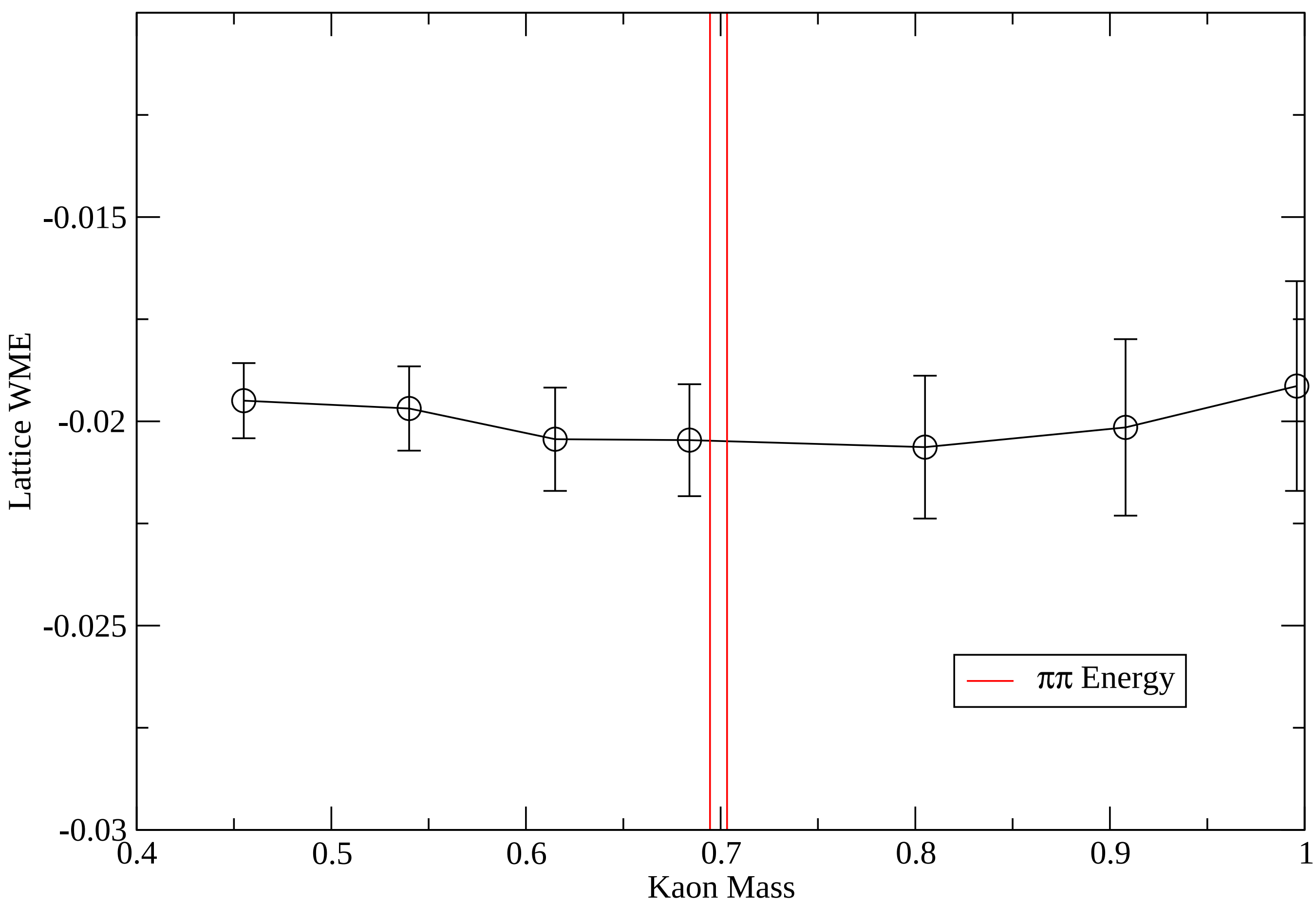
$$\lim_{t_{\pi\pi} \gg 0} G_{(\pi\pi)\,(\pi\pi)}(t_{\pi\pi}) \;\longrightarrow\; \langle 0|P_{\pi\pi}|\pi^+\pi^+\rangle\langle\pi^+\pi^+|P_{\pi\pi}|0\rangle e^{-E_{\pi\pi}(t_{\pi\pi})}$$

$$\lim_{t_K > 0} G_{K\,K}(t_K) \;\longrightarrow\; \langle K^+|P_K|0\rangle\langle 0|P_K|K^+\rangle e^{-m_K(t_K)}$$

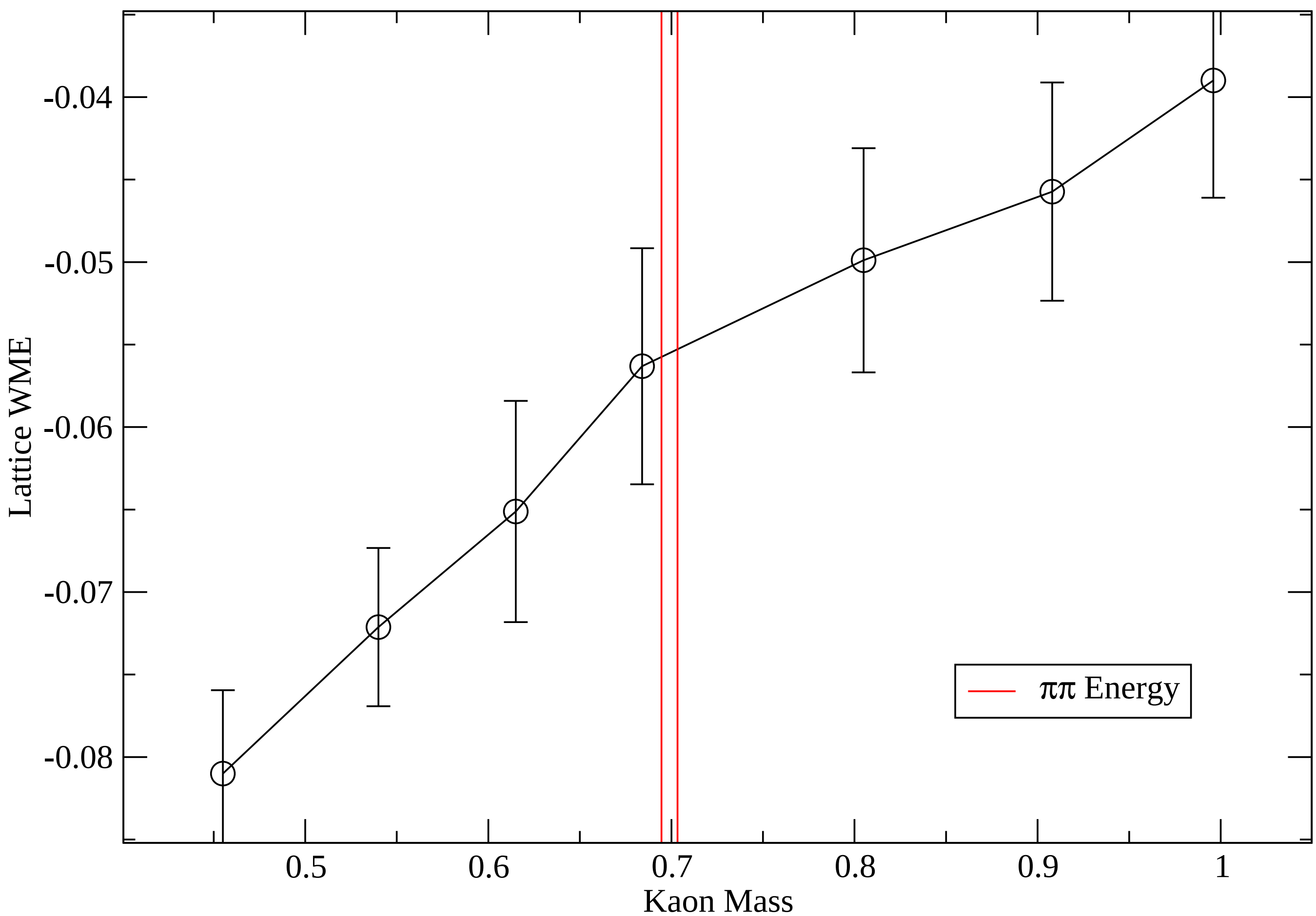
Effective mass difference from $G_{\pi\pi O27K}$



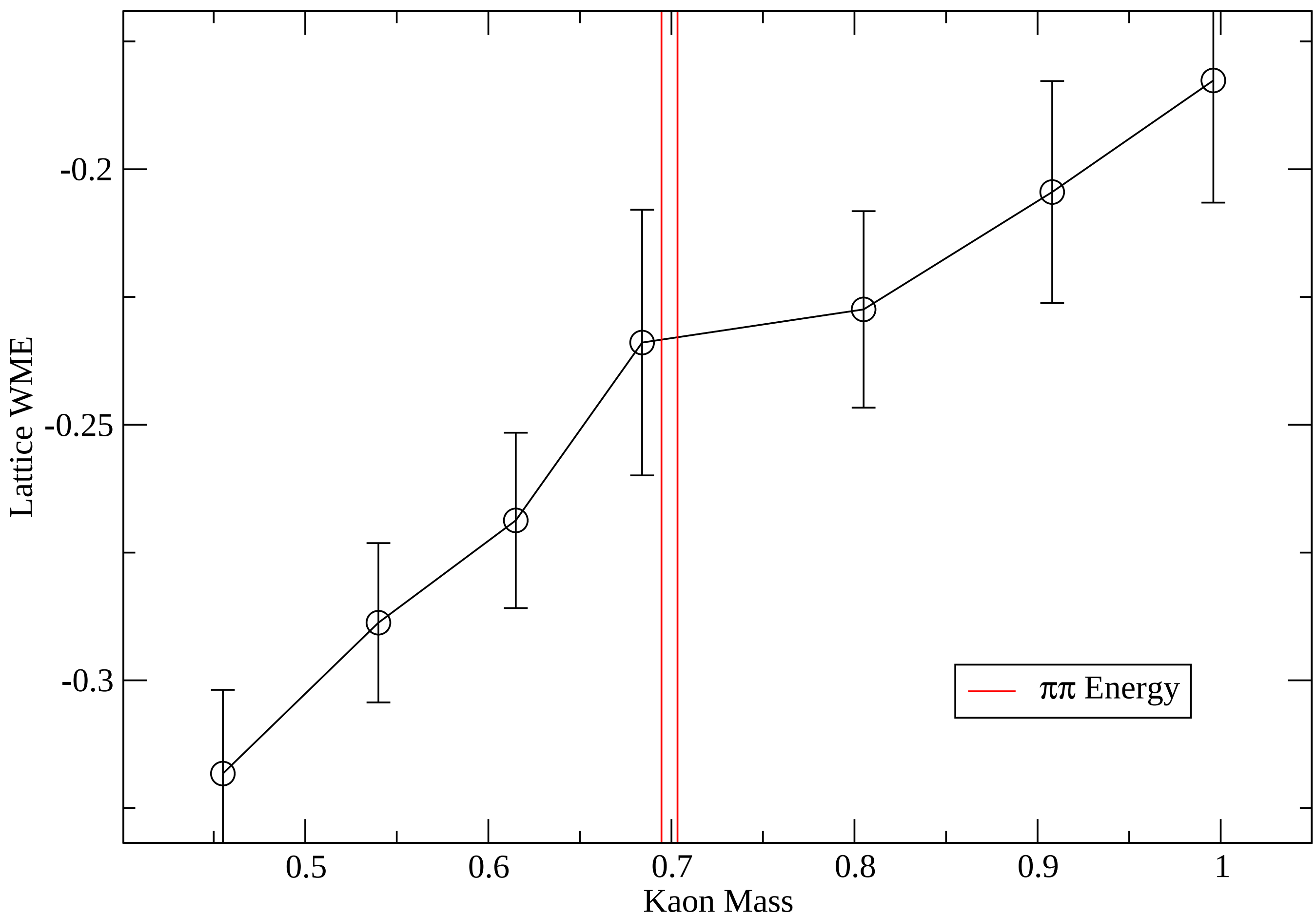
O^{27} matrix element versus Kaon mass



$O^{(8,8)}$ matrix element versus Kaon mass



$O_m^{(8,8)}$ matrix element versus Kaon mass



Preliminary lattice matrix elements

	O^{27}	$O^{(8,8)}$	$O_m^{(8,8)}$
0	**-8.379e-3(4.34e-4)**	**-5.267e-2(3.43e-3)**	**-1.928e-1(1.19e-2)**
1	**-2.048e-2(1.42e-3)**	**-5.553e-2(7.11e-3)**	**-2.331e-1(2.52e-2)**
2	**-2.299e-2(3.35e-3)**	**-4.696e-2(2.09e-2)**	**-1.687e-1(1.00e-1)**
3	**-1.772e-2(5.16e-3)**	**-3.546e-2(3.01e-2)**	**-2.321e-1(8.28e-2)**

Remaining steps

- **Apply Lellouch-Lüscher finite-volume correction (done).**
- **Compute renormalization matrix for $1/a = 1.3$GeV case (done).**
- **Evaluate needed Wilson coefficients (underway).**
- **Extract physically normalized matrix elements (soon).**

Conclusion and Outlook

- **Calculation of $\Delta I = 3/2$ amplitudes is practical with an on-shell π-π final state. For physical parameters:**
  - $1/a = 2$ **GeV,** $a = 0.1$ **fm.**
  - $L = 64$**,** ≈ 6 **fm,** $\approx 4/m_\pi$**.**
  - **Impose anti-periodic conditions on each face.**

- **Calculation of $\Delta I = 1/2$ amplitudes is possible using G-parity boundary conditions.**
  1. **Quenched calculations are not possible because zero-momentum, η'-η' states will dominate.**
  2. **Charge conjugation of the gauge fields on the boundary requires special configurations.**
  3. **Decay to the vacuum is allowed and must be subtracted.**

- **Using a K-meson with $\vec{p} \neq 0$ (Rummukainen-Gottlieb) would address 2. and 3. above.**