Multiparty Entanglement Routing in Quantum Networks

Vaisakh Mannalath*
Department of Physics and Materials Science & Engineering, Jaypee Institute of Information Technology, A-10, Sector 62, Noida, UP-201309, India
Vigo Quantum Communication Center, University of Vigo, Vigo E-36310, Spain and Escuela de Ingeniería de Telecomunicación, Department of Signal Theory and Communications, University of Vigo, Vigo E-36310, Spain

Anirban Pathak
Department of Physics and Materials Science & Engineering, Jaypee Institute of Information Technology, A-10, Sector 62, Noida, UP-201309, India

(Dated: May 21, 2023)

INTRODUCTION

A fundamental requirement for realizing a quantum internet is to develop algorithms for managing the entanglement present in the network and, thus, to distribute entangled states among two or more specific nodes (users) [1–3]. This easy-to-state algorithm designing problem is fundamental and challenging, and under the chosen condition, it leads to a set of related problems of interest. For example, in Ref. [4], the authors investigated whether a given multipartite state can be transformed into a set of Bell states between specific network nodes using operations restricted to single-qubit Clifford operations, single-qubit Pauli measurements, and classical communication. They showed that this specific problem is `NP-Complete`. This result highlights the difficulty of the problem at hand and the crucial need to devise better-performing protocols, at least for some specific instances relevant to multipartite schemes.

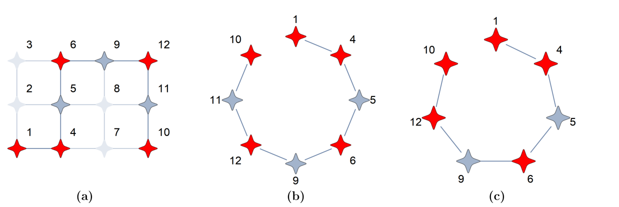

Figure 1: Isolating the desired repeater line. **(a)** A 3×4 grid network. The highlighted path connects the vertices $1, 4, 6, 12, 10$, which are to be part of the final GHZ state. Z-measurement on the vertices $3, 2, 8, 7$ isolates this path from the rest of the graph. **(b)** The isolated path. The vertex 11 is not required for the protocol, and we can remove it using an X-measurement. **(c)** The repeater line as required for applying our protocol. It contains the five nodes of the final GHZ state and extra nodes between the intermediate nodes $12, 6, 4$.

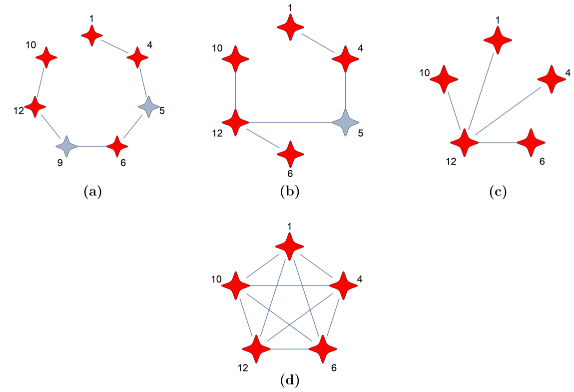

Figure 2: Extracting GHZ5 state **(a)** The isolated repeater line from Figure 1. **(b)** X-measurement on vertex 9. **(c)** X-measurement on vertex 5. **(d)** The final state obtained by performing local complementation on vertex 12.

A helpful tool used in the study of quantum networks is the notion of graph states [5, 6]. They have been employed to realize several tasks in quantum information processing, including quantum metrology [7], quantum error correcting codes [8] and one-way quantum computing [9]. Furthermore, a strong interplay between the graph theory and quantum entanglement is known, and the same has been investigated from various perspectives [6, 10]. Graph states can be generated in a network when the nodes, sharing maximally entangled pairs with nearby nodes, perform suitable entanglement-generating operations locally. Alternatively, a graph state could be prepared at one node, and subsequently the qubits may be distributed with the other nodes of the network in a manner that each node receives a qubit. Graph states have been studied extensively in the context of quantum networks [11, 12], with much of the research focused on generating them in a quantum network with varying assumptions [13, 14].

* vaisakhmannalath@gmail.com

$(\mathbf{3},\mathbf{3},\mathbf{0},\mathbf{0},\mathbf{0},\mathbf{0}) \succ (\mathbf{2},\mathbf{1},\mathbf{1},\mathbf{2},\mathbf{0},\mathbf{0}) \succ (\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1},\mathbf{1})$

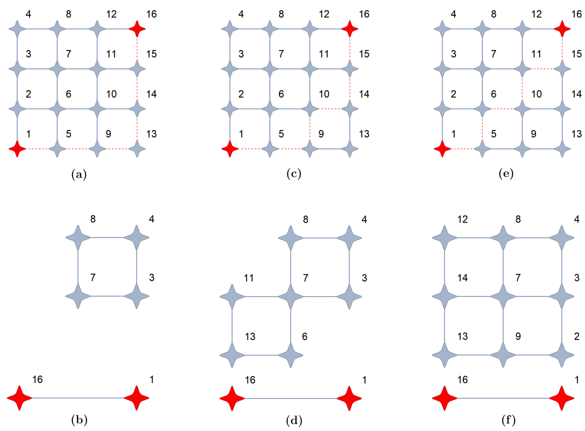

Figure 3: Entanglement routing paths and their end-products. **(a)-(b)** X protocol is performed along the path $1, 5, 9, 13, 14, 15, 16$. The path is equivalently represented by the vector $(3, 3, 0, 0, 0, 0)$. In addition to the desired Bell pair between $1, 16$, we also obtain a 2×2 grid state. **(c)-(d)** A different path that requires a lesser amount of measurements than the previous path. This is evident from the higher number of connected vertices left in the graph. **(e)-(f)** The optimal path to perform the X protocol. The path vector $(1, 1, 1, 1, 1, 1)$ corresponding to this path is majorized by every other path. We prove that this path maximizes the amount of entanglement left in the graph.

In our work, we define a new protocol for extracting maximally entangled states for any number of parties. The protocol only requires local measurements performed by the network users with access to a single qubit memory. In order to achieve this, we extensively use graph-theoretic tools in the graph state formalism of quantum networks. Our protocol can be viewed as a generalization of the results in [13], where a criterion was laid out for the extraction of four partite GHZ states. We improve upon their results and provide a criterion that works for n partite GHZ states. Moreover, we improve upon the results of *Ref.* [13] by providing a more efficient routine for establishing connections between two distant nodes of a network. We use the concept of majorization to establish a hierarchy among different paths in a network based on their efficiency. This concept utilizes the symmetry of the underlying graph state to obtain better-performing algorithms.

[1] Lu, H. *et al.* Experimental quantum network coding. *npj Quantum Information* **5**, 89 (2019). URL `https://doi.org/10.1038/s41534-019-0207-2`.

[2] Bäuml, S., Azuma, K., Kato, G. *et al.* Linear programs for entanglement and key distribution in the quantum internet. *Communications Physics* **3**, 55 (2020). URL `https://doi.org/10.1038/s42005-020-0318-2`.

[3] Mannalath, V. & Pathak, A. Entanglement routing and bottlenecks in grid networks (2023). 2211.12535.

[4] Dahlberg, A., Helsen, J. & Wehner, S. Transforming graph states to Bell-pairs is NP-Complete. *Quantum* **4**, 348 (2020). URL `https://doi.org/10.22331/q-2020-10-22-348`.

[5] Schlingemann, D. Cluster states, algorithms and graphs. *ArXiv e-prints* (2003). URL `https://doi.org/10.48550/arXiv.quant-ph/0305170`. quant-ph/0305170.

[6] Hein, M., Eisert, J. & Briegel, H. J. Multiparty entanglement in graph states. *Physical Review A* **69**, 062311 (2004). URL `https://doi.org/10.1103/PhysRevA.69.062311`.

[7] Shettell, N. & Markham, D. Graph States as a Resource for Quantum Metrology. *Physical Review Letters* **124**, 110502 (2020). URL `https://doi.org/10.1103/PhysRevLett.124.110502`.

[8] Schlingemann, D. & Werner, R. F. Quantum error-correcting codes associated with graphs. *Physical Review A* **65**, 012308 (2001). URL `https://doi.org/10.1103/PhysRevA.65.012308`.

[9] Raussendorf, R. & Briegel, H. J. A One-Way Quantum Computer **86**, 5188–5191 (2001). URL `https://doi.org/10.1103/PhysRevLett.86.5188`.

[10] Adcock, J. C., Morley-Short, S., Dahlberg, A. *et al.* Mapping graph state orbits under local complementation. *Quantum* **4**, 305 (2020). URL `https://doi.org/10.22331/q-2020-08-07-305`.

[11] Epping, M., Kampermann, H. & Bruß, D. Large-scale quantum networks based on graphs. *New Journal of Physics* **18**, 053036 (2016). URL `https://doi.org/10.1088/1367-2630/18/5/053036`.

[12] Pirker, A. & Dür, W. A quantum network stack and protocols for reliable entanglement-based networks. *New Journal of Physics* **21**, 033003 (2019). URL `https://doi.org/10.1088/1367-2630/ab05f7`.

[13] Hahn, F., Pappa, A. & Eisert, J. Quantum network routing and local complementation. *npj Quantum Information* **5**, 76 (2019). URL `https://doi.org/10.1038/s41534-019-0191-6`.

[14] Pirker, A., Wallnöfer, J. & Dür, W. Modular architectures for quantum networks. *New Journal of Physics* **20**, 053054 (2018). URL `https://doi.org/10.1088/1367-2630/aac2aa`.