Chiral Perturbation Theory at non-zero lattice spacing

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Matching Light Quarks to Hadrons
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Introduction: Chiral extrapolation

Some numbers I heard at Lattice 2004:

Collaboration	Fermion type	Nf	smallest m_π/m_ρ
UKQCD	Wilson	2	~0.44
SPQcdR	Wilson	2	~0.66
qq+q	Wilson	2	~0.47
CP-PACS	Wilson	2	~0.35
CP-PACS	Wilson	2+1	~0.62
MILC	Staggered	2+1	~0.3
RBC	Domain Wall	2	~0.53

Lattice 2004 Talk/poster by:

Tarantino

Scholz

Namekawa

Ishikawa

Bernard

Izubuchi

physical value: 0.18

Still needed: rather long extrapolation in the light quark mass

Chiral Perturbation Theory (ChPT)

Analytic guidance is required for the extrapolation

$\Longrightarrow$ provided by ChPT

ChPT: Effective theory for low-energy QCD

Weinberg, Gasser, Leutwyler

$\Longrightarrow$ quark mass dependence of observables

Example:

$$m_\pi^2 = M_0^2 \left[1 + \frac{M_0^2}{16\pi^2 N_f f^2} \ln M_0^2 - \frac{8LM_0^2}{f^2} \right]$$

$M_0^2 = (m_u + m_d)B$ $\qquad$ f, B, L: low-energy constants

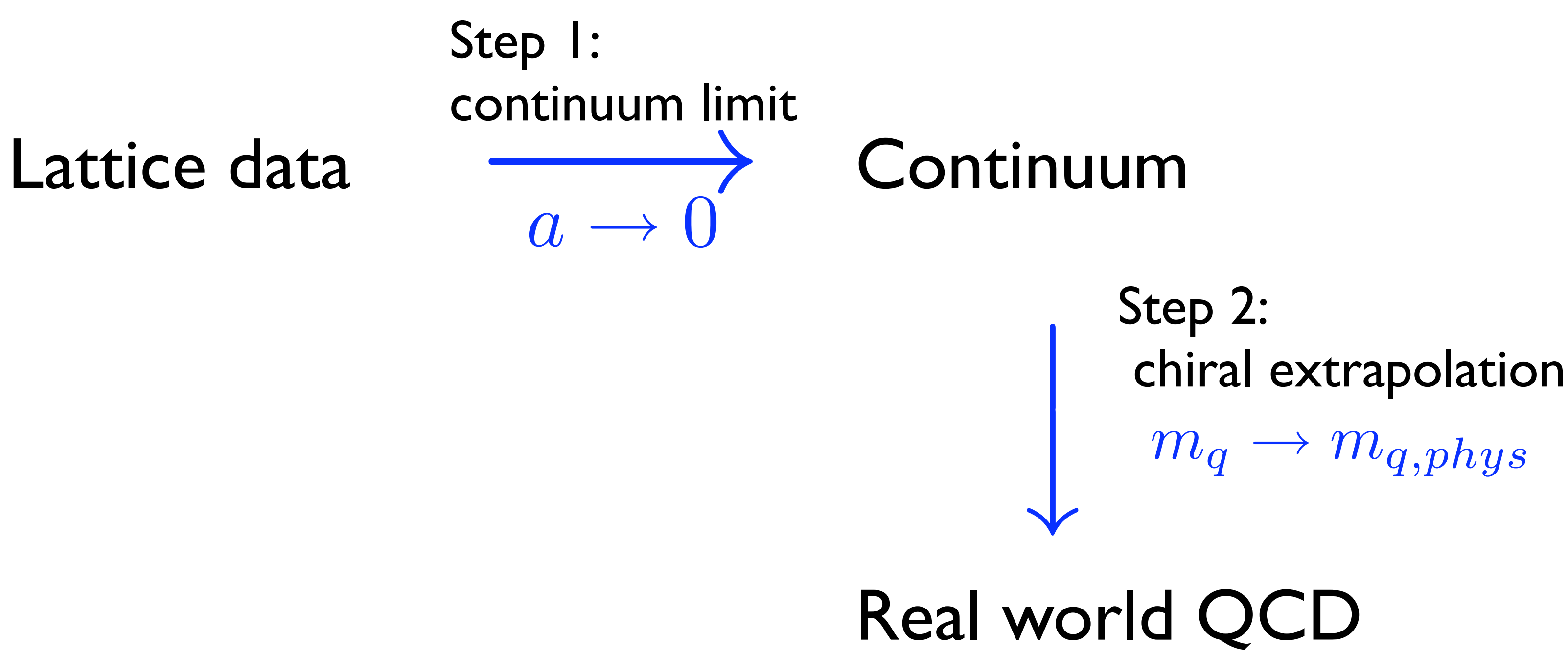
Question: Does ChPT describe the lattice data ?

Smoking gun: curvature due to chiral logs

Panel discussion
Lattice 2002, Boston

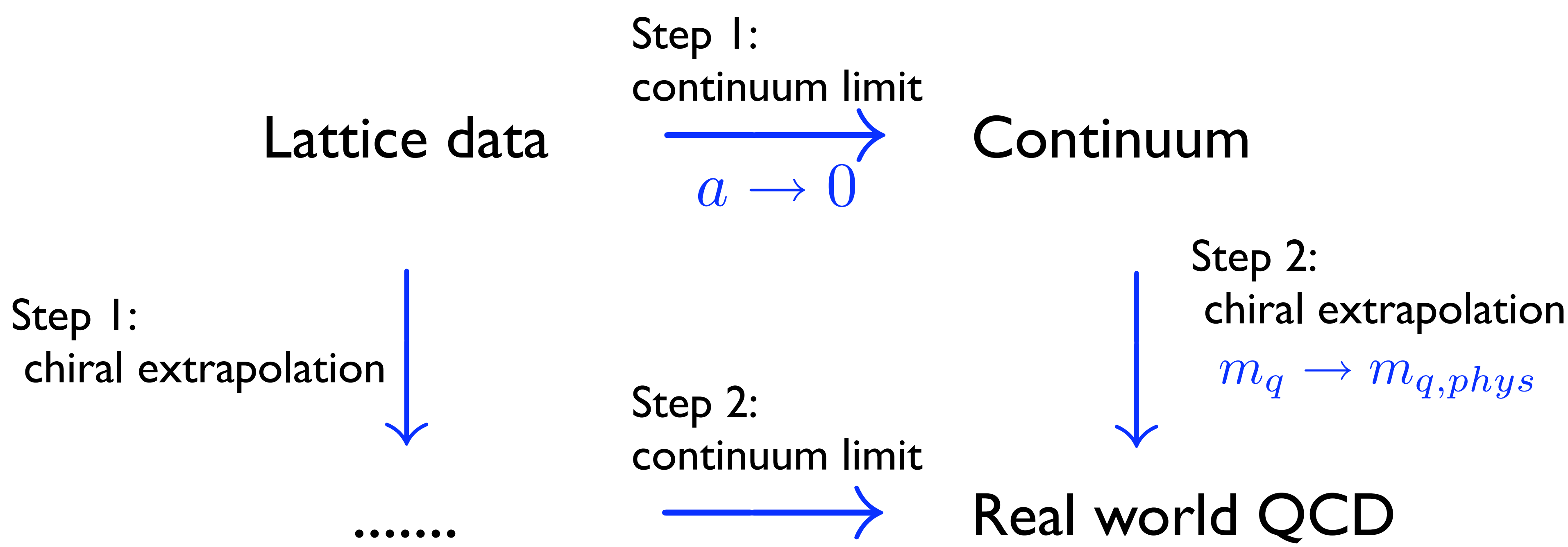
A potential problem

ChPT formulae derived for $a = 0$ only !



A potential problem

ChPT formulae derived for $a = 0$ only !



What about the opposite order ?

Desirable: ChPT for lattice theories at non-zero lattice spacing

ChPT at non-zero lattice spacing: Main strategy

Two-step matching to effective theories:

Lee, Sharpe '98
Sharpe, Singleton '98

1. Lattice theory $\longrightarrow$ Symanzik's effective theory

   continuum theory making the a-dependence explicit

2. Symanzik's effective theory $\longrightarrow$ ChPT

   including the a-dependence

$\Longrightarrow$ Chiral expressions for $m_\pi,\ f_\pi$ ... with explicit a-dependence

Symanzik's action for Lattice QCD with Wilson fermions

Locality and symmetries of the lattice theory

$$\Longrightarrow \qquad S_{eff} = S_{QCD} + a\, c \int \overline{\psi}\, i\sigma_{\mu\nu} G_{\mu\nu} \psi + \mathcal{O}(a^2)$$

- At $O(a)$ only one additional operator (making use of EOM)
- c : unknown coefficient (”low-energy constant”)
- $O(a^2)$: dim-6 operators: - fermion bilinears
  - 4-fermion operators

Sheikholeslami, Wohlert

O(4) rotation invariance is broken at this order

Reminder: Chiral Lagrangian

Fields: $$\Sigma(x) = \exp\left(\frac{2i}{F}\pi^a(x)T^a\right)$$ T^a: Group generators

Lagrangian: $$\mathcal{L}_{eff}[\Sigma, M] = \mathcal{L}_{eff}[\Sigma', M']$$ M: Quark mass matrix

$$\Sigma' = L\Sigma R^\dagger \qquad M' = LMR^\dagger$$ L, R : Left, Right transformations

Expand in powers of derivatives and masses: $\mathcal{L}_{eff} = \mathcal{L}_2 + \mathcal{L}_4 + \ldots$

$$\mathcal{L}_2 = \frac{f^2}{4}\mathrm{tr}\left[\partial_\mu\Sigma\partial_\mu\Sigma^\dagger\right] - \frac{f^2B}{2}\mathrm{tr}\left[\Sigma^\dagger M + M^\dagger\Sigma\right]$$

f, B : undetermined low-energy constants

Chiral Lagrangian including a

$$S_{eff} = S_{QCD} + a\, c \int \overline{\psi}\, i\sigma_{\mu\nu} G_{\mu\nu} \psi + \mathcal{O}(a^2)$$

Pauli term breaks the chiral symmetry exactly like the mass term in S_{QCD}

$\Longrightarrow$ a enters chiral Lagrangian exactly like the mass term

$$\Longrightarrow \qquad \mathcal{L}_2 = \frac{f^2}{4} \mathrm{tr}\left[\partial_\mu \Sigma \partial_\mu \Sigma^\dagger\right] - \frac{f^2 B}{2} \mathrm{tr}\left[\Sigma^\dagger M + M^\dagger \Sigma\right] - \frac{f^2 W_0}{2}\, a\, \mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]$$

Sharpe, Singleton '98
Rupak, Shoresh '02

W_0 : new undetermined low-energy constant

includes $c = c(g_0^2)$ not really a constant (weak a dependence)

$\mathcal{L}_4$-Lagrangian:

$$\mathcal{L}_4 = \mathcal{L}_4(p^4, p^2 m, m^2) \ + \ \mathcal{L}_4(p^2 a, ma) \ + \ \mathcal{L}_4(a^2)$$

Gasser, Leutwyler '85 — Rupak, Shoresh '02 — Rupak, Shoresh, OB '03, Aoki '03

- No O(4) symmetry breaking terms in $\mathcal{L}_4(a^2)$ (start at $\mathcal{O}(a^2 p^4)$)
- Total number of low-energy constants: 10 L_i + (5 + 3) W_i = 18 !

L_i are physical ChPT parameters $\longrightarrow$ values are interesting
(independent of a)

Role of W_i : parameterize a dependence $\longrightarrow$ values are less interesting

Note: W_i are not universal (depend on the lattice action)

Power counting

The power counting is non-trivial because of

1. the additive mass renormalization $\propto \dfrac{1}{a}$

2. two symmetry breaking parameters $a\,,\ m_{\rm quark}$

   $\Longrightarrow$ their relative size matters

Leading order pion mass (degenerate case)

$$M_0^2 = 2mB + 2aW_0 \qquad m = m_u = m_d$$

Leading a-effect: Shift in the pion mass

But: This shift might already be absorbed in m_c

e.g. $m_\pi^2 = 0$ for $m' = Z_m(m_0 - m_{\rm cr}) = 0$

$\Longrightarrow$ Express the observables $m_\pi, \ f_\pi$ in terms of m'

Note: additional shifts enter at a^2, ...

Two expansion parameters: $\frac{2Bm'}{(4\pi f)^2}$ $\frac{2W_0 a}{(4\pi f)^2}$

Both need to be small

Relative size determines which terms are LO, NLO etc.

Example:

$$\mathcal{L}_2(p^2, m') \downarrow \qquad \mathcal{L}_4(a^2) \downarrow$$

	$f^2 B\, m' \mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]$	$W\, a^2 \left(\mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]\right)^2$
$m' \gg a^2$	LO	NLO
$m' \approx a^2$	LO	LO

The proper power counting depends on the relative size of m' and a

Different power countings have been discussed:

If $m' \gg a^2 \rightarrow$ continuum like ChPT + small $\mathcal{O}(a^n)$ corrections

Rupak, Shoresh, OB '03

If $m' \approx a^2 \rightarrow$ qualitatively different !

Non-trivial phase diagram — Sharpe, Singleton '98

Modification of chiral logs — Aoki '03

Non-trivial phase diagram

Potential energy: Sharpe, Singleton '98

$(\mathrm{N_f} = 2)$

$$V = -c_1 m' \mathrm{tr}\left[\Sigma + \Sigma^\dagger\right] + c_2 a^2 \left(\mathrm{tr}\left[\Sigma + \Sigma^\dagger\right]\right)^2$$

$c_1(f, B)$

$c_2(f, B, W_i)$

A: $\mathrm{sign}\, c_2 = +1 \quad \Longrightarrow \quad \Sigma_{\mathrm{vacuum}} \neq \pm 1$

Aoki phase — Aoki '85
flavor and parity are broken
massless pions at $a \neq 0$

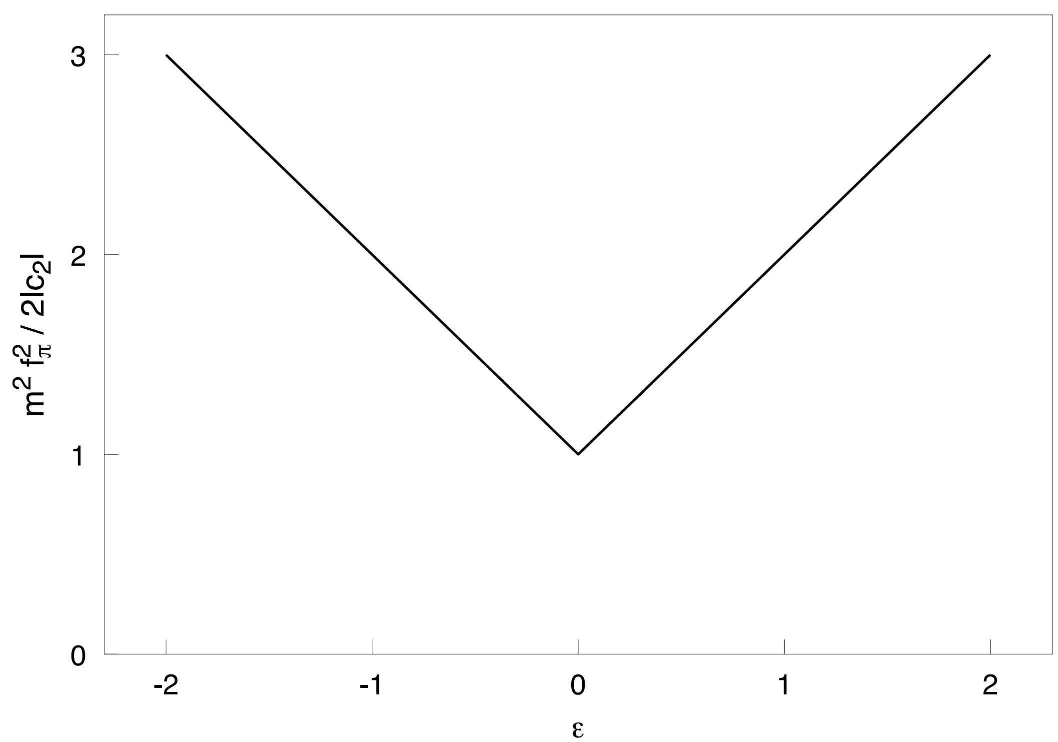
B: $\mathrm{sign}\, c_2 = -1 \quad \Longrightarrow \quad \Sigma_{\mathrm{vacuum}} = \pm 1$

no flavor/parity breaking
no massless pions

Question: Scenario A or B ? Depends on the underlying lattice action

Plaquette action + unimproved Wilson: Scenario B

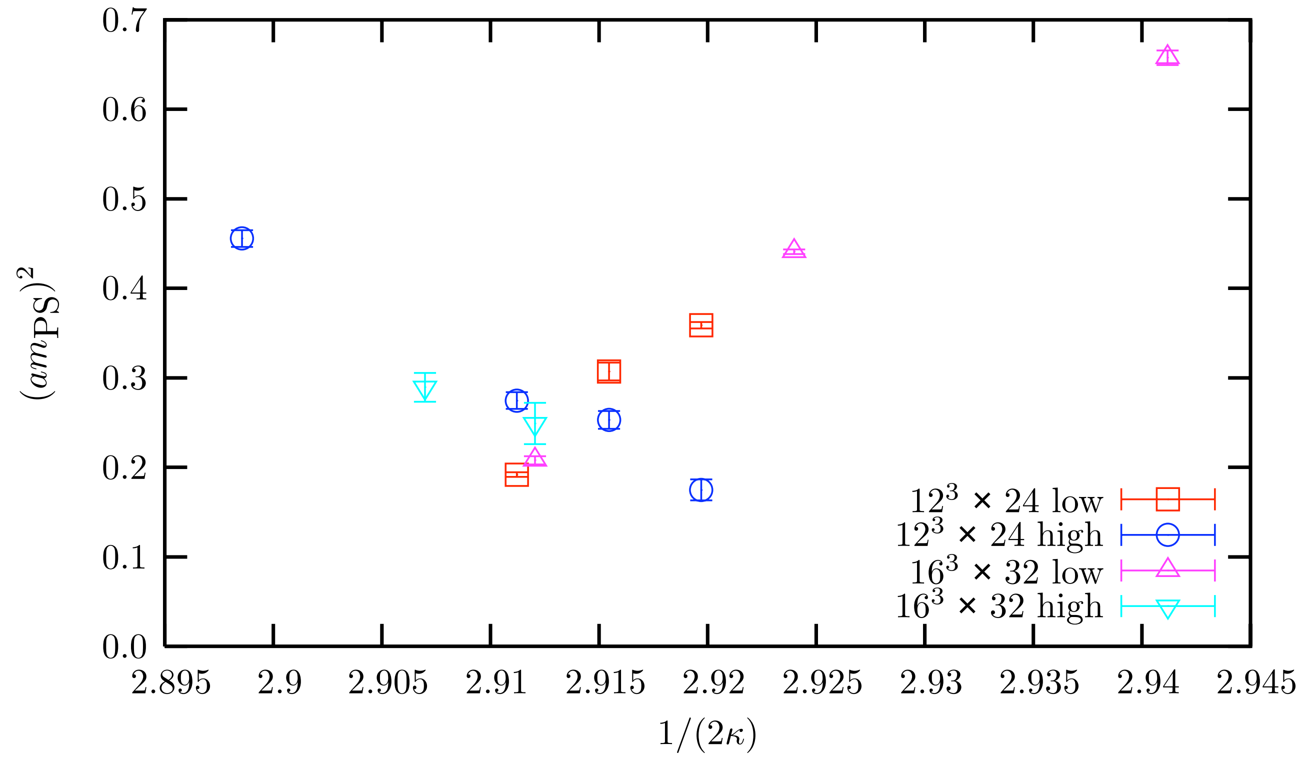
Theory:



Sharpe, Singleton '98

Data:

$\beta = 5.2$



F. Farchioni et.al. '04

similar findings by
Ilgenfritz et. al. '03

Questions:

- Which scenario is realized for improved fermions ?
- Can we find a lattice action with very small c_2 ?
- What are the consequences for overlap fermions ?

Pseudo-scalar mass to 1-loop (2 deg. flavors) Aoki '03

$$m_\pi^2 = 2Bm' \left[1 + \frac{m'(2B + w_1 a)}{32\pi^2 f^2} \log \frac{2Bm'}{\Lambda^2} + \frac{w_0 a^2}{32\pi^2 f^2} \log \frac{2Bm'}{\Lambda^2} \right]$$

$$+ \text{analytic}$$

- $a \to 0$ gives the correct continuum expression
- w_0, w_1 : combinations of low-energy constants W_i
- The coefficient of $m' \ln m'$ is no longer universal at non-zero a
- The lattice spacing generates an $a^2 \ln m'$ contribution

  $\Longrightarrow$ This term dominates [....] for small m'

  But: A resummation of the $(\ln m')^n$ terms can be performed ....

Resummend Pseudo-scalar mass

Aoki '03

$$m_\pi^2 = 2\tilde{B}m' \left[1 + \frac{m'(2B + w_1 a)}{32\pi^2 f^2} \log \frac{2Bm'}{\Lambda^2}\right] \left\{\log \frac{2Bm'}{\tilde{\Lambda}^2}\right\}^{\tilde{w}_0 a^2/32\pi^2 f^2}$$

$$+ \text{analytic}$$

Expansion of $\{\ldots\}^{a^2}$ gives result on previous slide

Assumption: Aoki phase scenario

Similar modifications for f_π and m_{AWI}

Feature in WChPT:

The coefficients of the chiral logs can be altered by O(a) corrections

Question: What does $m \approx a^2$ precisely mean for a given lattice action ?

Crude dimensional argument: $m \approx a^2 \Lambda_{\rm QCD}^3$

$$a = 0.15\,\text{fm} \quad \Lambda_{\rm QCD} = 300\,\text{MeV} \quad \Longrightarrow \quad m \approx 15\,\text{MeV}$$

Current lattice simulations do not satisfy $m \gg a^2$

Applying to numerical data

Two groups analyzed their data using WChPT:

1. CP-PACS :

   Namekawa et.al '04

   Tadpole improved clover quark action

   $$a = 0.2 fm \ , \qquad 0.35 \leq \frac{M_\pi}{M_\rho} \leq 0.8$$ (8 Sea quark masses)

   $\longrightarrow$ Good fits with WChPT assuming Aoki power counting, i.e. $m \approx a^2$

2. qq+q :

   Farchioni et.al. '03/'04

   Unimproved Wilson quarks

   $$a \approx 0.2 fm \ , \qquad 0.47 \leq \frac{M_\pi}{M_\rho} \leq 0.76$$ (4 Sea quark masses)

   $\longrightarrow$ Good fits with continuum ChPT

Current numerical data is not conclusive

Staggered fermions

+ Fast to simulate

+ Exact U(1) symmetry for massless quarks

– Fermion doubling: Each flavor comes with 4 tastes

Taste reduction on the lattice: $\det D \to \sqrt[4]{\det D}$ "fourth root trick"

1. The $\sqrt[4]{\det D}$-theory has no local lattice action: Universality ?

2. The $\sqrt[4]{\det D}$-theory has no local Symanzik action: How to include a in ChPT ?

Staggered Chiral Perturbation Theory (SChPT)

Bernard '02
Aubin, Bernard '03

Main strategy:

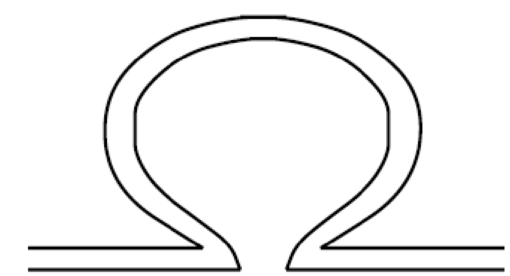
1. Lattice theory with N_f staggered fermions $\longrightarrow$ Symanzik's effective theory with $4N_f$ fermions

2. Symanzik's effective theory $\longrightarrow$ ChPT

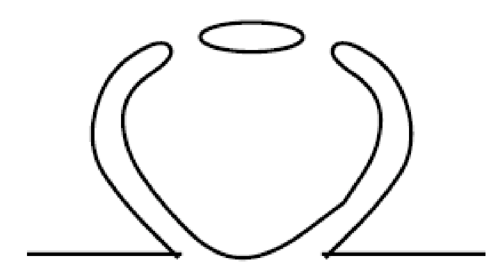
3. Compute physical observables like $M_\pi^2, f_\pi, \ldots$

4. Adjust by hand to one taste per flavor = include factors of 1/4 for sea quark loops

No factor $\frac{1}{4}$



One factor $\frac{1}{4}$



"quark flow" diagrams

Pseudo-scalar mass to 1-loop (3 deg. flavor)

Aubin, Bernard '02/03

In the Continuum limit :

$$\frac{(m_{\pi_5^+})^2}{2Bm} = 1 + \frac{1}{24\pi^2 f^2} m_{\pi_I}^2 \ln \frac{m_{\pi_I}^2}{\Lambda^2} + \text{analytic} \quad \longrightarrow \quad \text{Continuum expression}$$

$$+ \frac{4}{24\pi^2 f^2}\left[m_{\eta_V'}^2 \ln \frac{m_{\eta_V'}^2}{\Lambda^2} - m_{\pi_V}^2 \ln \frac{m_{\pi_V}^2}{\Lambda^2}) \right] \quad \longrightarrow \quad 0$$

$$+ \frac{4}{24\pi^2 f^2}\left[m_{\eta_A'}^2 \ln \frac{m_{\eta_A'}^2}{\Lambda^2} - m_{\pi_A}^2 \ln \frac{m_{\pi_A}^2}{\Lambda^2}) \right] + a^2 C \quad \longrightarrow \quad 0$$

- Reduces to the continuum expression for $a \rightarrow 0$ (not easy to see here)
- Non-zero a : Additional log contributions involving other particles
  Continuum log behaviour may be changed significantly !
- MILC data strongly suggests the presence of these contributions

$\longrightarrow$ talk by Claude Bernard

Question:

Is the 4th-root trick really legitimate ?

- Can we analytically understand why it works ?
- Can we find additional cross checks ?
  (Besides comparing with experimental results)

tmQCD

Twisted mass term: $m' e^{i\omega\gamma_5\tau_3} = m + i\mu\gamma_5\tau_3$ ω: twist angle

Why tmQCD:

- No exceptional configurations
- Automatic O(a) improvement at maximal twist $\omega = \dfrac{\pi}{2}$ Frezzotti, Rossi '03

But: Seems to work only if $m \gg a^2$ Why ?

Twisted mass term on the lattice

Mass term + Wilson term on the lattice :

$$\bar{\psi}(x)\left[\left(-a\frac{r}{2}\sum_{\mu}\nabla^{\star}_{\mu}\nabla_{\mu}+M_{\rm cr}(r)\right)+m_q\exp(iw\gamma_5\tau_3)\right]\psi(x)$$

$$m_q = m_0 - M_{\rm cr}(r)$$

m_0: bare quark mass, $M_{\rm cr}(r)$: critical quark mass

Field redefinition:

$$\psi_{\rm ph} = \exp(i\frac{\omega}{2}\gamma_5\tau_3)\psi,$$

$$\bar{\psi}_{\rm ph} = \bar{\psi}\exp(i\frac{\omega}{2}\gamma_5\tau_3)$$

$$\bar{\psi}_{\rm ph}(x)\left[\left(-a\frac{r}{2}\sum_{\mu}\nabla^{\star}_{\mu}\nabla_{\mu}+M_{\rm cr}(r)\right)\exp(-iw\gamma_5\tau_3)+m_q\right]\psi_{\rm ph}(x)$$

Wilson average and O(a) improvement

The Wilson average $$\langle O\rangle^{WA}(r,m_q,\omega)\equiv\frac{1}{2}\Big[\langle O\rangle(r,m_q,\omega)+\langle O\rangle(-r,m_q,\omega)\Big]$$

can be shown to be O(a) improved: $$=\langle O\rangle^{\mathrm{cont}}(m_q)+O(a^2)$$

Crucial assumption: $$M_{\mathrm{cr}}(-r)=-M_{\mathrm{cr}}(r)$$

Automatic O(a) improvement at maximal twist

Consider the twist average:

$$\langle O\rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) \equiv \frac{1}{2}\Big[\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(r, m_q, \omega = -\frac{\pi}{2})\Big]$$

$$\exp(-i\frac{\pi}{2}\gamma_5\tau_3) = -\exp(i\frac{\pi}{2}\gamma_5\tau_3)$$

$$= \frac{1}{2}\Big[\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(-r, m_q, \omega = \frac{\pi}{2})\Big]$$

For observables even in ω (e.g. masses):

$$\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) = \langle O\rangle^{TA}(r, m_q, \omega = \frac{\pi}{2}) = \langle O\rangle^{\text{cont}}(m_q) + O(a^2)$$

O(a) improvement without taking an average !

Using WChPT you can explicitly show (in the Aoki phase scenario with $c_2 > 0$)

$$M_{\rm cr}(r) = \frac{r}{a} M_1(ra) + a^2 r^2 M_2(ra)$$

$M_{1,2}(ra)$:

even polynomials in ra !

$$\Longrightarrow \qquad M_{\rm cr}(r) \neq -M_{\rm cr}(-r)$$

$$\langle O\rangle(r, m_q, \omega = \frac{\pi}{2})^{TA} = \frac{1}{2}\Big[\langle O\rangle(r, m_q, \omega = \frac{\pi}{2}) + \langle O\rangle(-r, m_q', \omega = \frac{\pi}{2} + \omega')\Big]$$

$$m_q' = \sqrt{m_q^2 + (2a^2r^2M_2(ar))^2} \qquad \tan\omega' = \frac{2a^2r^2M_2(ar)}{m_q}$$

$\Longrightarrow$ Automatic O(a) improvement only if $m_q \gg a^2$

New definition for the twist angle

Define:

$$\overline{M}_{\rm cr}(r) = \frac{M_{\rm cr}(r) - M_{\rm cr}(-r)}{2} = -\overline{M}_{\rm cr}(-r)$$

$$\Delta M_{\rm cr}(r) = \frac{M_{\rm cr}(r) + M_{\rm cr}(-r)}{2} = \Delta M_{\rm cr}(-r)$$

$\Longrightarrow$ New definition for the twist angle:

$$\bar{\psi}_{\rm ph}(x)\left[-\left(-a\frac{r}{2}\sum_{\mu}\nabla^{\star}_{\mu}\nabla_{\mu} + \overline{M}_{\rm cr}(r)\right)\exp(-iw\gamma_5\tau_3) + m_q + \Delta M_{\rm cr}(r)\right]\psi_{\rm ph}(x)$$

You can show:

Automatic O(a) improvement at maximal twist without restrictions on m_q