Chiral Perturbation Theory at non-zero lattice spacing

Oliver Baer
University of Tsukuba

Matching Light Quarks to Hadrons
Workshop at the BENASQUE CENTER FOR SCIENCE
27 July 2004
# Introduction: Chiral extrapolation

Some numbers I heard at Lattice 2004:

<table>
<thead>
<tr>
<th>Collaboration</th>
<th>Fermion type</th>
<th>Nf</th>
<th>smallest $m_\pi/m_\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UKQCD</td>
<td>Wilson</td>
<td>2</td>
<td>~0.44</td>
</tr>
<tr>
<td>SPQcdR</td>
<td>Wilson</td>
<td>2</td>
<td>~0.66</td>
</tr>
<tr>
<td>qq+q</td>
<td>Wilson</td>
<td>2</td>
<td>~0.47</td>
</tr>
<tr>
<td>CP-PACS</td>
<td>Wilson</td>
<td>2</td>
<td>~0.35</td>
</tr>
<tr>
<td>CP-PACS</td>
<td>Wilson</td>
<td>2+1</td>
<td>~0.62</td>
</tr>
<tr>
<td>MILC</td>
<td>Staggered</td>
<td>2+1</td>
<td>~0.3</td>
</tr>
<tr>
<td>RBC</td>
<td>Domain Wall</td>
<td>2</td>
<td>~0.53</td>
</tr>
</tbody>
</table>

physical value: 0.18

Still needed: rather long extrapolation in the light quark mass
Chiral Perturbation Theory (ChPT)

Analytic guidance is required for the extrapolation
⇒ provided by ChPT

ChPT: Effective theory for low-energy QCD
⇒ quark mass dependence of observables

Example:

\[ m_\pi^2 = M_0^2 \left[ 1 + \frac{M_0^2}{16\pi^2 N_f f^2} \ln M_0^2 - \frac{8LM_0^2}{f^2} \right] \]

\[ M_0^2 = (m_u + m_d)B \quad f, B, L: \text{low-energy constants} \]

Question: Does ChPT describe the lattice data?

Smoking gun: curvature due to chiral logs

Weinberg, Gasser, Leutwyler

Panel discussion
Lattice 2002, Boston
A potential problem

ChPT formulae derived for $a = 0$ only!

\[ Lattice \text{ data} \quad \xrightarrow{a \rightarrow 0} \quad \text{Continuum} \]

Step 1: continuum limit

Step 2: chiral extrapolation

\[ m_q \rightarrow m_{q,\text{phys}} \]

Real world QCD
A potential problem

ChPT formulae derived for $a = 0$ only!

Lattice data $\xrightarrow{a \to 0}$ Continuum

Step 1: chiral extrapolation

....... $\xrightarrow{\text{continuum limit}}$ Real world QCD

Step 2: chiral extrapolation

$mq \to mq,\text{phys}$

What about the opposite order?

Desirable: ChPT for lattice theories at non-zero lattice spacing
ChPT at non-zero lattice spacing: Main strategy

Two-step matching to effective theories:

1. Lattice theory $\rightarrow$ Symanzik’s effective theory
   continuum theory making the $a$-dependence explicit

2. Symanzik’s effective theory $\rightarrow$ ChPT
   including the $a$-dependence

$\Rightarrow$ Chiral expressions for $m_\pi$, $f_\pi$ ... with explicit $a$-dependence
Symanzik's action for Lattice QCD with Wilson fermions

Locality and symmetries of the lattice theory

\[ S_{\text{eff}} = S_{\text{QCD}} + a c \int \overline{\psi} i \sigma_{\mu\nu} G_{\mu\nu} \psi + O(a^2) \]

- At \( O(a) \) only one additional operator (making use of EOM)
- \( c \) : unknown coefficient ("low-energy constant")
- \( O(a^2) \) : dim-6 operators: - fermion bilinears - 4-fermion operators

\( O(4) \) rotation invariance is broken at this order

Sheikholeslami, Wohlert
Reminder: Chiral Lagrangian

Fields: \[ \Sigma(x) = \exp \left( \frac{2i}{F} \pi^a(x) T^a \right) \]

\[ T^a : \text{Group generators} \]

Lagrangian: \[ \mathcal{L}_{eff} [\Sigma, M] = \mathcal{L}_{eff} [\Sigma', M'] \]
\[ M: \text{Quark mass matrix} \]
\[ \Sigma' = L \Sigma R^\dagger \quad M' = L M R^\dagger \]
\[ L, R: \text{Left, Right transformations} \]

Expand in powers of derivatives and masses: \[ \mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots \]

\[ \mathcal{L}_2 = \frac{f^2}{4} \text{tr} \left[ \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \right] - \frac{f^2 B}{2} \text{tr} \left[ \Sigma^\dagger M + M^\dagger \Sigma \right] \]

\[ f, B: \text{undetermined low-energy constants} \]
Chiral Lagrangian including $\alpha$

$$S_{\text{eff}} = S_{QCD} + a \, c \int \overline{\psi} i\sigma_{\mu\nu} G_{\mu\nu} \psi + \mathcal{O}(a^2)$$

Pauli term breaks the chiral symmetry exactly like the mass term in $S_{QCD}$

$\Rightarrow$ $\alpha$ enters chiral Lagrangian exactly like the mass term

$\Rightarrow$

$$L_2 = \frac{f^2}{4} \text{tr} \left[ \partial_\mu \Sigma \partial_\mu \Sigma^\dagger \right] - \frac{f^2 B}{2} \text{tr} \left[ \Sigma^\dagger M + M^\dagger \Sigma \right]$$

$$- \frac{f^2 W_0}{2} a \text{tr} \left[ \Sigma + \Sigma^\dagger \right]$$

$W_0$ : new undetermined low-energy constant

includes $c = c(g_0^2)$ not really a constant (weak $\alpha$ dependence)

Sharpe, Singleton '98
Rupak, Shoresh '02
\( \mathcal{L}_4 \)-Lagrangian:

\[
\mathcal{L}_4 = \mathcal{L}_4(p^4, p^2m, m^2) + \mathcal{L}_4(p^2a, ma) + \mathcal{L}_4(a^2)
\]

- No \( O(4) \) symmetry breaking terms in \( \mathcal{L}_4(a^2) \) ( start at \( O(a^2p^4) \) )
- Total number of low-energy constants: \( 10 L_i + (5 + 3) W_i = 18 \)

\( L_i \) are physical ChPT parameters \( \rightarrow \) values are interesting

( independent of \( a \) )

Role of \( W_i \): parameterize \( a \) dependence \( \rightarrow \) values are less interesting

Note: \( W_i \) are not universal ( depend on the lattice action )
Power counting

The power counting is non-trivial because of

1. the additive mass renormalization \( \propto \frac{1}{a} \)

2. two symmetry breaking parameters \( a, m_{\text{quark}} \)

\( \Rightarrow \) their relative size matters
Leading order pion mass (degenerate case)

\[ M_0^2 = 2mB + 2aW_0 \]

\[ m = m_u = m_d \]

Leading \( a \)-effect: Shift in the pion mass

But: This shift might already be absorbed in \( m_c \)

\[ m^2_\pi = 0 \quad \text{for} \quad m' = Z_m(m_0 - m_{cr}) = 0 \]

⇒ Express the observables \( m_\pi, f_\pi \) in terms of \( m' \)

Note: additional shifts enter at \( a^2, ... \)
Two expansion parameters:

\[
\frac{2Bm'}{(4\pi f)^2} \quad \frac{2W_0a}{(4\pi f)^2}
\]

Both need to be small

Relative size determines which terms are LO, NLO etc.

Example:

\[
\mathcal{L}_2(p^2, m') \quad \mathcal{L}_4(a^2)
\]

\[
\begin{array}{|c|c|c|}
\hline
 & f^2 B m' \text{tr} \left[ \Sigma + \Sigma^\dagger \right] & W a^2 \left( \text{tr} \left[ \Sigma + \Sigma^\dagger \right] \right)^2 \\
\hline
m' \gg a^2 & \text{LO} & \text{NLO} \\
\hline
m' \approx a^2 & \text{LO} & \text{LO} \\
\hline
\end{array}
\]

The proper power counting depends on the relative size of \( m' \) and \( a \)
Different power countings have been discussed:

If \( m' \gg a^2 \rightarrow \) continuum like ChPT + small \( O(a^n) \) corrections

\[ Rupak, Shoresh, OB '03 \]

If \( m' \approx a^2 \rightarrow \) qualitatively different!

- Non-trivial phase diagram
- Modification of chiral logs

\[ Sharpe, Singleton '98 \]
\[ Aoki '03 \]
Non-trivial phase diagram

Potential energy: 
\( (N_f = 2) \)

\[
V = -c_1 m' \text{tr} \left[ \Sigma + \Sigma^\dagger \right] + c_2 a^2 \left( \text{tr} \left[ \Sigma + \Sigma^\dagger \right] \right)^2
\]

Sharpe, Singleton '98

\( c_1(f, B) \)

\( c_2(f, B, W_i) \)

Aoki phase for and parity are broken at
massless pions at \( a \neq 0 \)

\( (N_f = 2) \)

Aoki '85

A: \( \text{sign} \ c_2 = +1 \) \quad \Rightarrow \quad \Sigma_{\text{vacuum}} \neq \pm 1 \)

\( \Sigma_{\text{vacuum}} \neq \pm 1 \)

B: \( \text{sign} \ c_2 = -1 \) \quad \Rightarrow \quad \Sigma_{\text{vacuum}} = \pm 1 \)

\( \Sigma_{\text{vacuum}} = \pm 1 \)

no flavor/parity breaking
no massless pions

Question: Scenario A or B? Depends on the underlying lattice action
Plaquette action + unimproved Wilson: Scenario B

Theory:

Data:

Questions:

• Which scenario is realized for improved fermions?
• Can we find a lattice action with very small $C_2$?
• What are the consequences for overlap fermions?
Pseudo-scalar mass to 1-loop (2 deg. flavors)  

\[ m_\pi^2 = 2Bm' \left[ 1 + \frac{m'(2B + w_1 a)}{32\pi^2 f^2} \log \frac{2Bm'}{\Lambda^2} + \frac{w_0 a^2}{32\pi^2 f^2} \log \frac{2Bm'}{\Lambda^2} \right] + \text{analytic} \]

- \( a \to 0 \) gives the correct continuum expression
- \( w_0, w_1 \) : combinations of low-energy constants \( W_i \)
- The coefficient of \( m' \ln m' \) is no longer universal at non-zero \( a \)
- The lattice spacing generates an \( a^2 \ln m' \) contribution
  \[ \Rightarrow \text{This term dominates} \ldots \text{for small } m' \]
  But: A resummation of the \( (\ln m')^n \) terms can be performed ....
Resummend Pseudo-scalar mass

\[ m_\pi^2 = 2\tilde{B}m' \left[ 1 + \frac{m'(2B + w_1 a)}{32\pi^2 f^2} \log \frac{2Bm'}{\Lambda^2} \right] \left\{ \log \frac{2Bm'}{\tilde{\Lambda}^2} \right\} \tilde{w}_0 a^2 / 32\pi^2 f^2 \]

Expansion of \( \{ \ldots \} a^2 \) gives result on previous slide

Assumption: Aoki phase scenario

Similar modifications for \( f_\pi \) and \( m_{\text{AWI}} \)
Feature in WChPT:

The coefficients of the chiral logs can be altered by $O(a)$ corrections

Question: What does $m \approx a^2$ precisely mean for a given lattice action?

Crude dimensional argument: $m \approx a^2 \Lambda_{QCD}^3$

$a = 0.15\text{fm}$

$\Lambda_{QCD} = 300\text{MeV}$

$\Rightarrow m \approx 15\text{MeV}$

Current lattice simulations do not satisfy $m \gg a^2$
**Applying to numerical data**

Two groups analyzed their data using WChPT:

1. **CP-PACS** :
   - Tadpole improved clover quark action
   - \( a = 0.2 \, fm \), \( 0.35 \leq \frac{M_\pi}{M_\rho} \leq 0.8 \) (8 Sea quark masses)
   - Good fits with WChPT assuming Aoki power counting, i.e. \( m \approx a^2 \)

2. **qq+q** :
   - Unimproved Wilson quarks
   - \( a \approx 0.2 \, fm \), \( 0.47 \leq \frac{M_\pi}{M_\rho} \leq 0.76 \) (4 Sea quark masses)
   - Good fits with continuum ChPT

Current numerical data is not conclusive
Staggered fermions

+ Fast to simulate
+ Exact U(1) symmetry for massless quarks
- Fermion doubling: Each flavor comes with 4 tastes

Taste reduction on the lattice: \( \det D \rightarrow 4\sqrt{\det D} \) “fourth root trick”

1. The \( 4\sqrt{\det D} \)-theory has no local lattice action: Universality ?

2. The \( 4\sqrt{\det D} \)-theory has no local Symanzik action: How to include \( a \) in ChPT ?
Staggered Chiral Perturbation Theory (SChPT)

Main strategy:

1. Lattice theory with \( N_f \) staggered fermions \( \rightarrow \) Symanzik’s effective theory with \( 4N_f \) fermions

2. Symanzik’s effective theory \( \rightarrow \) ChPT

3. Compute physical observables like \( M_{\pi}^2, f_{\pi}, \ldots \)

4. Adjust by hand to one taste per flavor
   = include factors of 1/4
   for sea quark loops

Bernard ‘02
Aubin, Bernard ‘03

\( \frac{1}{4} \) No factor

\( \frac{1}{4} \) One factor

“quark flow” diagrams
In the Continuum limit:

\[
\frac{(m_{\pi^+}^2)^2}{2Bm} = 1 + \frac{1}{24\pi^2 f^2} m_{\pi I}^2 \ln \frac{m_{\pi I}^2}{\Lambda^2} + \text{analytic} \quad \longrightarrow \quad \text{Continuum expression}
\]

\[
+ \frac{4}{24\pi^2 f^2} \left[ m_{\eta'V}^2 \ln \frac{m_{\eta'V}^2}{\Lambda^2} - m_{\pi V}^2 \ln \frac{m_{\pi V}^2}{\Lambda^2} \right] \quad \longrightarrow \quad 0
\]

\[
+ \frac{4}{24\pi^2 f^2} \left[ m_{\eta'_A}^2 \ln \frac{m_{\eta'_A}^2}{\Lambda^2} - m_{\pi A}^2 \ln \frac{m_{\pi A}^2}{\Lambda^2} \right] + a^2 C \quad \longrightarrow \quad 0
\]

- Reduces to the continuum expression for \( a \to 0 \) (not easy to see here)

- Non-zero \( a \): Additional log contributions involving other particles
  Continuum log behaviour may be changed significantly!

- MILC data strongly suggests the presence of these contributions

---- talk by Claude Bernard ----
Question:

Is the 4th-root trick really legitimate?

- Can we analytically understand why it works?

- Can we find additional cross checks?
  (Besides comparing with experimental results)
Twisted mass term:

\[ m' e^{i \omega \gamma_5 \tau_3} = m + i \mu \gamma_5 \tau_3 \]

\( \omega \): twist angle

Why tmQCD:

- No exceptional configurations
- Automatic O(a) improvement at maximal twist \( \omega = \frac{\pi}{2} \)

But: Seems to work only if \( m \gg a^2 \)

Frezzotti, Rossi '03
Twisted mass term on the lattice

Mass term + Wilson term on the lattice:

$$\bar{\psi}(x) \left[ \left( -a \frac{r}{2} \sum_{\mu} \nabla^*_\mu \nabla_\mu + M_{cr}(r) \right) + m_q \exp(i\omega \gamma_5 \tau_3) \right] \psi(x)$$

$$m_q = m_0 - M_{cr}(r)$$

Field redefinition:

$$\psi_{ph} = \exp(i \frac{\omega}{2} \gamma_5 \tau_3) \psi,$$

$$\bar{\psi}_{ph} = \bar{\psi} \exp(i \frac{\omega}{2} \gamma_5 \tau_3)$$

$$\bar{\psi}_{ph}(x) \left[ \left( -a \frac{r}{2} \sum_{\mu} \nabla^*_\mu \nabla_\mu + M_{cr}(r) \right) \exp(-i\omega \gamma_5 \tau_3) + m_q \right] \psi_{ph}(x)$$
Wilson average and $O(a)$ improvement

The Wilson average
\[
\langle O \rangle^{WA}(r, m_q, \omega) \equiv \frac{1}{2} \left[ \langle O \rangle(r, m_q, \omega) + \langle O \rangle(-r, m_q, \omega) \right]
\]

can be shown to be $O(a)$ improved:
\[
= \langle O \rangle^{\text{cont}}(m_q) + O(a^2)
\]

Crucial assumption:
\[
M_{cr}(-r) = -M_{cr}(r)
\]
Automatic $O(a)$ improvement at maximal twist

Consider the twist average:

$$\langle O \rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) \equiv \frac{1}{2} \left[ \langle O \rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O \rangle(r, m_q, \omega = -\frac{\pi}{2}) \right]$$

$$= \frac{1}{2} \left[ \langle O \rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O \rangle(-r, m_q, \omega = \frac{\pi}{2}) \right]$$

For observables even in $\omega$ (e.g. masses):

$$\langle O \rangle(r, m_q, \omega = \frac{\pi}{2}) = \langle O \rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) = \langle O \rangle^{cont}(m_q) + O(a^2)$$

$O(a)$ improvement without taking an average!
Using WChPT you can explicitly show (in the Aoki phase scenario with $c_2 > 0$)

$$M_{cr}(r) = \frac{r}{a} M_1(ra) + a^2 r^2 M_2(ra)$$

$M_{1,2}(ra) :$ even polynomials in $ra$!

$$\Rightarrow M_{cr}(r) \neq -M_{cr}(-r)$$

$$\langle O\rangle(r, m_q, \omega = \frac{\pi}{2})^{TA} = \frac{1}{2} \left[ \langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(-r, m'_q, \omega = \frac{\pi}{2} + \omega') \right]$$

$$m'_q = \sqrt{m_q^2 + (2a^2 r^2 M_2(ar))^2} \quad \tan \omega' = \frac{2a^2 r^2 M_2(ar)}{m_q}$$

$$\Rightarrow \text{Automatic O}(a) \text{ improvement only if } m_q \gg a^2$$
New definition for the twist angle

Define:

\[
\overline{M_{cr}}(r) = \frac{M_{cr}(r) - M_{cr}(-r)}{2} = -\overline{M_{cr}}(-r)
\]

\[
\Delta M_{cr}(r) = \frac{M_{cr}(r) + M_{cr}(-r)}{2} = \Delta M_{cr}(-r)
\]

\[\Rightarrow\] New definition for the twist angle:

\[
\bar{\psi}_{ph}(x) \left[ - \left( -a \frac{r}{2} \sum_{\mu} \nabla_\mu^* \nabla_\mu + \overline{M_{cr}}(r) \right) \exp(-i\omega \gamma_5 \tau_3) + m_q + \Delta M_{cr}(r) \right] \psi_{ph}(x)
\]

You can show:

**Automatic O(a) improvement at maximal twist without restrictions on** \(m_q\)