Neutrino masses from unified fermion mixing

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with R. Gastmans and T.T. Wu

- Introduction
- Mathematics, generalized hypergeom functions, ${}_2F_2$
- The model
- Fits, masses

The LNW model

$$M = \begin{pmatrix} 0 & d & 0 \\ d & c & b \\ 0 & b & a \end{pmatrix}$$

Four parameters

$$b^2 = 8\,c^2$$

arguments based on S(3) symmetry

Three parameters ⇔ Three masses

(sign ambiguity)

Lehmann, Newton, Wu, 1996

Mathematical prelude

Three-flavour MSW effect, exponential density

$$M^2 = \begin{bmatrix} M^2_{11} & M^2_{12} & M^2_{13} \\ M^2_{21} & M^2_{22} & M^2_{23} \\ M^2_{31} & M^2_{32} & M^2_{33} \end{bmatrix}$$

real and symmetric: $M^2_{ji} = M^2_{ij}$

$M^2_{ij} \equiv (M^2)_{ij} \neq (M_{ij})^2$

$$i\frac{\mathrm{d}}{\mathrm{d}r}\begin{bmatrix} \phi_1(r) \\ \phi_2(r) \\ \phi_3(r) \end{bmatrix}$$

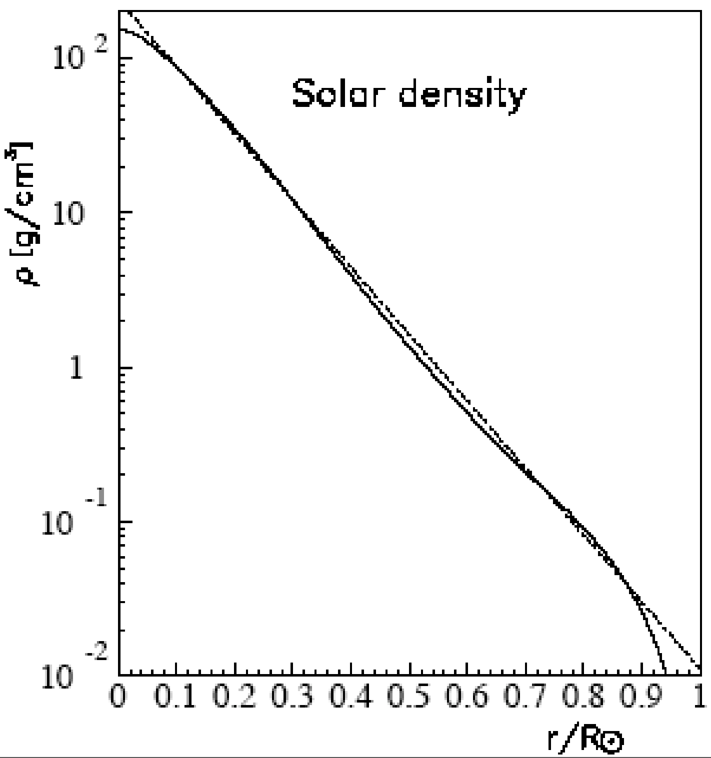
$$= \left(\begin{bmatrix} D(r) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2p} \begin{bmatrix} M^2_{11} & M^2_{12} & M^2_{13} \\ M^2_{21} & M^2_{22} & M^2_{23} \\ M^2_{31} & M^2_{32} & M^2_{33} \end{bmatrix} \right) \begin{bmatrix} \phi_1(r) \\ \phi_2(r) \\ \phi_3(r) \end{bmatrix}$$

Solar density: $D(r) = \sqrt{2} G_{\mathrm{F}} N_e(r)$

Boundary conditions: $\begin{bmatrix} \phi_1(0) \\ \phi_2(0) \\ \phi_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

For exponential density, solution in terms of ${}_2F_2(a_1, a_2; c_1, c_2; z)$

(generalized confluent hypergeometric function)



Bahcall 1998

$N_e(r) = N_e(0)\, e^{-r/r_0}$

$r_0 \simeq 0.1 \times R_\odot$

New radial variable: $u = r/r_0 + u_0$

Rotation:

$$\begin{bmatrix} \cos\theta_0 & -\sin\theta_0 \\ \sin\theta_0 & \cos\theta_0 \end{bmatrix} \frac{r_0}{2p} \begin{bmatrix} M_{22}^2 & M_{23}^2 \\ M_{32}^2 & M_{33}^2 \end{bmatrix} \begin{bmatrix} \cos\theta_0 & \sin\theta_0 \\ -\sin\theta_0 & \cos\theta_0 \end{bmatrix}$$

$$= \begin{bmatrix} \omega_2 & 0 \\ 0 & \omega_3 \end{bmatrix}$$

$$i\frac{\mathrm{d}}{\mathrm{d}u} \begin{bmatrix} \psi_1(u) \\ \psi_2(u) \\ \psi_3(u) \end{bmatrix} = \begin{bmatrix} \omega_1 + e^{-u} & \chi_2 & \chi_3 \\ \chi_2 & \omega_2 & 0 \\ \chi_3 & 0 & \omega_3 \end{bmatrix} \begin{bmatrix} \psi_1(u) \\ \psi_2(u) \\ \psi_3(u) \end{bmatrix}$$

Let μ_1, μ_2 and μ_3 be the eigenvalues of the 3×3 matrix

$$\begin{bmatrix} \omega_1 & \chi_2 & \chi_3 \\ \chi_2 & \omega_2 & 0 \\ \chi_3 & 0 & \omega_3 \end{bmatrix}$$

μ's are the squares of the neutrino masses multiplied by $r_0/(2p)$

New variable: $z = ie^{-u}$

$$\left[\left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\mu_1\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\mu_2\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\mu_3\right) - z\left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\omega_2\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z} - i\omega_3\right)\right]\psi = 0$$

$$\psi^{(1)} = e^{-i\mu_1 u}\,{}_2F_2\left[\begin{matrix} -i(\omega_2-\mu_1), & -i(\omega_3-\mu_1) \\ 1-i(\mu_2-\mu_1), & 1-i(\mu_3-\mu_1)\end{matrix}\middle| ie^{-u}\right]$$

$$\psi^{(2)} = e^{-i\mu_2 u}\,{}_2F_2\left[\begin{matrix} -i(\omega_2-\mu_2), & -i(\omega_3-\mu_2) \\ 1-i(\mu_1-\mu_2), & 1-i(\mu_3-\mu_2)\end{matrix}\middle| ie^{-u}\right]$$

$$\psi^{(3)} = e^{-i\mu_3 u}\,{}_2F_2\left[\begin{matrix} -i(\omega_2-\mu_3), & -i(\omega_3-\mu_3) \\ 1-i(\mu_1-\mu_3), & 1-i(\mu_2-\mu_3)\end{matrix}\middle| ie^{-u}\right]$$

Solutions in terms of ${}_2F_2$

Problem: Parameters (and argument) are large, ~ 20000

One (three) ${}_2F_2$ calculable by stationary phase approximation

Convert to o.d.e. for ${}_3F_1$ by $\hat{z} = z^{-1}$:

$$\left[\left(\hat{z}\,\frac{d}{d\hat{z}} - \alpha_1\right)\left(\hat{z}\,\frac{d}{d\hat{z}} - \alpha_2\right) + \hat{z}\left(\hat{z}\,\frac{d}{d\hat{z}} - \beta_1\right)\left(\hat{z}\,\frac{d}{d\hat{z}} - \beta_2\right)\left(\hat{z}\,\frac{d}{d\hat{z}} - \beta_3\right)\right] f = 0$$

Exact treatment of MSW effect for 3 flavours

⇨ Fast scan of parameter space of some model

Limitation: Exponential density

Ref: P. Osland and T. T. Wu: PRD [hep-ph/9912540]

see also E. Torrente Lujan: PRD [hep-ph/9505209]

The model

H. Lehmann, C. Newton and T. T. Wu,
Phys. Lett. **B 384** (1996) 249

"A new variant of symmetry breaking for quark mass matrices"

Quarks:

$$M(d) = \begin{pmatrix} 0 & d(d) & 0 \\ d(d) & c(d) & b(d) \\ 0 & b(d) & a(d) \end{pmatrix}$$

$$M(u) = \begin{pmatrix} 0 & id(u) & 0 \\ -id(u) & c(u) & b(u) \\ 0 & b(u) & a(u) \end{pmatrix}$$

Note *i*

➪ CP viol.

$$b^2(d) = 8c^2(d), \qquad b^2(u) = 8c^2(u)$$

The *i* can be introduced in either M(d) or M(u) - equivalent

Quarks, cont.:

Diagonalize:

$$M(d) = R(d) M_{\text{diag}}(d) R^{\text{T}}(d)$$

$$M(u) = \text{diag}(-i, 1, 1) R(u) M_{\text{diag}}(u) R^{\text{T}}(u) \text{diag}(i, 1, 1)$$

Kobayashi-Maskawa matrix:

$$V_{\text{KM}} = R^{\text{T}}(u) \text{diag}(i, 1, 1) R(d)$$

Jarlskog determinant:

$$J = \text{Im}[V_{\text{KM}}(1,1) V_{\text{KM}}(3,3) V^*_{\text{KM}}(1,3) V^*_{\text{KM}}(3,1)]$$

Quarks, cont.:

CKM matrix, PDG 2002:

$$\begin{pmatrix} 0.9742 \text{ to } 0.9757 & 0.219 \text{ to } 0.226 & 0.002 \text{ to } 0.005 \\ 0.219 \text{ to } 0.225 & 0.9734 \text{ to } 0.9749 & 0.037 \text{ to } 0.043 \\ 0.004 \text{ to } 0.014 & 0.035 \text{ to } 0.043 & 0.9990 \text{ to } 0.9993 \end{pmatrix}$$

Reproduced by LNW (1996) with

LNW solution was based on expansion in these small ratios

$$\frac{m_d}{m_s} = 0.05, \qquad \frac{m_s}{m_b} = 0.025$$

$$\frac{m_u}{m_c} = 0.004, \qquad \frac{m_c}{m_t} = 0.005$$

and $|J_{CP}| = 2.6 \times 10^{-5}$ ← Result of model

$(3.0 \pm 1.3) \times 10^{-5}$ ← BABAR

Neutrinos:

$$M(\nu) = \begin{pmatrix} 0 & d(\nu) & 0 \\ d(\nu) & c(\nu) & b(\nu) \\ 0 & b(\nu) & a(\nu) \end{pmatrix}$$

Eigenvalues given by: m_1, m_2, m_3, with $m_1 \leq m_3$

Diagonalization leads to equations:

$$\begin{aligned} a + c &= +S_1 \equiv \quad m_3 - m_2 + m_1, \\ 8c^2 + d^2 - ac &= -S_2 \equiv \quad m_3 m_2 - m_3 m_1 + m_2 m_1, \\ ad^2 &= -S_3 \equiv \quad m_1 m_2 m_3 \end{aligned}$$

Cubic equation for the parameter a

$$9a^3 - 17 S_1 a^2 + (8S_1^2 + S_2)a - S_3 = 0$$

Condition: a real and positive

Sign of *d* undetermined

Consider simple limit:

$$m_1 = m_2 = 0$$

$$9a^3 - 17m_3a^2 + 8m_3^2a = 0$$

$$a = m_3, \quad \frac{8}{9}m_3, \quad 0.$$

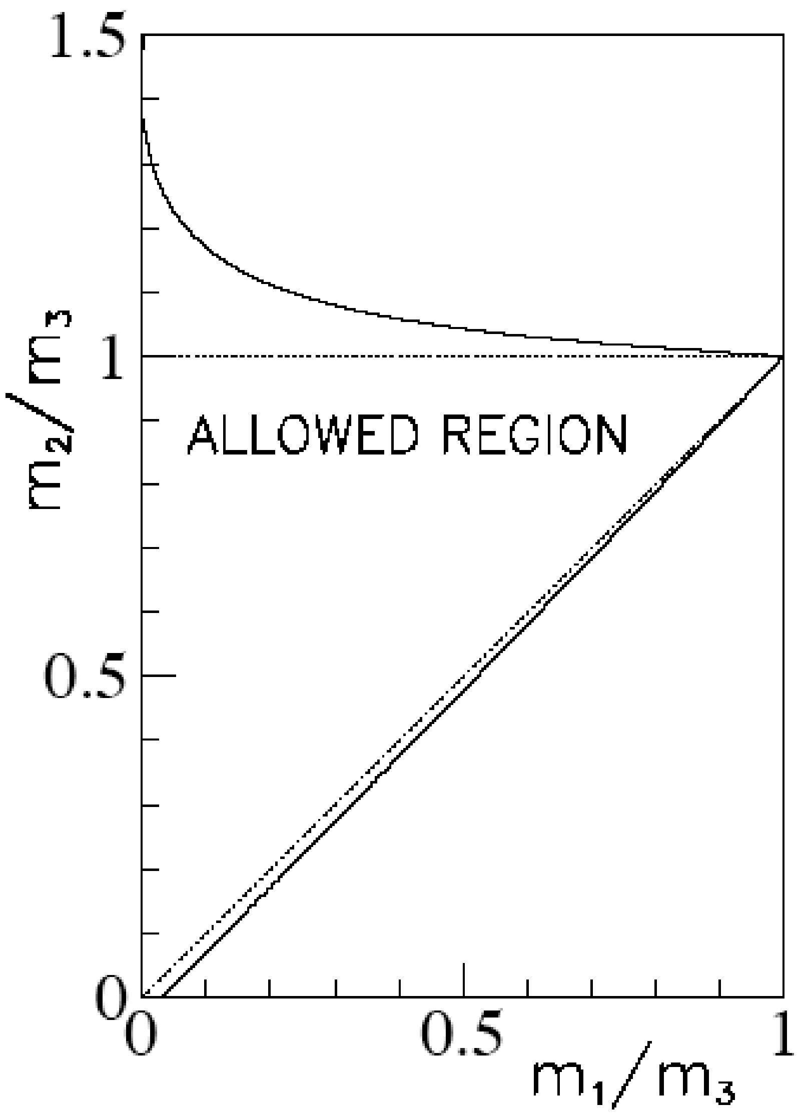
Solution 1:

$$M(\nu) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} m_3$$

Solution 2:

$$M(\nu) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{9} & \frac{2\sqrt{2}}{9} \\ 0 & \frac{2\sqrt{2}}{9} & \frac{8}{9} \end{pmatrix} m_3.$$

Allowed region:



Inspect coefficients:

a positive

⇔3 real solutions of cubic eqn

2 positive solutions for *a*

Solution 1
Solution 2

$$a \Rightarrow b, c, |d|$$

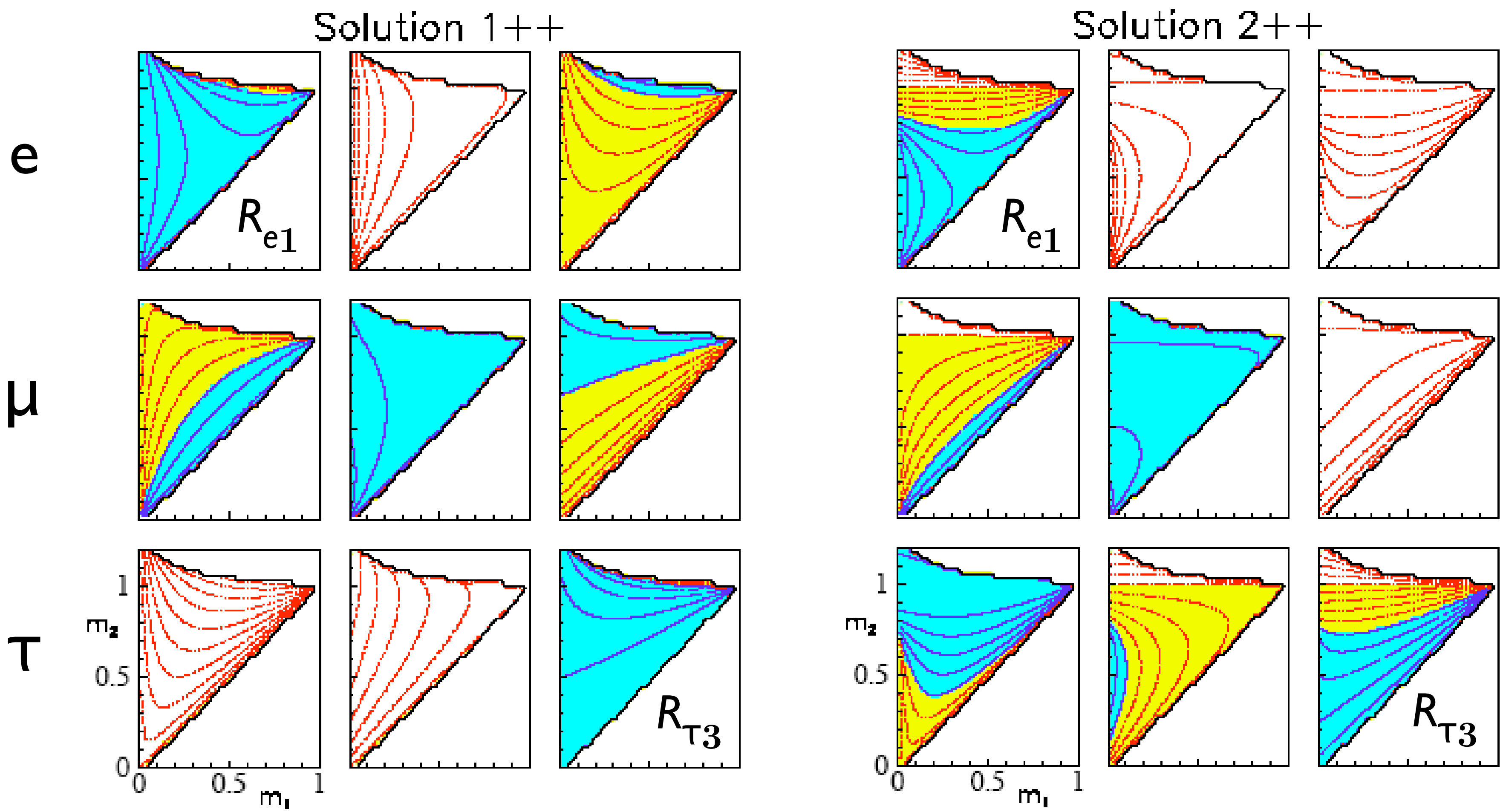
Sign of *b* undetermined

Neutrino mixing matrix:

$$U = \begin{bmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu1} & U_{\mu2} & U_{\mu3} \\ U_{\tau1} & U_{\tau2} & U_{\tau3} \end{bmatrix} = R(\nu)$$

Rotation matrix $R_{\alpha i}$ for neutrino case

Two Solutions, also sign ambiguities of *b* and *d* (consider ++ below)



blue: 0.5 < R < 1

yellow: 0 < R < 0.5

white: -1 < R < 0

Complication: need also charged-lepton mixing

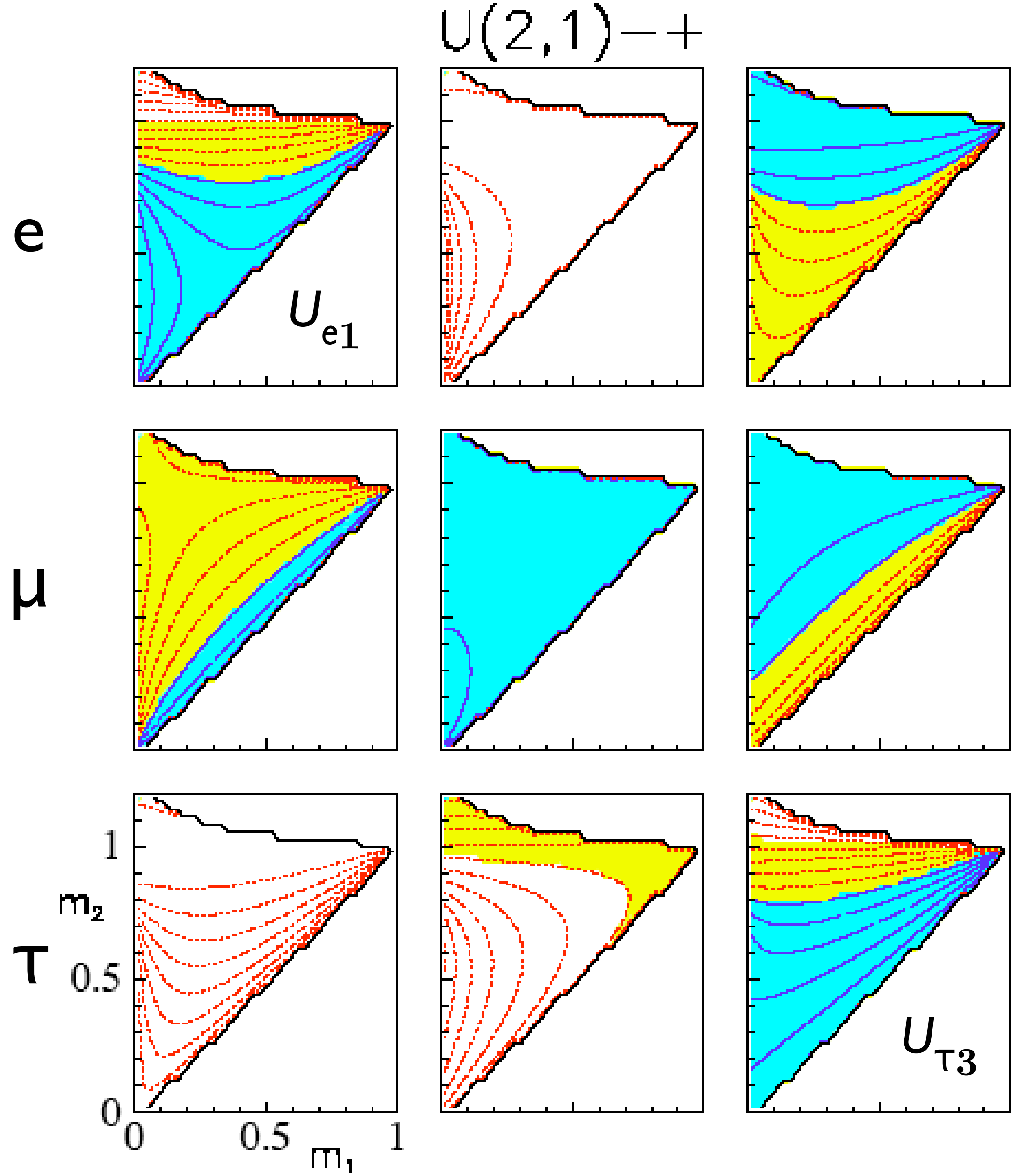
$U = R^T(\ell)\,\mathrm{diag}(\epsilon, 1, 1)\,R(\nu)$

ϵ=i, CP violation
ϵ=1, no CP violation

Two possible solutions for $R(\ell)$ and two for $R(\nu)$

Note:
U given by mass *ratios*

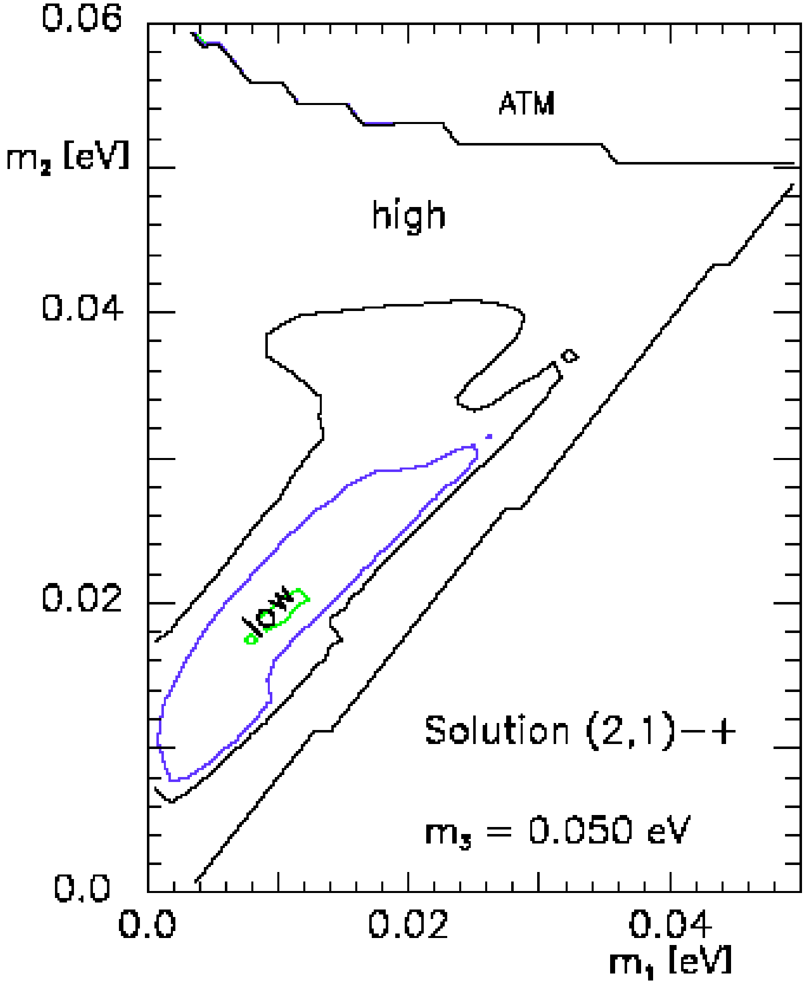
Quark case: Solution 1 considered by LNW

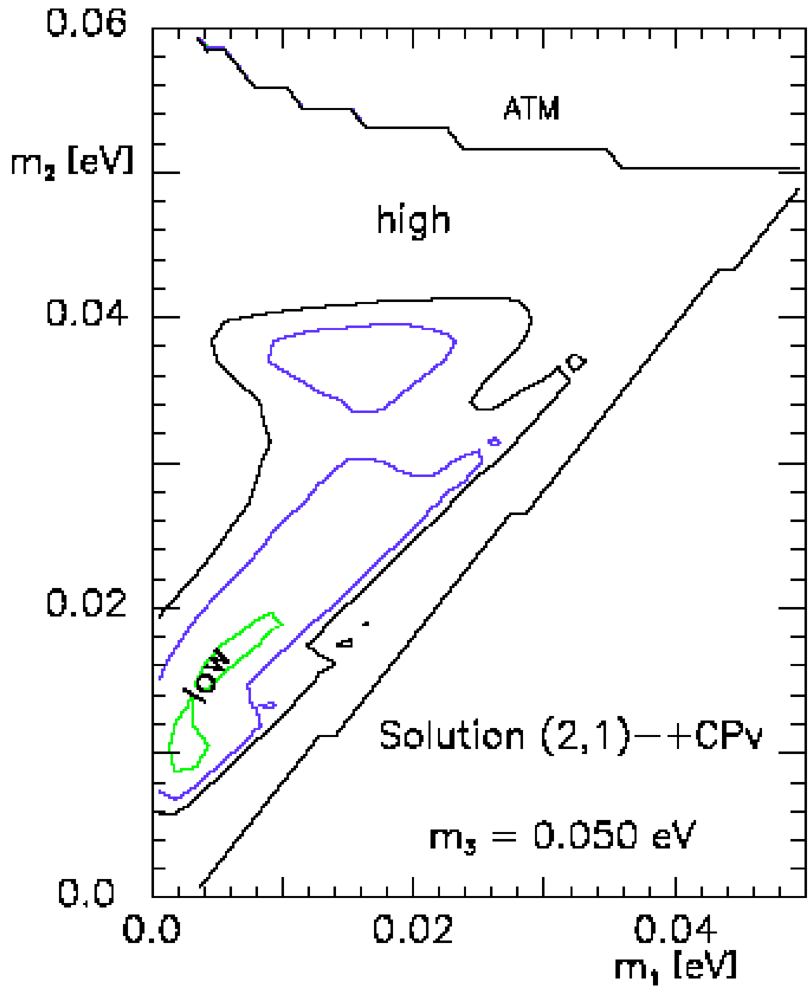


Fits to data

$$\chi^2 = \chi^2_{\rm atm} + \chi^2_{\rm solar} + \cdots$$

atmospheric data (SK: 2002) – m_3 fixed



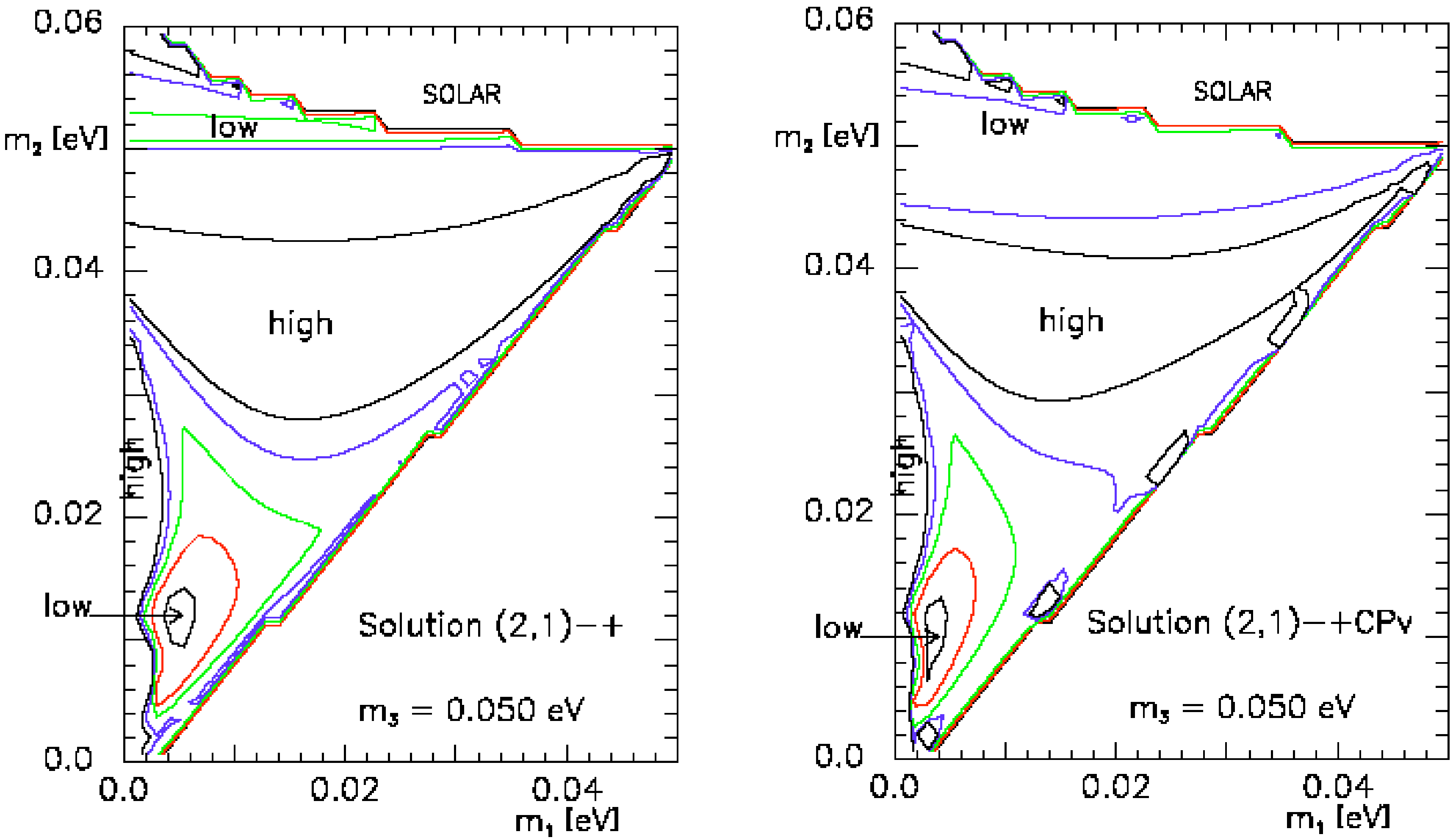


Contours are given at $\chi^2 = 5, 10, 15, 20, 25$.

solar data (2002) – m_3 fixed

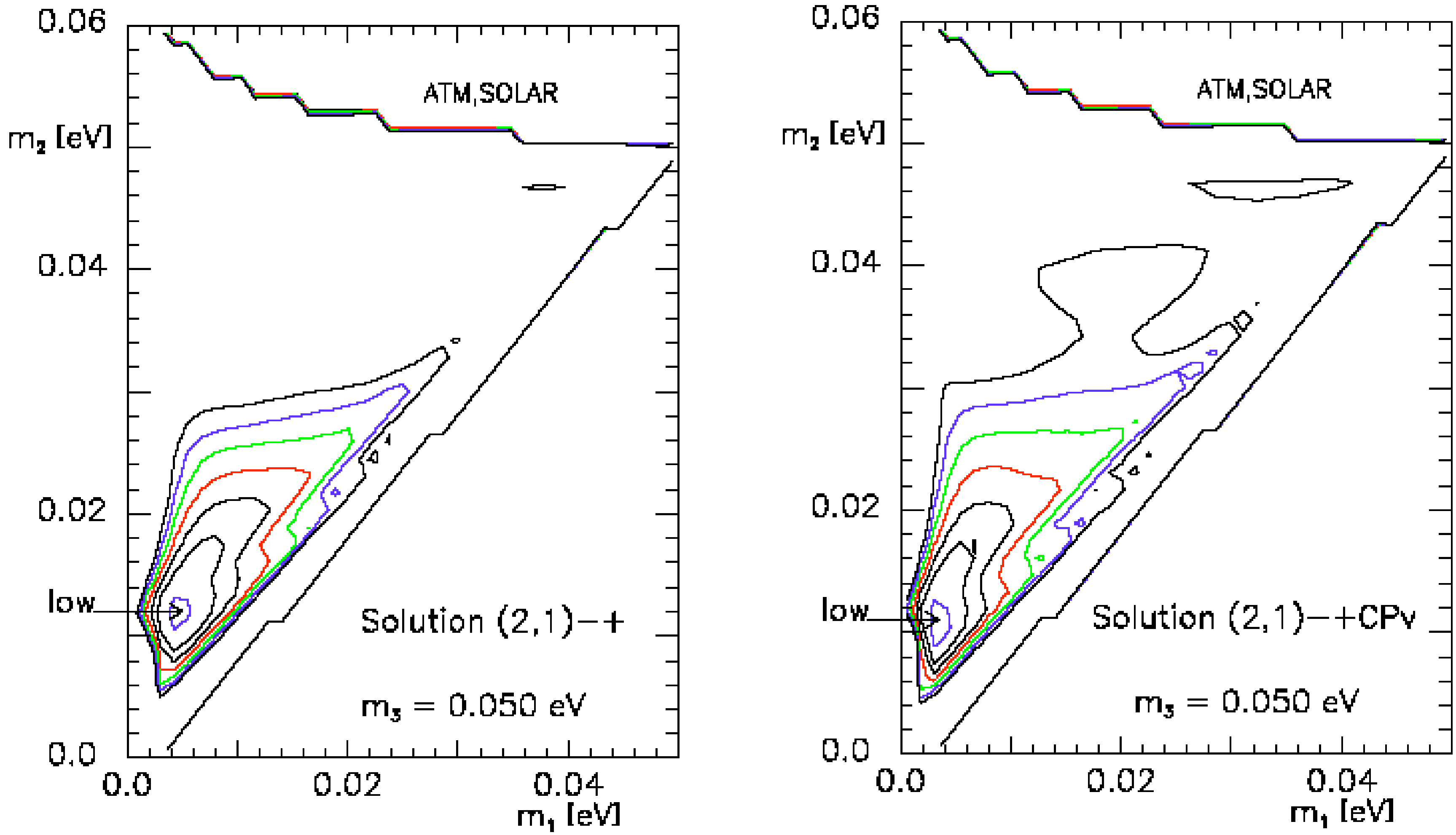
Cl, Ga, SK, SNO

- point Sun
- exponential density



Contours are given at $\chi^2 = 5, 10, 15, 20, 25$.

Sum: $\chi^2 = \chi^2_{\rm atm} + \chi^2_{\rm solar}$



Rather good fit!

TABLE III: Best fits to the atmospheric and solar neutrino data from the fitting procedure A. The number of degrees of freedom is 18, masses are in eV. R. Gastmans, P. Osland, T. T. Wu, hep-ph/0210260

Solution	m_1	m_2	m_3	$\chi^2_{\rm atm}$	$\chi^2_{\rm solar}$	χ^2	
$(1,1)-+$	0.0049	0.013	0.038	19.2	3.8	23.0	CP cons.
same	0.0036	0.012	0.037	18.0	3.7	21.7	CP non-cons.
$(1,1)--$	0.0027	0.011	0.038	17.1	3.7	20.8	CP cons.
same	0.0036	0.012	0.038	17.7	3.7	21.4	CP non-cons.
$(1,2)++$	0.0047	0.008	0.052	14.9	5.8	20.7	CP cons.
same	0.0030	0.007	0.052	15.3	5.7	21.0	CP non-cons.
$(1,2)+-$	0.0022	0.006	0.052	16.5	6.1	22.6	CP cons.
same	0.0030	0.007	0.052	16.3	5.9	22.1	CP non-cons.
$(2,1)++$	0.0045	0.008	0.052	15.1	6.1	21.2	CP cons.
same	0.0028	0.006	0.052	15.4	5.3	20.8	CP non-cons.
$(2,1)+-$	0.0021	0.006	0.052	16.7	6.0	22.7	CP cons.
same	0.0028	0.007	0.052	16.6	5.3	21.9	CP non-cons.
same	0.0029	0.007	0.095	16.3	4.5	20.8	CP non-cons.
$(2,1)-+$	0.0045	0.013	0.052	14.5	3.6	18.1	CP cons.
same	0.0035	0.011	0.053	13.8	3.6	17.3	CP non-cons.
$(2,1)--$	0.0026	0.010	0.052	12.5	3.8	16.3	CP cons.
same	0.0034	0.011	0.052	13.1	3.7	16.8	CP non-cons.

m_1 “poorly” determined

Two fitting procedures:

Procedure A: Hata & Langacker
Procedure B: ignore correlations

TABLE IV: Best fits to the atmospheric and solar neutrino data from the fitting procedure B. The number of degrees of freedom is 16, masses are in eV.

Solution	m_1	m_2	m_3	$\chi^2_{\rm atm}$	$\chi^2_{\rm solar}$	χ^2	
$(2,1)-+$	0.0047	0.011	0.054	16.2	0.8	17.0	CP cons.
same	0.0034	0.010	0.054	15.0	0.8	15.8	CP non-cons.
$(2,1)--$	0.0025	0.010	0.054	15.0	0.8	15.8	CP cons.
same	0.0035	0.010	0.054	15.3	0.8	16.1	CP non-cons.

CHOOZ data have little impact (on masses)

TABLE V: Best fits from the fitting procedure A, including the CHOOZ data [34]. Masses are in eV.

Solution	m_1	m_2	m_3	$\chi^2_{\rm atm}$	$\chi^2_{\rm solar}$	$\chi^2_{\rm CHOOZ}$	χ^2	
$(2,1)-+$	0.0044	0.013	0.052	14.5	3.6	2.9	21.0	CP cons.
same	0.0034	0.011	0.052	13.6	3.7	3.1	20.4	CP non-cons.
$(2,1)--$	0.0026	0.010	0.053	12.6	3.8	3.9	20.2	CP cons.
same	0.0034	0.011	0.052	13.1	3.6	3.1	19.9	CP non-cons.

Fit is already constrained to be in region of small U_{e3}

Mixing matrix elements

TABLE VI: Mixing matrix elements $U_{\alpha j}$ ($\alpha = e, \mu, \tau$) for the best fits, given as (modulus, phase/π). Masses are in eV.

Solution	m_1	m_2	m_3	$U_{\alpha 1}$	$U_{\alpha 2}$	$U_{\alpha 3}$	J_{CP}
$(2,1)-+$	0.0044	0.013	0.052	$(0.87, 0.00)$ $(0.39, 0.00)$ $(0.31, 1.00)$	$(0.48, 1.00)$ $(0.81, 0.00)$ $(0.33, 1.00)$	$(0.13, 0.00)$ $(0.44, 0.00)$ $(0.89, 0.00)$	0.00
$(2,1)-+$	0.0034	0.011	0.052	$(0.86, -0.49)$ $(0.43, 0.04)$ $(0.28, 1.00)$	$(0.50, 0.47)$ $(0.79, -0.01)$ $(0.34, 1.00)$	$(0.08, -0.38)$ $(0.44, 0.00)$ $(0.90, 0.00)$	0.0060
$(2,1)--$	0.0026	0.010	0.053	$(0.85, 0.00)$ $(0.46, 1.00)$ $(0.26, 0.00)$	$(0.52, 0.00)$ $(0.77, 0.00)$ $(0.36, 1.00)$	$(0.03, 1.00)$ $(0.44, 0.00)$ $(0.90, 0.00)$	0.00
$(2,1)--$	0.0034	0.011	0.052	$(0.86, -0.51)$ $(0.43, 0.96)$ $(0.28, 0.00)$	$(0.51, -0.47)$ $(0.79, 0.01)$ $(0.34, 1.00)$	$(0.08, 0.38)$ $(0.43, 0.00)$ $(0.90, 0.00)$	-0.0060

Sign convention for mixing matrix:

- main diagonal (real case) positive when $m_1 < m_2 < m_3$
- continuity

Comments

- Solution (2,1) preferred (ν: 2, charged leptons: 1)
- 'b-parity' negative
- 'd-parity': no preference
- CP: no preference

Degeneracies

For Solution 1 (which is preferred for charged leptons), if mass ratios are small, R=1

When $R(\ell)=1$, then $U=R(\nu)$, and oscillation probabilities are invariant under $d \leftrightarrow -d$ and $b \leftrightarrow -b$

Refinements

- Point source (sun)
- Exponential solar density
- More data (SNO, KamLand, etc.)

Extended sun, non-exponential (Bahcall) density: with G.Vigdel

Solution	m_1	m_2	m_3	χ^2				
				atm	solar	CZ	total	
Point sun	masses in meV							
$(2,1)++$	4.4	8	52	15.2	5.7	3.0	23.9	CP c
same	2.7	6	52	15.4	5.0	3.6	24.0	CP nc
$(2,1)-+$	4.3	12	52	14.4	3.6	2.9	20.9	CP c
same	3.2	11	52	13.5	3.6	3.2	20.3	CP nc
$(2,1)--$	2.5	10	52	12.5	3.6	3.9	20.0	CP c
same	3.3	11	52	13.0	3.6	3.2	19.8	CP nc
Extended sun								
$(2,1)++$	4.1	7	52	15.0	4.8	3.1	22.8	CP c
same	2.5	6	52	15.1	4.2	3.7	23.0	CP nc
$(2,1)-+$	4.1	11	52	15.2	4.1	2.9	22.2	CP c
same	2.9	10	52	13.6	4.1	3.4	21.1	CP nc
$(2,1)--$	2.1	10	52	12.4	3.9	4.0	20.3	CP c
same	3.0	10	52	13.3	4.2	3.3	20.8	CP nc

Very little effect

Including more recent data (in progress):

Solution	m_1	m_2	m_3	χ^2						
				atm	solar	CZ	KL	salt	total	
$(2,1) -+$	4.1	10	52	16.2	3.6	3.0	5.0		27.8	CP c
same	3.0	10	52	14.0	3.9	3.4	4.9		26.2	CP nc
$(2,1) --$	3.0	10	52	13.8	3.9	3.4	4.9		25.9	CP c
same	3.0	10	52	13.8	3.9	3.4	4.9		26.0	CP nc

Summary

Lehmann-Newton-Wu mass matrix

$$M = \begin{pmatrix} 0 & d & 0 \\ d & c & b \\ 0 & b & a \end{pmatrix} \qquad b^2 = 8\,c^2$$

- describes CKM matrix

- provides economical description of ν oscillations

Accident??

m_1 = 0.003-0.004 eV, m_2 = 0.010 eV, m_3 = 0.052 eV,