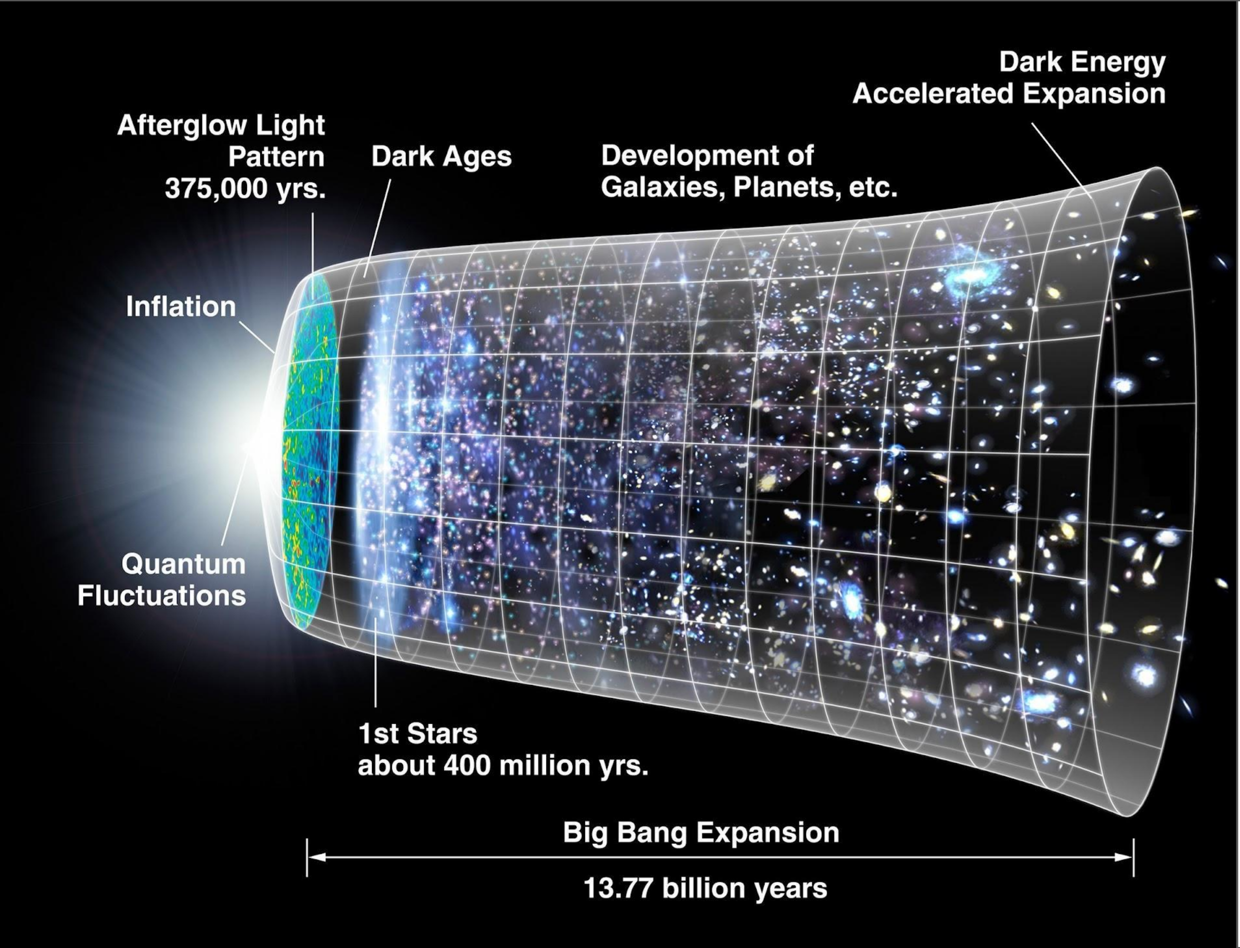
Cosmology

TAE 2022, Benasque

David Alonso - University of Oxford

Cosmology



Cosmology:
"Study of the origin, evolution, and fate of the Universe on large scales"

- Extreme physical systems (large scales, high-energies)
- Fundamental physics problems:
  * Dark matter
  * Dark energy
  * Inflation
- <u>Data-driven science</u> since ~2000
  Confronted with astrophysical questions more and more often.
  We dabble in astrophysics!

Cosmology

Lesson 1: background cosmology and Newtonian perturbations

a) **Homogeneous cosmology.** The FRW metric. Distances and redshift. The Friedman Equation.
b) **Newtonian perturbations.** The perturbation equations. Linear theory. Jeans equation and growth.
c) **Relativistic perturbations.** Gauge-invariant PT. Hydrodynamic perturbations. Qualitative behavior.

Lesson 2: Inflation

a) **Inflation.** The curvature and horizon problems. Scalar fields. Slow roll.
b) **Perturbations from inflation.** Inflationary PT. Random fields. The primordial power spectrum.

Lesson 3: cosmological probes of structure

a) **The CMB.** Recombination. Temperature anisotropies. Scattering and polarization.
b) **The matter power spectrum.** The linear power spectrum. Non-linearities
c) **Gravitational lensing.** Geodesics. Galaxy lensing. CMB lensing.

Lecture notes: https://www.overleaf.com/read/gdndjchkksnq

Books:

- Mukhanov: "*Physical foundations of cosmology*"
- Dodelson: "*Modern cosmology*"
- Mo, van den Bosch & White: "*Galaxy formation and evolution*"

Outline

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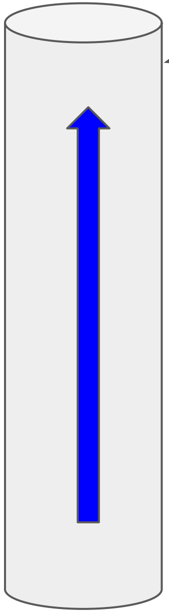
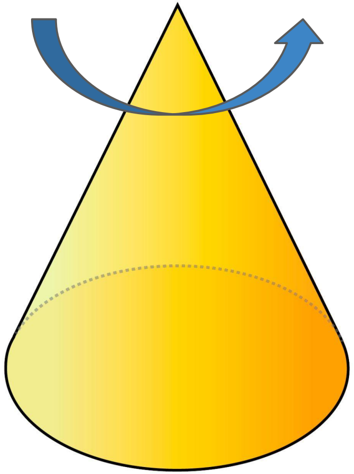
Lesson 1 a) Homogeneous cosmology

The cosmological principle:

"On sufficiently large scales, the Universe is <u>homogeneous</u> and <u>isotropic</u>"





Lesson 1 a) Homogeneous cosmology

The cosmological principle:

"On sufficiently large scales, the Universe is <u>homogeneous</u> and <u>isotropic</u>"

In math: on large scales the Universe has maximally-symmetric time slices.

$$d\tau^2 = b^2(t)dt^2 - a^2(t)S_{ij}dx^i dx^j\,,$$

Proof: Weinberg "Gravitation and Cosmology"

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$$\boxed{\mathrm{sinn}(\chi)} = \begin{cases} k^{-1/2}\sin(\sqrt{k}\chi) & k > 0 \\ \chi & k = 0 \\ |k|^{-1/2}\sinh(\sqrt{-k}\chi) & k < 0 \end{cases}$$

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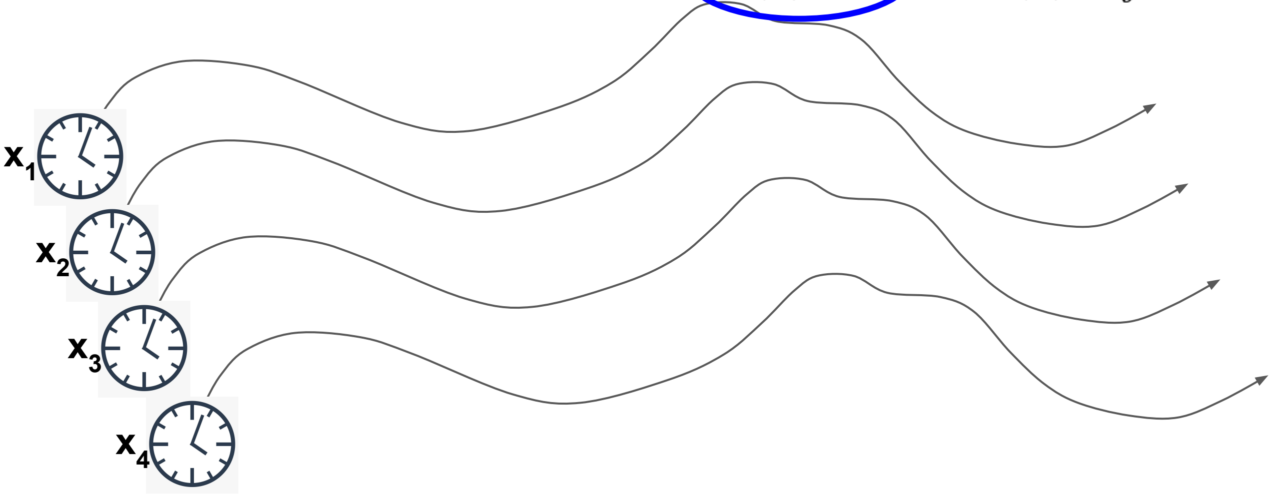

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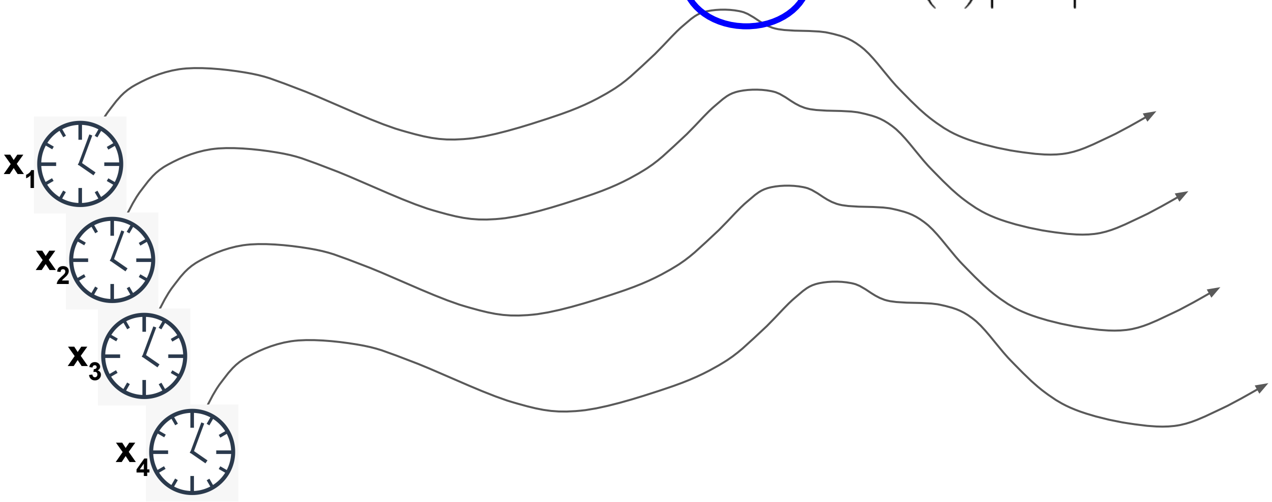

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Scale factor

Radial comoving distance

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$$d\tau^2 = a^2(\eta)\left(d\eta^2 - |d\mathbf{x}|^2\right).$$

$$\dot{a} \equiv \frac{da}{dt}, \quad a' \equiv \frac{da}{d\eta}$$

Conformal time

Lesson 1 a) Homogeneous cosmology

Photon propagation in an expanding Universe.

Geodesic equation:

$$\frac{dp^\mu}{d\lambda} + \Gamma^\mu_{\nu\sigma} p^\nu p^\sigma = 0 \qquad p^\mu \equiv \frac{dx^\mu}{d\lambda}$$

For μ=0:

$$\frac{dp^0}{d\lambda}\left(\frac{dt}{d\lambda}\right) = -a\dot{a}\left(p^\chi\right)^2$$

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For photons dτ^2=0, which implies:

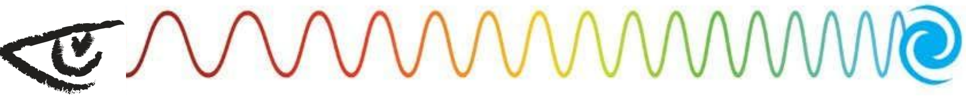
$$p^0 = ap^\chi$$

Therefore:

$$\frac{dp^0}{d\lambda} = -Hp^0\frac{dt}{d\lambda}, \qquad H \equiv \frac{\dot{a}}{a}$$

Integrating:

$$p^0 \propto \nu \propto a^{-1}$$



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Expansion rate

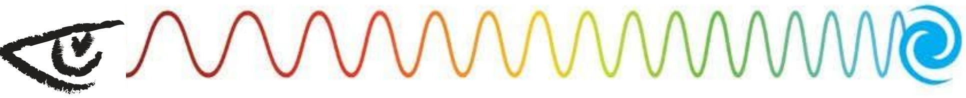
Integrating: $p^0 \propto \nu \propto a^{-1}$

Defining redshift:

$$z \equiv \frac{\lambda - \lambda_{\rm em}}{\lambda_{\rm em}}$$

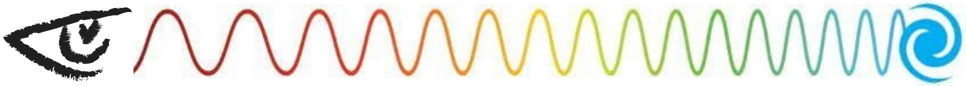
Relation to scale factor:

$$a(t) = \frac{1}{1+z}$$



Lesson 1 a) Homogeneous cosmology

Radial photon geodesics:



$$ds^2 = 0 \quad \longrightarrow \quad \chi = \int \frac{dt}{a} = \int_0^z \frac{dz}{H(z)}$$

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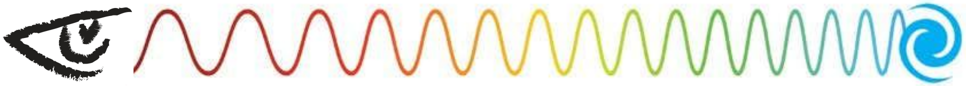

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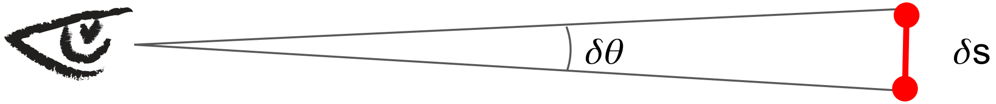
Standard rulers:



$$\delta s = d_A \, \delta\theta \rightarrow d_A(z) = \frac{\chi(z)}{1+z}$$

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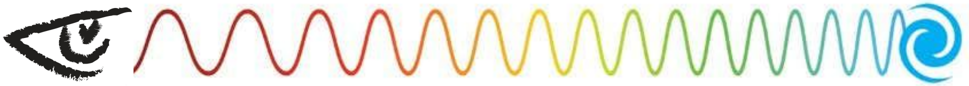

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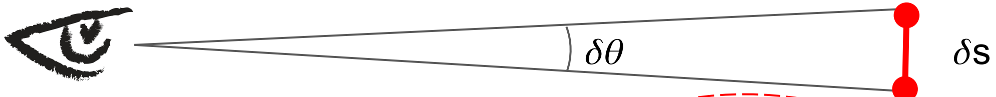

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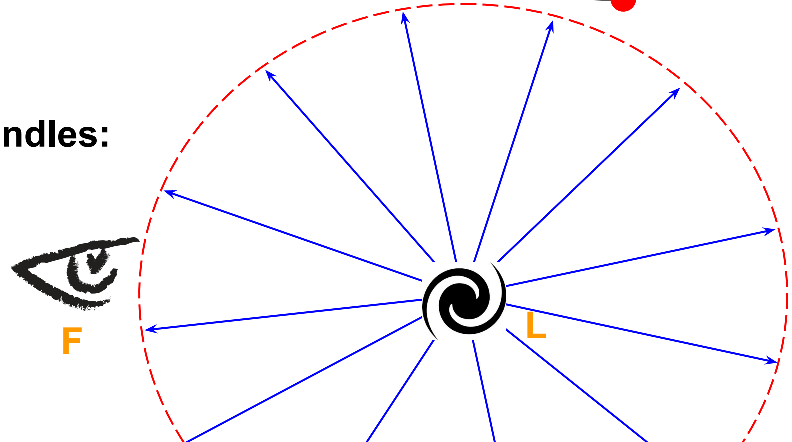
Standard candles:



$$F = \frac{L}{4\pi d_L^2} \longrightarrow d_L = (1+z)\chi(z)$$

Lesson 1 a) Homogeneous cosmology

Ideal fluid: $$T_{\mu\nu} = (\rho + p)U_\mu U_\nu - p g_{\mu\nu}, \qquad G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

Fluid velocity. In comoving coords: $U_\mu = (1, 0, 0, 0)$

Pressure in comoving frame: $p = \frac{T_{\mu\nu}}{3}\left[U^\mu U^\nu - g^{\mu\nu}\right]$

Energy density in comoving frame: $\rho \equiv T_{\mu\nu} U^\mu U^\nu$

In comoving coords: $$T^\mu_\nu = \mathrm{diag}(\rho, -p, -p, -p)$$

Discards heat conduction, shear and bulk viscosity.

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Energy-momentum conservation: $$\nabla_\mu T^\mu_\nu = 0$$

- Holds for any non-interacting species
- And for the overall fluid
- With ν=0, energy conservation $\longrightarrow$ $\dot{\rho} + 3H(\rho + p) = 0$

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- Non-relativistic matter (dust): $w_M = 0 \;\rightarrow\; \rho_M \propto a^{-3}$ $\longleftarrow$ dilution
- Relativistic matter (radiation): $w_R = 1/3 \;\rightarrow\; \rho_R \propto a^{-4}$ $\longleftarrow$ dilution + redshifting
- Cosmological constant (vacuum): $\rho_\Lambda = \text{const.} \;\rightarrow\; w_\Lambda = -1$

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Natural scenario:

- R dominates at early times
- Then M takes over
- Finally Λ dominates over everything else.

Lesson 1 a) Homogeneous cosmology

The (0,0) component of the Einstein equations yields the **1st Friedmann eq.**:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$H^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}$$

- Equation analogous to expansion of a gas in Newtonian gravity.
- Expansion (or "Hubble") rate: $H \equiv \frac{\dot{a}}{a}$

Later we will also use:

$$\mathcal{H} \equiv \frac{a'}{a} = aH$$

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- Critical density: $k = 0 \;\rightarrow\; \rho_c = \frac{3H^2}{8\pi G}$
- **Cosmological parameters:** $\Omega_i \equiv \frac{\rho_{i,0}}{\rho_c}, \quad \Omega_k = -\frac{k}{H_0^2}$

Since k~0, ϱ_{tot} ~ ϱ_c

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- Critical density: $k = 0 \;\rightarrow\; \rho_c = \frac{3H^2}{8\pi G}$ Since k~0, $\varrho_{tot} \sim \varrho_c$
- **Cosmological parameters:** $\Omega_i \equiv \frac{\rho_{i,0}}{\rho_c}, \quad \Omega_k = -\frac{k}{H_0^2}$
- Using energy conservation, Friedman eq. reads:

$$H^2 = H_0^2 \sum_i \Omega_i \left(1+z\right)^{3(1+w_i)}$$

- **Specific solutions:**
  Radiation domination: $a \propto t^{1/2} \propto \eta$
  Matter domination: $a \propto t^{2/3} \propto \eta^2$
  Dark-energy domination: $a \propto e^{Ht}$ "de-Sitter" Universe

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Lesson 1 a) Homogeneous cosmology

The spatial components lead to **2nd Friedmann eq.**:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)$$

- Not independent of 1st Eq. + conservation of energy.

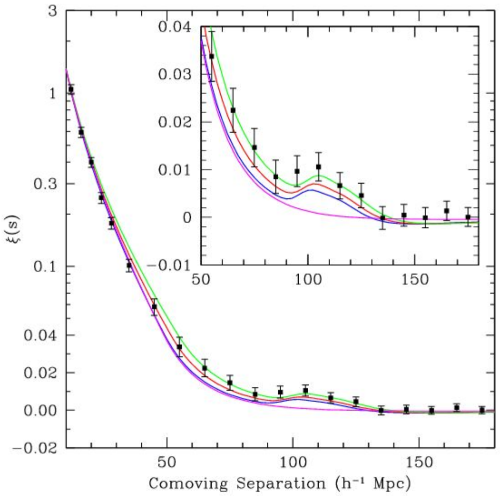
- Interesting consequence: $w < -1/3 \ \rightarrow \ \ddot{a} > 0$

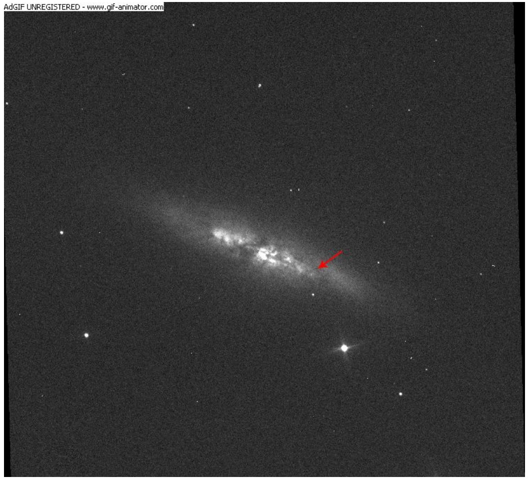
Lesson 1 a) Homogeneous cosmology

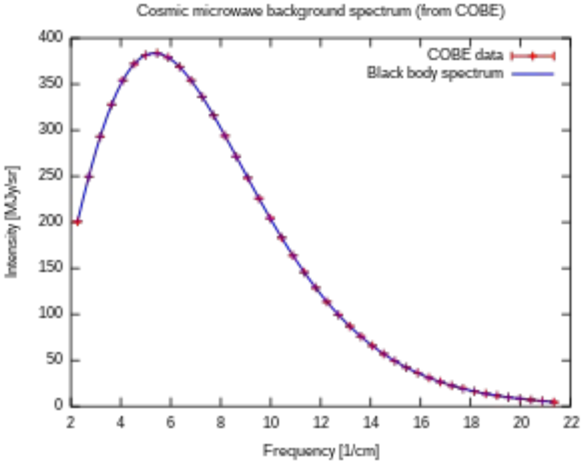
Background cosmology data:

- BAO (standard ruler)
- SNe (standard candles)
- BBN (baryon abundance)
- T_{CMB} (from CMB spectrum)

$$\Omega_M \sim 0.3, \;\; \Omega_\Lambda \sim 0.7, \;\; \Omega_b \sim 0.05$$
$$\Omega_R \sim 8 \times 10^{-5}, \;\; \Omega_k \leq 10^{-3}, \;\; H_0 \sim 70\,\mathrm{km/s/Mpc}$$







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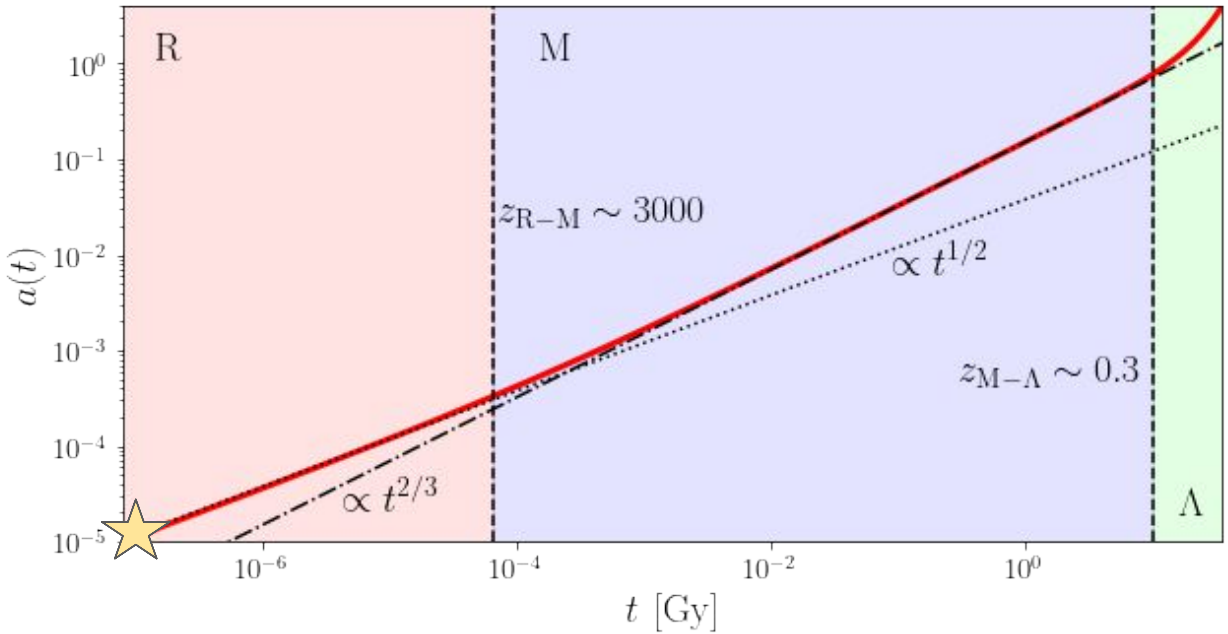

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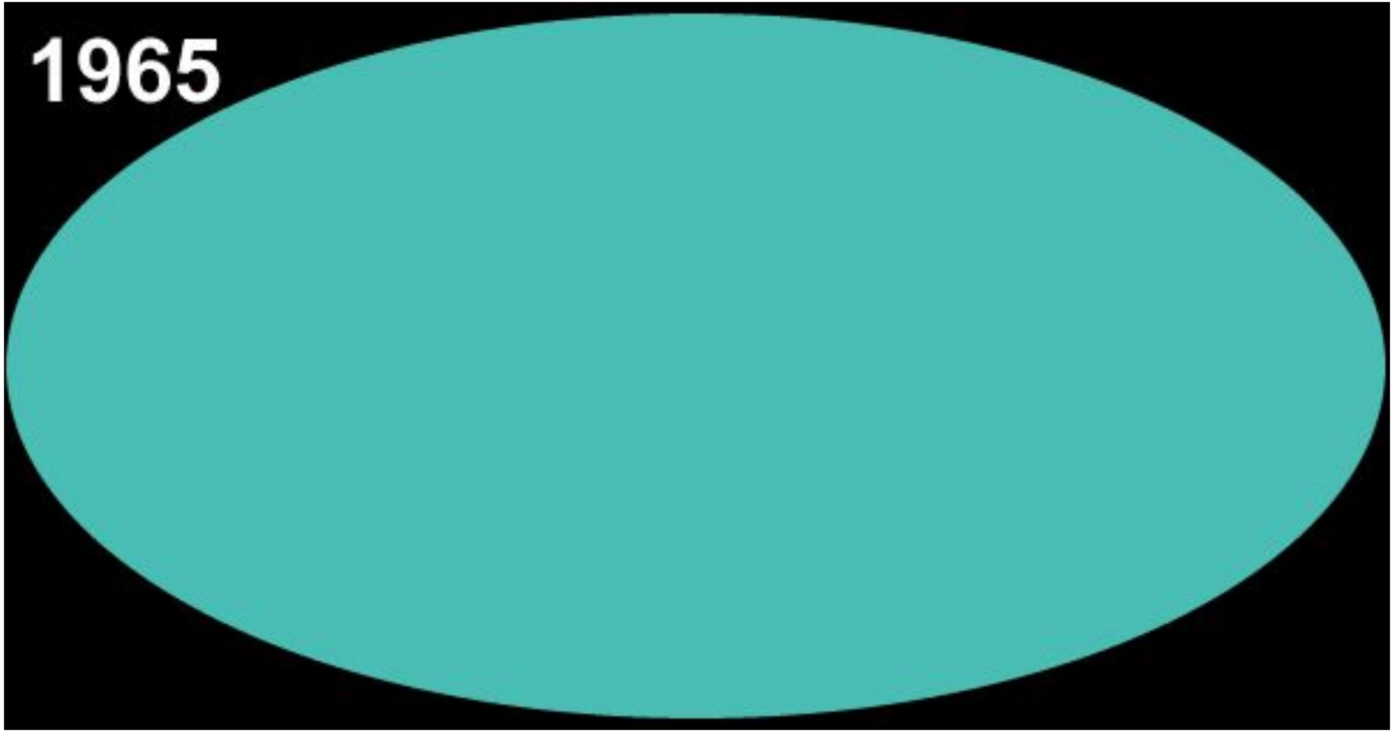
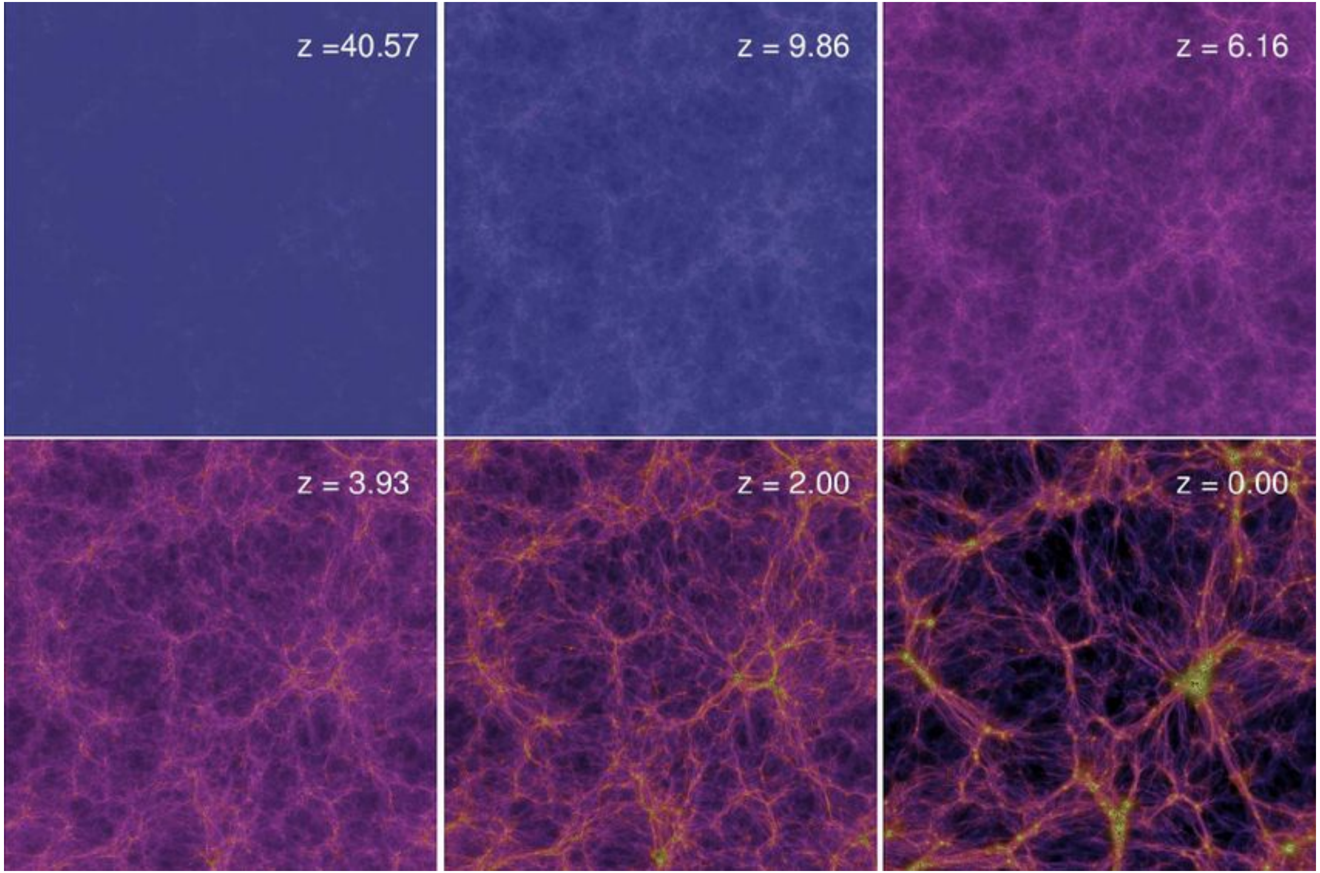
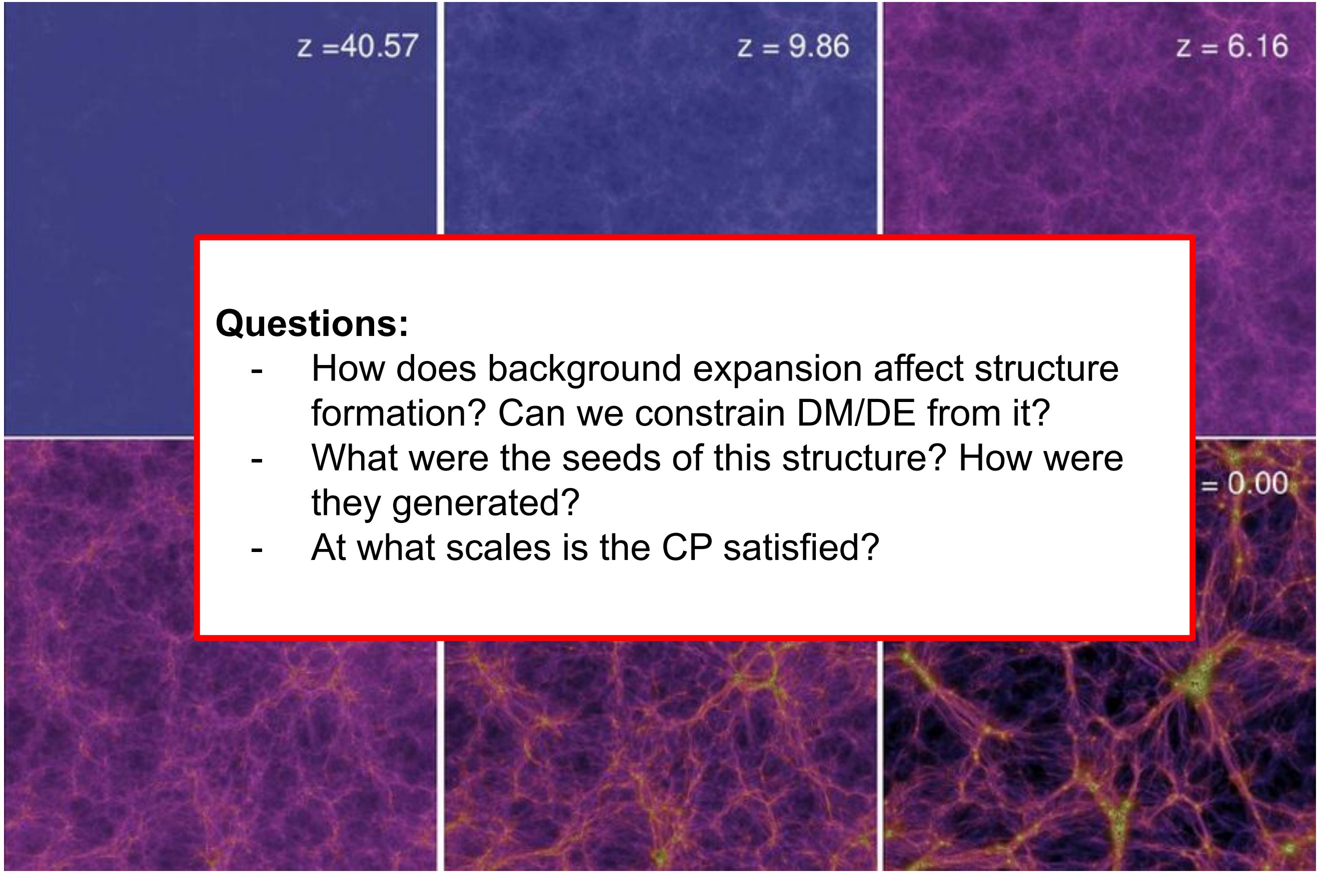


Lesson 1 b) Newtonian perturbations



Credit: Zhao et al. 2012

Lesson 1 b) Newtonian perturbations



Questions:

- How does background expansion affect structure formation? Can we constrain DM/DE from it?
- What were the seeds of this structure? How were they generated?
- At what scales is the CP satisfied?

Credit: Zhao et al. 2012

Lesson 1 b) Newtonian perturbations

Newtonian perturbation theory:

- Simplified treatment that forgoes all complications associated with GR.
- Idea: perturbations in non-relativistic fluid in an expanding background.
- Not valid when
  - Perturbations in relativistic fluid (e.g. radiation at early times).
  - Scales comparable to the horizon.
- Good approximation when studying structure at late times on most scales!

Lesson 1 b) Newtonian perturbations

Newtonian fluid characterised by a density $\rho(\mathbf{r}, t)$ and velocity field $\mathbf{V}(\mathbf{r}, t)$ in Eulerian coordinates **r**.

Evolution governed by 2 equations of motion:

- Conservation of mass (continuity eq.): $$\partial_t \rho + \nabla_r \cdot (\rho \mathbf{V}) = 0$$
- 2nd Newton's law (Euler eq.): $$\partial_t \mathbf{V} + (\mathbf{V} \cdot \nabla_{\mathbf{r}})\mathbf{V} + \frac{\nabla_{\mathbf{r}} p}{\rho} + \nabla_{\mathbf{r}} \Psi = 0$$

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Relation between density and gravity (Poisson's eq.):

$$\nabla_\mathbf{r}^2\Psi = 4\pi G\rho.$$

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$$\nabla^2_{\mathbf{r}} \Psi = 4\pi G \rho.$$

Relation between pressure and density (eq. of state):

$$w \equiv p/\rho, \quad c_s^2 \equiv dp/d\rho$$

Sound speed

Lesson 1 b) Newtonian perturbations

Introducing **background expansion**:

1. Change to comoving coordinates

$$\mathbf{r} = a(t)\mathbf{x}, \quad \nabla_r = a^{-1}\nabla_x, \quad \partial_t|_r = \partial_t|_x - H\mathbf{x}\cdot\nabla_x$$

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$$\mathbf{r} = a(t)\mathbf{x}, \quad \nabla_r = a^{-1}\nabla_x, \quad \partial_t|_r = \partial_t|_x - H\mathbf{x}\cdot\nabla_x$$

2. Split fields into background and perturbations:

$$\rho(\mathbf{x},t) = \bar{\rho}(t)\left[1+\delta(\mathbf{x},t)\right], \quad \mathbf{V}(\mathbf{x},t) = \dot{a}\mathbf{x} + \mathbf{v}$$

$$\Psi = \bar{\Psi} + \psi(\mathbf{x},t), \;\; p = \bar{p}(t) + c_s^2\bar{\rho}(t)\delta(\mathbf{x},t)$$

Background follows Friedmann eqs.

Lesson 1 b) Newtonian perturbations

Introducing **background expansion**:

1. Change to comoving coordinates

$$\mathbf{r} = a(t)\mathbf{x}, \quad \nabla_r = a^{-1}\nabla_x, \quad \partial_t|_r = \partial_t|_x - H\mathbf{x}\cdot\nabla_x$$

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   Background follows Friedmann eqs.

3. Substitute and isolate contribution from perturbations:

$$\dot{\delta} + a^{-1}\nabla\cdot((1+\delta)\mathbf{v}) = 0$$

$$\dot{\mathbf{v}} + H\mathbf{v} + \frac{c_s^2}{a}\nabla\delta + \frac{1}{a}\nabla\psi = 0$$

$$\nabla^2\psi = 4\pi G a^2\bar{\rho}\delta$$

Lesson 1 b) Newtonian perturbations

In comoving coords we can now take advantage of translational invariance. **Fourier transform**:

$$f_{\mathbf{k}}(t) \equiv \int d^3x\, e^{-i\mathbf{k}\cdot\mathbf{x}} f(\mathbf{x},t), \quad f(\mathbf{x},t) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} f_{\mathbf{k}}(t)$$

Makes gradients easier:

$$\nabla f(\mathbf{x},t) \;\rightarrow\; i\mathbf{k} f(\mathbf{k},t)$$

Lesson 1 b) Newtonian perturbations

Consider small perturbations and linearise. Keep only terms linear in δ, $\mathbf{v}$, and ϕ

$$\dot{\delta}_{\mathbf{k}} + \frac{i}{a}\mathbf{k} \cdot \mathbf{v}_{\mathbf{k}} = 0$$

$$\dot{\mathbf{v}}_{\mathbf{k}} + H\mathbf{v}_{\mathbf{k}} + \frac{i}{a}\mathbf{k}\left[c_s^2\delta_{\mathbf{k}} + \psi_{\mathbf{k}}\right] = 0$$

$$k^2\psi_{\mathbf{k}} = -4\pi G a^2\bar{\rho}\delta_{\mathbf{k}}$$

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Vorticity

Split Euler equation into longitudinal and transverse modes:

$$\mathbf{v}_{\mathbf{k}} = v_{||}\hat{\mathbf{k}} + \mathbf{v}_{\perp}, \quad \hat{\mathbf{k}}\cdot\mathbf{v}_{\perp} = 0$$

Gradient-like

Curl-like

Lesson 1 b) Newtonian perturbations

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Vorticity

Split Euler equation into longitudinal and transverse modes:

$$\mathbf{v}_{\mathbf{k}} = v_{\parallel}\hat{\mathbf{k}} + \mathbf{v}_{\perp}, \quad \hat{\mathbf{k}}\cdot\mathbf{v}_{\perp} = 0$$

Gradient-like   Curl-like

<u>No source for transverse modes at linear level:</u>

$$\dot{\mathbf{v}}_{\perp} + H\mathbf{v}_{\perp} = 0, \quad \rightarrow \quad \mathbf{v}_{\perp} \propto \frac{1}{a}$$

We can disregard transverse modes and focus only on $v_{\parallel}$

Non-linear evolution will create vorticity.

Lesson 1 b) Newtonian perturbations

Jeans equation

Take divergence of Euler eq. and substitute Poisson eq.

$$\dot{\delta}_{\mathbf{k}} + \frac{i}{a}\mathbf{k} \cdot \mathbf{v}_{\mathbf{k}} = 0$$

$$\boxed{\begin{aligned} &\dot{\mathbf{v}}_{\mathbf{k}} + H\mathbf{v}_{\mathbf{k}} + \frac{i}{a}\mathbf{k}\left[c_s^2\delta_{\mathbf{k}} + \psi_{\mathbf{k}}\right] = 0 \\ &k^2\psi_{\mathbf{k}} = -4\pi G a^2 \bar{\rho}\delta_{\mathbf{k}} \end{aligned}}$$

Lesson 1 b) Newtonian perturbations

Jeans equation

Take divergence of Euler eq. and substitute Poisson eq.

$$\theta_{\mathbf{k}} \equiv i\mathbf{k} \cdot \mathbf{v}_{\mathbf{k}}$$

$$\dot{\delta}_{\mathbf{k}} + \frac{1}{a}\theta_{\mathbf{k}} = 0$$

$$\dot{\theta}_{\mathbf{k}} + H\theta_{\mathbf{k}} + \frac{1}{a}\left[4\pi G a^2 \bar{\rho} - c_s^2 k^2\right]\delta_{\mathbf{k}} = 0$$

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$$\theta_{\mathbf{k}} \equiv i\mathbf{k}\cdot\mathbf{v}_{\mathbf{k}}$$

Finally, sub in continuity eq.

Lesson 1 b) Newtonian perturbations

Jeans equation

Take divergence of Euler eq. and substitute Poisson eq., sub in continuity eq.

$$\ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} + \frac{c_s^2}{a^2}\left(k^2 - k_J^2\right)\delta = 0$$

<u>Jeans scale</u> $k_J \equiv \frac{a}{c_s}\sqrt{4\pi G\bar{\rho}}$ separates behaviour into two regimes:

1. Small scales ($k >> k_J$) $\quad \ddot{\delta}_{\mathbf{k}} + 2H\dot{\delta}_{\mathbf{k}} + \frac{c_s^2}{a^2}k^2\delta_{\mathbf{k}} = 0$

?? ??

Lesson 1 b) Newtonian perturbations

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Damped Oscillations

$$\delta_{\mathbf{k}} \propto \frac{1}{\sqrt{c_s a}}\exp\left[\pm ik\int\frac{dt}{a}c_s\right]$$

Remember this!!

Pressure waves!

Lesson 1 b) Newtonian perturbations

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2. Large scales or pressureless: $\frac{d}{da}\left(a^3 H \frac{d\delta}{da}\right) = \frac{3}{2}\Omega_M(a)\, a\, H(a)\,\delta$

   Scale-independent growth!
   Solutions in the form: $\delta(\mathbf{k},t) = \delta_+(\mathbf{k})D_+(a) + \delta_-(\mathbf{k})D_-(a)$

Growing mode

Decaying mode

Lesson 1 b) Newtonian perturbations

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Decaying mode

Lesson 1 b) Newtonian perturbations

Jeans equation

Examples:

1. **Matter domination**: $\frac{d^2\delta}{da^2} + \frac{3}{2a}\frac{d\delta}{da} - \frac{3}{2a^2}\delta = 0$

   Solution:

Lesson 1 b) Newtonian perturbations

Jeans equation

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Lesson 1 b) Newtonian perturbations

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2. **Λ domination:** $$\frac{d^2\delta}{da^2} + \frac{3}{a}\frac{d\delta}{da} = 0$$

   Solution: $\delta(a) = \delta_+ + \delta_- a^{-3}$ δ stalls

Lesson 1 b) Newtonian perturbations

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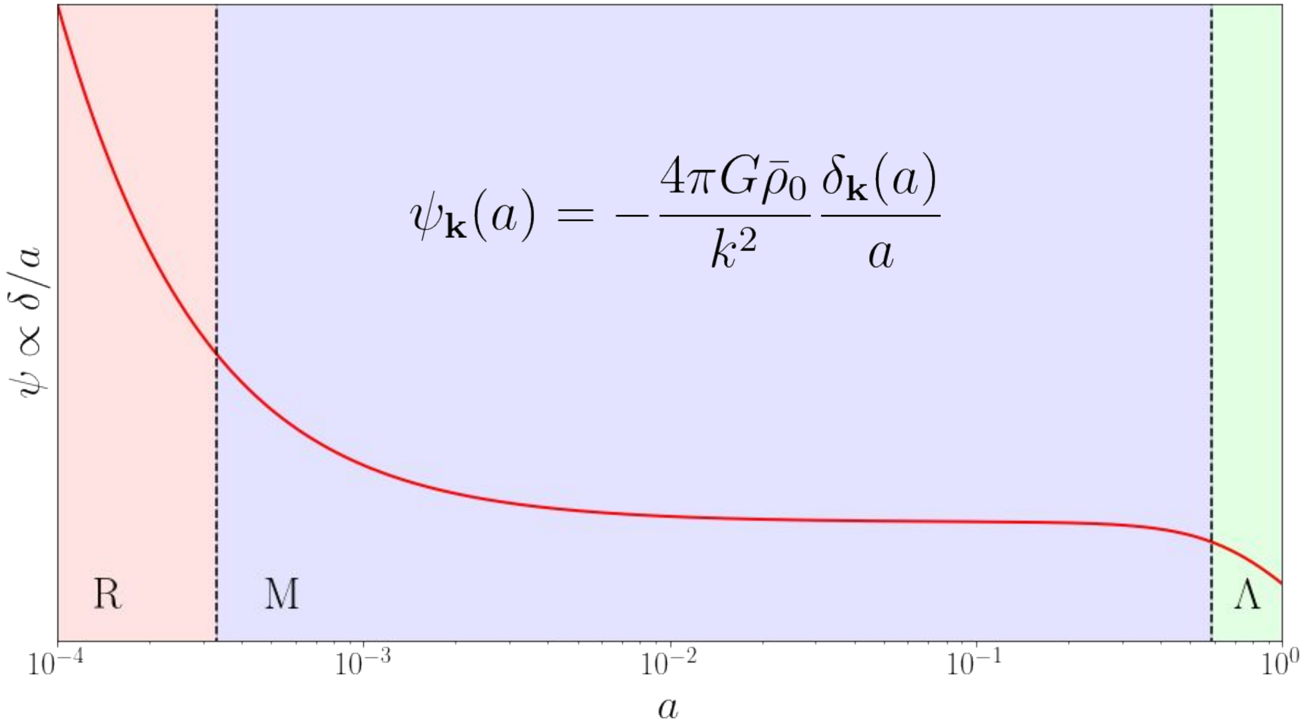
3. **Radiation domination** (Meszaros solution):

$$\delta \propto \begin{cases} A + B\log a & a \ll a_{\rm eq} \\ a & a \gg a_{\rm eq}, \end{cases}$$

δ also stalls

Lesson 1 b) Newtonian perturbations



Gravitational potential decays at early and late times, and stays constant during matter domination.

Outline

Lesson 1: background cosmology and Newtonian perturbations

a) **Homogeneous cosmology.** The FRW metric. Distances and redshift. The Friedman Equation.
b) **Newtonian perturbations.** The perturbation equations. Linear theory. Jeans equation and growth.
c) **Relativistic perturbations.** Gauge-invariant PT. Hydrodynamic perturbations. Qualitative behavior.

Lesson 2: Inflation

a) **Inflation.** The curvature and horizon problems. Scalar fields. Slow roll.
b) **Perturbations from inflation.** Inflationary PT. Random fields. The primordial power spectrum.

Lesson 3: cosmological probes of structure

a) **The CMB.** Recombination. Temperature anisotropies. Scattering and polarization.
b) **The matter power spectrum.** The linear power spectrum. Non-linearities
c) **Gravitational lensing.** Geodesics. Galaxy lensing. CMB lensing.

Lesson 1 c) Relativistic perturbations

Density fluctuations will perturb the FRW metric (and vice-versa)!
Problem: freedom to choose coordinates (no preferred frame, unlike in FRW).
General coordinate transformations can cause fictitious perturbations.

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Check the notes!

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In principle 10 perturbative d.o.f.s

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But 4 of them can be cancelled by coordinate transformations

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Result: 6 real perturbative d.o.f.s
- 2 scalar
- 2 vector
- 2 tensor

Defined wrt SO(3) (symmetry group of FRW background).
They evolve independently

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These are very interesting. GWs!
Unfortunately we won't have time to cover them

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They evolve independently

Lesson 1 c) Relativistic perturbations

The conformal Newtonian gauge

With a wise choice of coordinates, we can express general scalar perturbations as:

$$d\tau^2 = a^2 \left[(1+2\psi)d\eta^2 - (1-2\phi)|d\mathbf{x}|^2\right]$$

ϕ and ψ are closely related to the usual Newtonian potential.

Lesson 1 c) Relativistic perturbations

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Important notes:

- We use conformal time because it makes life simpler.

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  Gauge-dependent results for non-observable quantities.

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- Multitude of other gauges out there. Simpler equations for specific cases.

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  Gauge-dependent results for non-observable quantities.
- Multitude of other gauges out there. Simpler equations for specific cases.
- Einstein's equations for these perturbations:
  a) Linearised!
  b) Even so, tedious calculation. Worth doing at least once in your life!

Lesson 1 c) Relativistic perturbations

The conformal Newtonian gauge

The result is:

$$\nabla^2\phi - 3\mathcal{H}(\phi' + \mathcal{H}\psi) = 4\pi G a^2\,\delta T^0_0,$$
$$\partial_i(\phi' + \mathcal{H}\psi) = 4\pi G a^2\,\delta T^0_i,$$
$$\phi'' + \mathcal{H}(2\phi + \psi)' + (2\mathcal{H}' + \mathcal{H}^2)\psi + \frac{1}{3}\nabla^2(\psi - \phi) = -\frac{4\pi}{3} G a^2\,\delta T^i_i,$$
$$\partial_i\partial_j(\psi - \phi) = 8\pi G a^2\,\delta T^i_j \;\; (i \neq j).$$

Lesson 1 c) Relativistic perturbations

The conformal Newtonian gauge

Finding Einstein's equations for a perturbed FRW is lengthy (but worth doing once in your life!). The result is:

$$\nabla^2\phi - 3\mathcal{H}(\phi' + \mathcal{H}\psi) = 4\pi G a^2\, \delta T^0_0,$$

$$\partial_i(\phi' + \mathcal{H}\psi) = 4\pi G a^2\, \delta T^0_i,$$

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This looks like Poisson's equation + relativistic corrections.

If T_{ij} is diagonal, $\underline{\phi=\psi}$

Lesson 1 c) Relativistic perturbations

Perturbing T$_{\mu\nu}$

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu - p g_{\mu\nu},$$

In the background $U_\mu = (1, 0, 0, 0)$

A spatial component will be already a perturbation. The time component is fixed by normalisation:

$$U^\mu = \frac{1}{a}(1 - \psi, \mathbf{v})$$

Lesson 1 c) Relativistic perturbations

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This yields:

$$\delta T^0_0 = \bar{\rho}\delta, \;\; \delta T^0_i = (\bar{\rho} + \bar{p}) v_i, \;\; T^i_j = -\bar{\rho} c_s^2 \delta \, \delta^i_j$$

where **v** is a pure gradient.

Diagonal! ψ = ϕ

Lesson 1 c) Relativistic perturbations

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where **v** is a pure gradient.

Back to Einstein eq. and into Fourier space:

$$k^2\psi + 3\mathcal{H}(\psi' + \mathcal{H}\psi) = -4\pi G a^2\, \bar{\rho}\, \delta,$$

$$k^2(\psi' + \mathcal{H}\psi) = 4\pi G a^2\, (\bar{\rho} + \bar{p})\theta, \longleftarrow \theta_{\mathbf{k}} \equiv i\mathbf{k} \cdot \mathbf{v}_{\mathbf{k}}$$

$$\psi'' + 3\mathcal{H}\psi' + (2\mathcal{H}' + \mathcal{H}^2)\psi = 4\pi G a^2\, c_s^2 \bar{\rho}\, \delta,$$

Lesson 1 c) Relativistic perturbations

Example: Einstein deSitter

Matter domination: $c_s^2 = 0, \quad a \propto \eta^2, \quad \mathcal{H} = 2/\eta$

$$k^2\psi + 3\mathcal{H}(\psi' + \mathcal{H}\psi) = -4\pi G a^2\,\bar{\rho}\,\delta,$$

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Scale-independent growth

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Scale-independent growth

$$\psi'' + \frac{6}{\eta}\psi' = 0, \quad \rightarrow \quad \psi = C_1 + \frac{C_2}{\eta^5}$$

Potential stays constant (as we found in Newtonian case)

Lesson 1 c) Relativistic perturbations

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$$\delta_{\mathbf{k}} = \left[-\frac{(k\eta)^2}{6} - 2\right]\psi_{\mathbf{k}}$$

δ grows like a (as in Newtonian PT)

Lesson 1 c) Relativistic perturbations

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$$k^2\psi + 3\mathcal{H}(\psi' + \mathcal{H}\psi) = -4\pi G a^2\,\bar{\rho}\,\delta,$$

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$$\boxed{\psi'' + 3\mathcal{H}\psi' + (2\mathcal{H}' + \mathcal{H}^2)\psi = 4\pi G a^2\,\cancel{c_s^2}\bar{\rho}\,\delta,}$$

Scale-independent growth

$$\psi'' + \frac{6}{\eta}\psi' = 0, \quad \rightarrow \quad \psi = C_1 + \frac{C_2}{\eta^5}$$

$$\delta_{\mathbf{k}} = \left[-\frac{(k\eta)^2}{6} - 2\right]\psi_{\mathbf{k}}$$

$$k_H \sim 1/\eta \sim \mathcal{H}$$

δ grows like a (as in Newtonian PT) + relativistic horizon-sized correction

Lesson 1 c) Relativistic perturbations

Example: Einstein deSitter

Matter domination: $c_s^2 = 0, \quad a \propto \eta^2, \quad \mathcal{H} = 2/\eta$

$$k^2\psi + 3\mathcal{H}(\psi' + \mathcal{H}\psi) = -4\pi G a^2\,\bar{\rho}\,\delta,$$

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These actually depend on the gauge!

$$k_H \sim 1/\eta \sim \mathcal{H}$$

δ grows like a (as in Newtonian PT) + relativistic horizon-sized correction

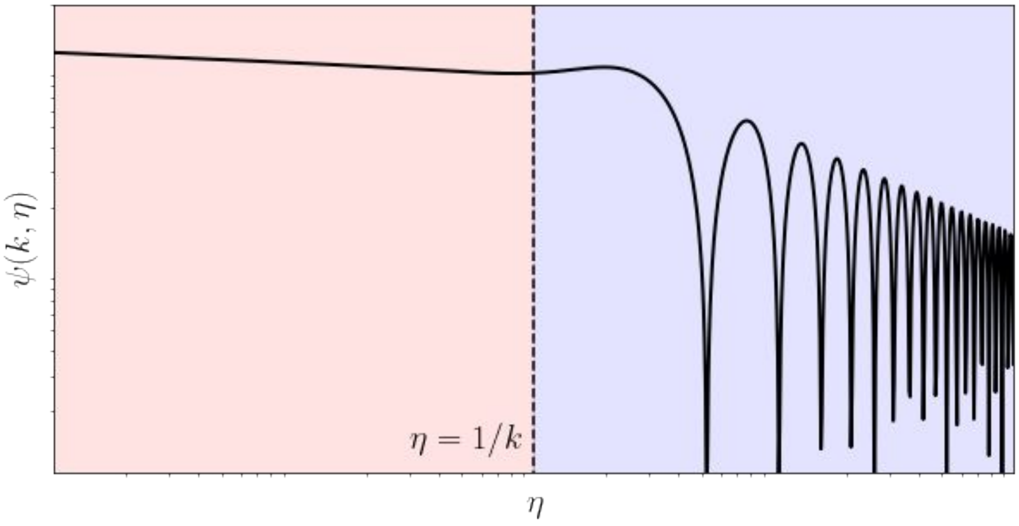
Lesson 1 c) Relativistic perturbations

General behavior

$$\psi(k,\eta) = \begin{cases} f(\eta) & k \ll 1/(c_s\eta) \\ g(\eta)e^{ic_s k\eta} & k \gg 1/(c_s\eta) \end{cases}$$

Where:

- $c_s\eta$ ~ sound horizon ~ horizon (e.g. c_s^2=⅓ for radiation).
- $f(\eta)$ is a slowly-varying (almost constant) function
- $g(\eta)$ is a decaying amplitude



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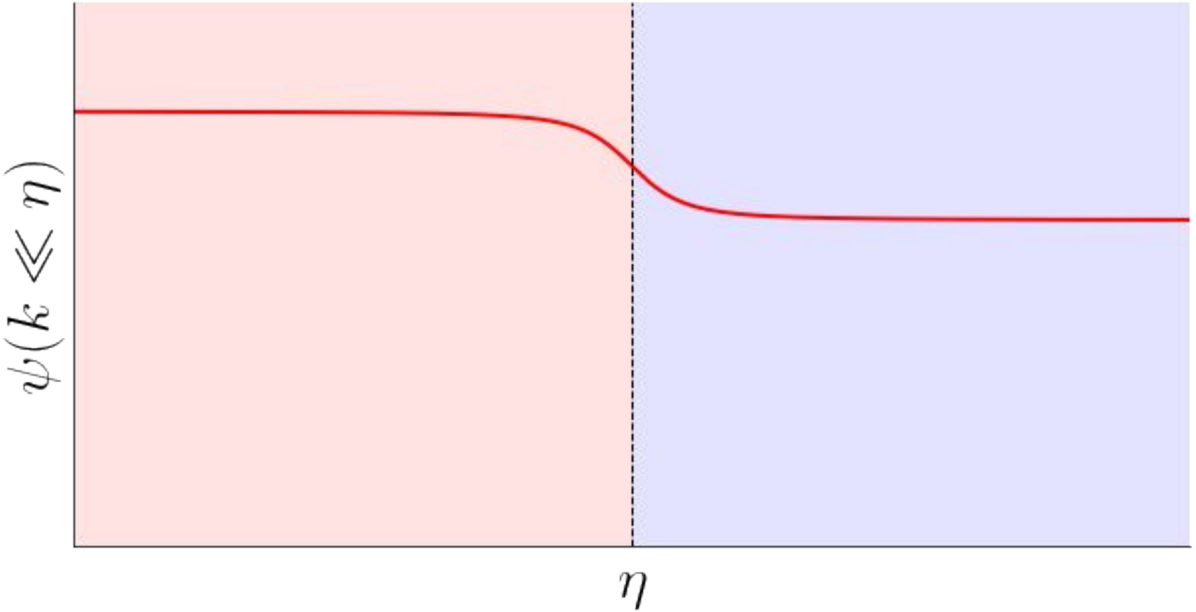
Note that ψ may vary on large scales in between epochs (e.g. radiation to matter domination).



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Note that ψ may vary on large scales in between epochs (e.g. radiation to matter domination).

However, the following quantity ("curvature perturbation"), is always constant on superhorizon modes:

$$\mathcal{R} \equiv -\psi - \frac{\mathcal{H}(\psi' + \mathcal{H}\psi)}{4\pi G a^2(\bar{\rho} + \bar{p})}$$

Lesson 1 c) Relativistic perturbations

Energy-momentum conservation

In the presence of perturbations, $\nabla_\mu T^\mu_\nu = 0$ yields

Relativistic continuity eq.

$$\nu = 0 : \quad \delta' = -(1+w)(\theta - 3\phi') - 3\mathcal{H}(c_s^2 - w)\delta,$$

$$\nu = i : \quad \theta' = -\mathcal{H}(1-3w)\theta - \frac{w'}{1+w}\theta + \frac{c_s^2}{1+w}k^2\delta + k^2\psi.$$

Relativistic Euler eq.

Lesson 1 c) Relativistic perturbations

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Relativistic Euler eq.

Reminder: these hold for the total $T_{\mu\nu}$, or for each independent component.

- When applied to the total fluid, these do not contain more information than the Einstein eqs.
- Additional information when applied to independent species.
- In the presence of interactions, momentum transfer terms must be added. (E.g. radiation-baryons before decoupling).

Outline

Lesson 1: background cosmology and Newtonian perturbations

a) **Homogeneous cosmology.** The FRW metric. Distances and redshift. The Friedman Equation.
b) **Newtonian perturbations.** The perturbation equations. Linear theory. Jeans equation and growth.
c) **Relativistic perturbations.** Gauge-invariant PT. Hydrodynamic perturbations. Qualitative behavior.

Lesson 2: Inflation

a) **Inflation.** The curvature and horizon problems. Scalar fields. Slow roll.
b) **Perturbations from inflation.** Inflationary PT. Random fields. The primordial power spectrum.

Lesson 3: cosmological probes of structure

a) **The CMB.** Recombination. Temperature anisotropies. Scattering and polarization.
b) **The matter power spectrum.** The linear power spectrum. Non-linearities
c) **Gravitational lensing.** Geodesics. Galaxy lensing. CMB lensing.

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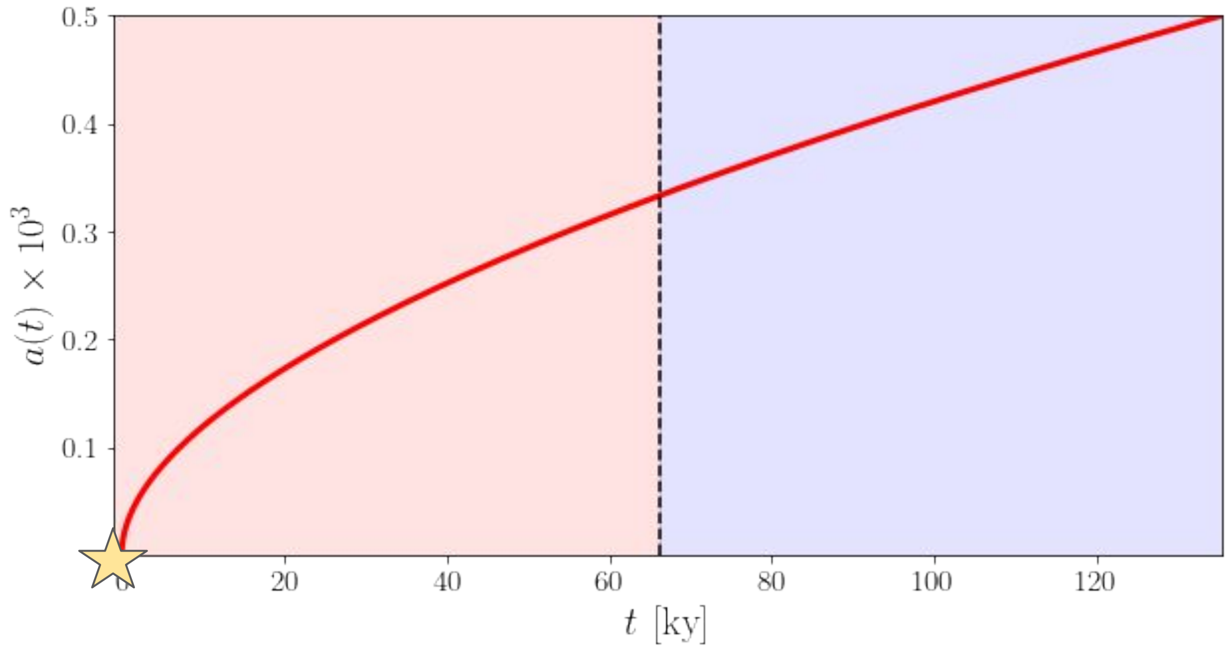
Why inflation?

The hot Big-Bang model (radiation domination at early times) predicts a singularity in the past. This is problematic when confronted with observations.



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The hot Big-Bang model (radiation domination at early times) predicts a singularity in the past.
This is problematic when confronted with observations.

Firstly, <u>the causal horizon is finite</u>!

$$\chi_H(t) = \int_0^t \frac{dt'}{a(t')}$$

The lower limit converges if $a \propto t^\alpha$ with α<1.
During radiation domination α=1/2.

Lesson 2 a) Inflation

Why inflation?

The hot Big-Bang model (radiation domination at early times) predicts a singularity in the past. This is problematic when confronted with observations.

Firstly, the causal horizon is finite!

Then, the causal horizon when the CMB was emitted (“photon decoupling”) is:

$$\chi_H = \int_{z_d}^{\infty} \frac{dz'}{H(z')} \simeq 250\,\mathrm{Mpc}$$

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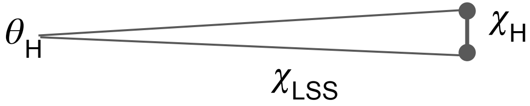
However, the distance to the last-scattering surface is:

$$\chi_{\rm LSS} = \int_0^{z_d} \frac{dz'}{H(z')} \simeq 14\,\mathrm{Gpc}$$



So the horizon subtends an angle:

$$\theta_H = \chi_H / \chi_{\rm LSS} \sim 1^\circ$$

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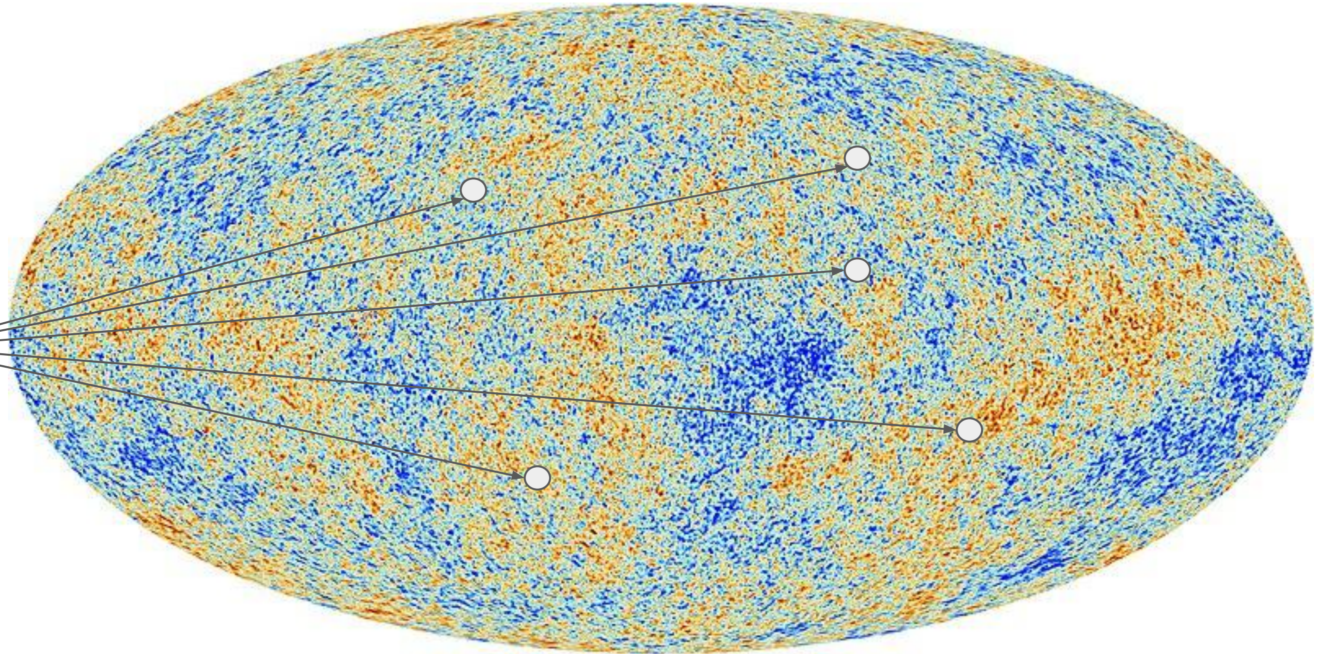
Why do so many causally disconnected patches have the same temperature (within 10^{-5})?



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We can solve this if there was an epoch before rad. dom. with $a \propto t^\alpha$ and α>1.

But α>1 means acceleration! This violates the Strong Energy Principle, and involves some exotic fluid.
Can we find more justification for something this bizarre?

Lesson 2 a) Inflation

Why inflation?

The hot Big-Bang model (radiation domination at early times) predicts a singularity in the past. This is problematic when confronted with observations.

Firstly, the causal horizon is finite!

Secondly, the Universe is very flat!

$$\Omega_k(t) = \frac{k}{(aH)^2} = \frac{k}{\dot{a}^2}$$

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In a decelerating Universe, $|\Omega_K| < 10^{-3}$, and therefore it must have been $|\Omega_K| < 10^{-17}$ during R. dom. Can we justify such a finely-tuned initial condition?

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An early period of large acceleration would increase $\dot{a}$, driving Ω_K to zero before radiation domination.

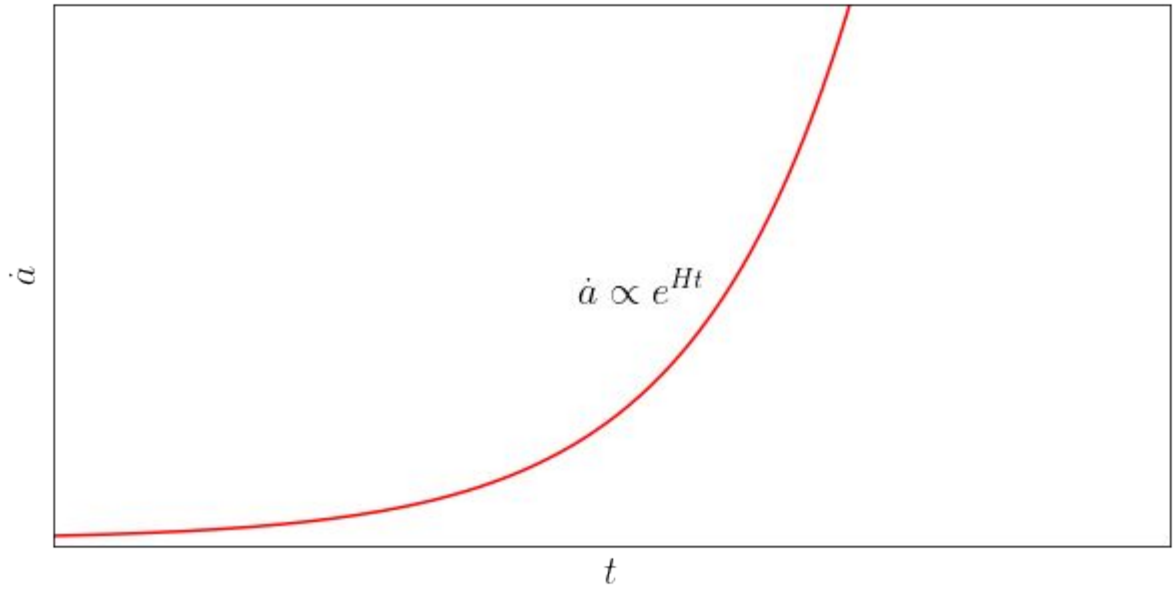
Inflation ($\ddot{a} > 0$ at early times) can therefore solve the horizon and curvature problems!

To solve them, the scale factor must expand by $\frac{a_{\rm end}}{a_{\rm start}} \gtrsim e^{60}$

Lesson 2 a) Inflation

How inflation?

The simplest accelerating model we've seen is a cosmological constant (de-Sitter universe)

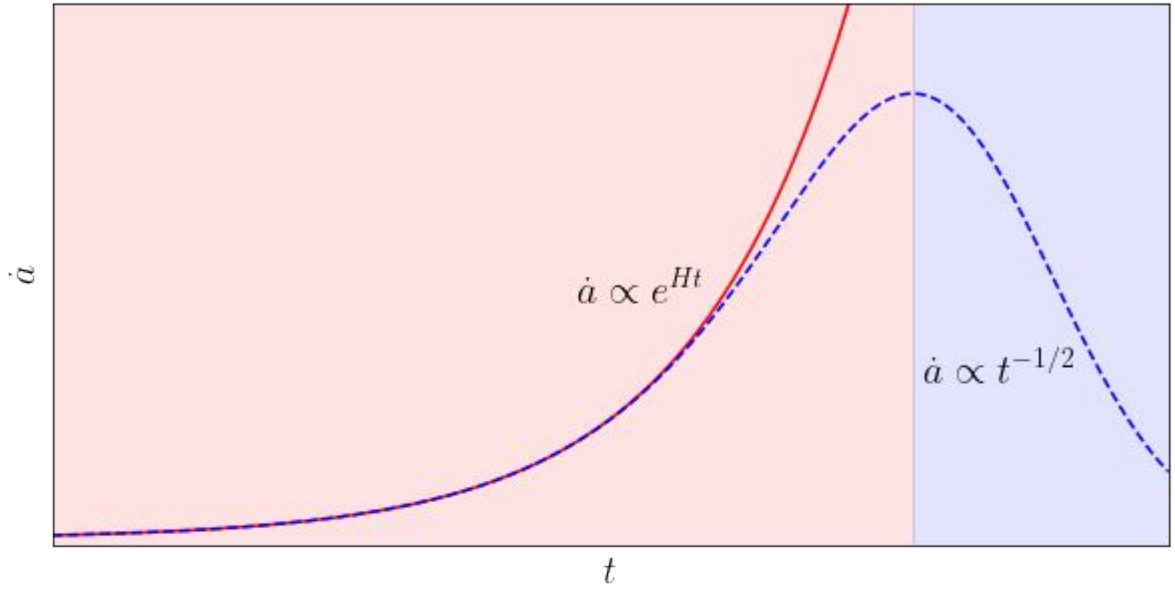


Lesson 2 a) Inflation

How inflation?

The simplest accelerating model we've seen is a cosmological constant (de-Sitter universe)

However, once vacuum dominates, it dominates forever. We need a "graceful exit"



Lesson 2 a) Inflation

Scalar fields

Next simplest model is a scalar field

$$\mathcal{L} = \frac{1}{2} g^{\mu\nu} \nabla_\mu \varphi \nabla_\nu \varphi - V(\varphi)$$

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In the homogeneous limit:

$$\mathcal{L} = \frac{1}{2}\dot{\varphi}^2 - V(\varphi), \quad T^\mu_\nu = \mathrm{diag}(\rho, -p, -p, -p)$$

$$\rho \equiv \frac{1}{2}\dot{\varphi}^2 + V, \quad p \equiv \frac{1}{2}\dot{\varphi}^2 - V$$

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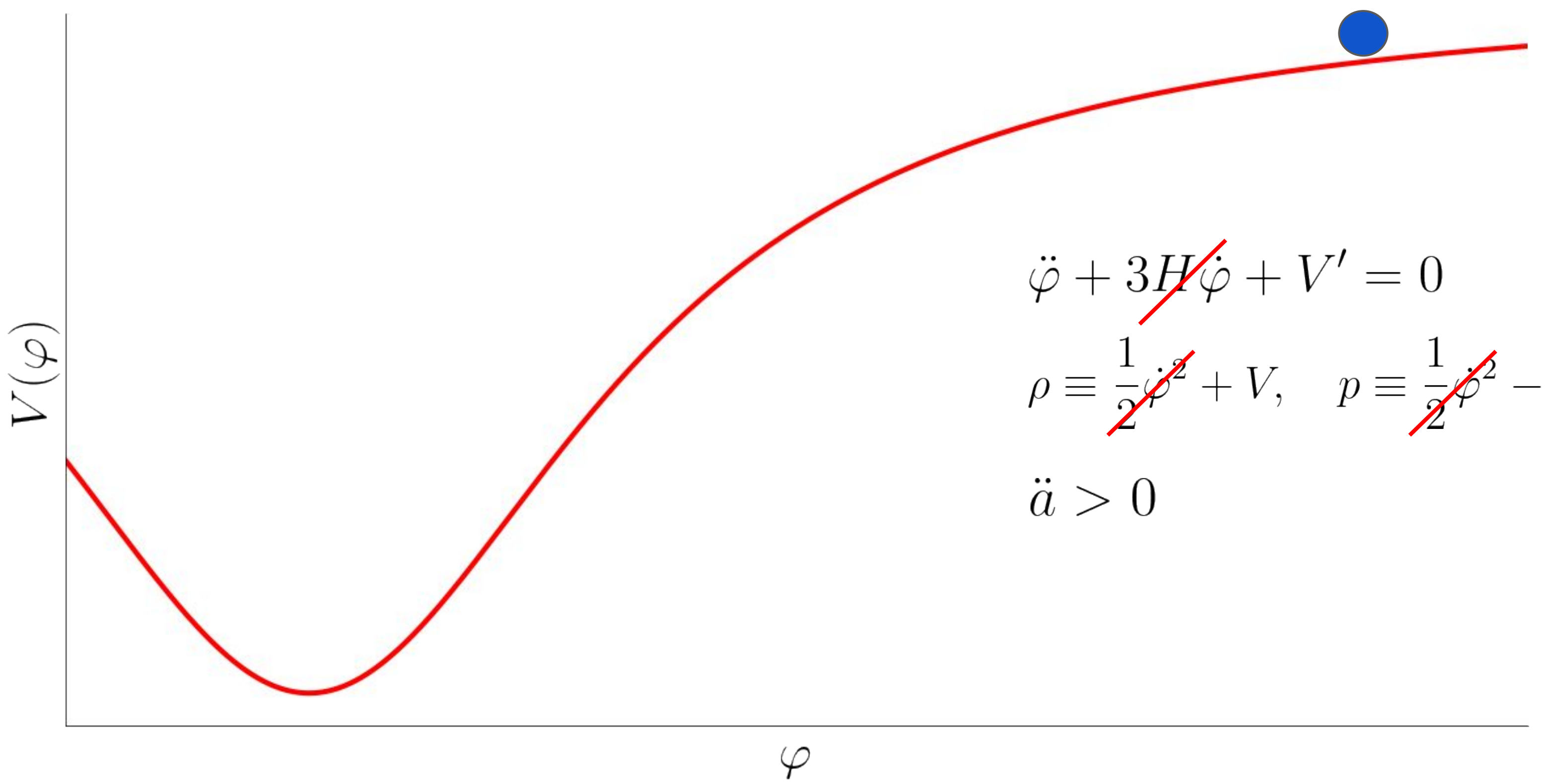
<u>Dynamics in an expanding Universe:</u>

$$\ddot{\varphi} + 3H\dot{\varphi} + V' = 0$$ Klein-Gordon equation

$$H^2 = \frac{8\pi G}{3}\left[\frac{1}{2}\dot{\varphi}^2 + V\right]$$ Friedmann equation

Lesson 2 a) Inflation

The slow-roll picture



$$\ddot{\varphi} + 3\cancel{H\dot{\varphi}} + V' = 0$$

$$\rho \equiv \cancel{\frac{1}{2}\dot{\varphi}^2} + V, \quad p \equiv \cancel{\frac{1}{2}\dot{\varphi}^2} - V$$

$$\ddot{a} > 0$$

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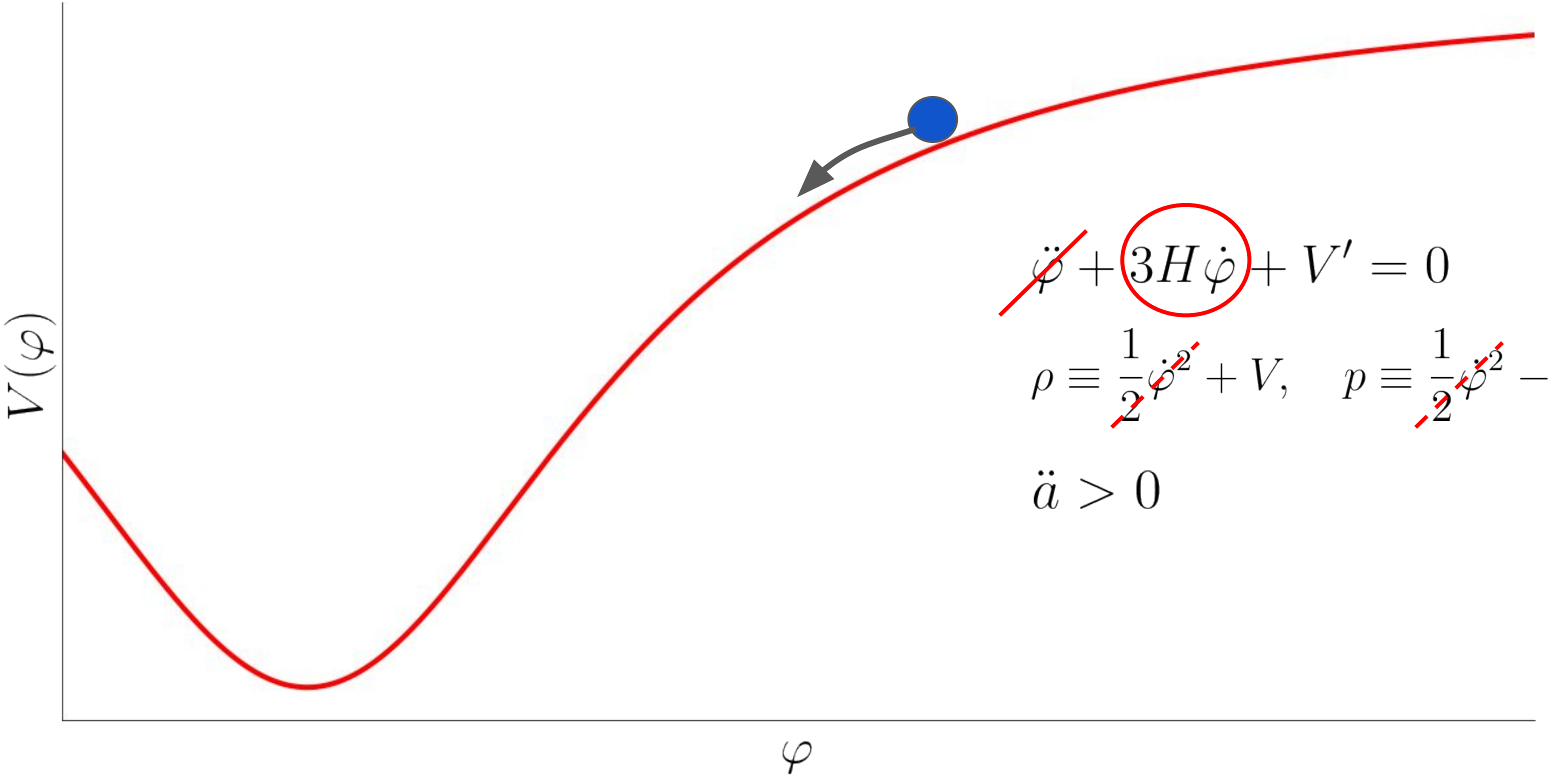

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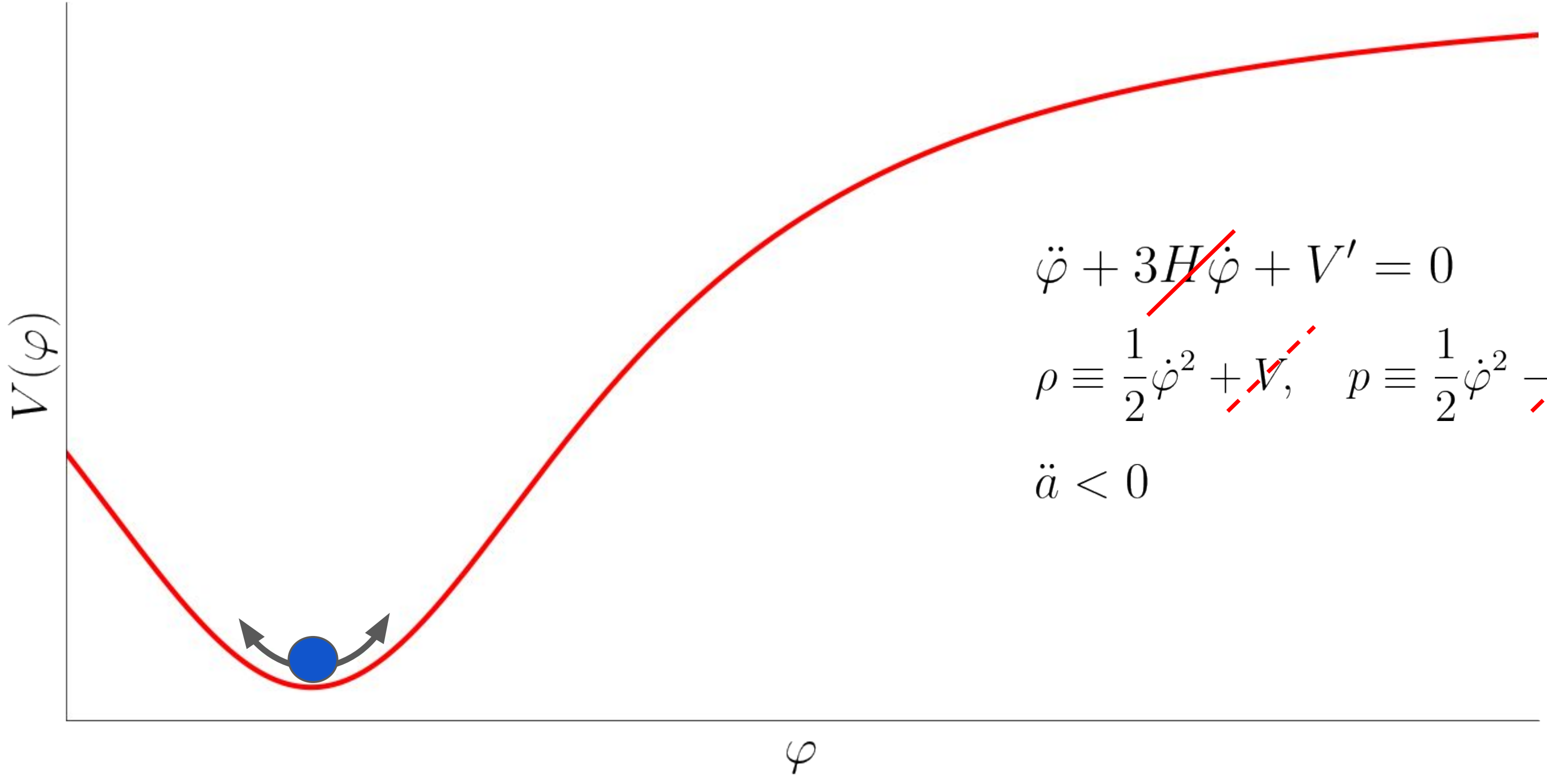

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$$\ddot{a} < 0$$

Lesson 2 a) Inflation

The slow-roll math

A successful inflaton model must achieve:

$$\dot{\varphi}^2 \ll V \qquad\qquad |\ddot{\varphi}| \ll 3H\dot{\varphi}$$

So we get acceleration

So inflation can last

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The slow-roll math

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It is common to define two parameters:

$$\varepsilon = -\frac{\dot{H}}{H^2} = 1 - \frac{\ddot{a}a}{\dot{a}^2} \ll 1, \quad \eta = \frac{d\log\varepsilon}{d\log a} \ll 1$$

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These can be related to model properties

$$\varepsilon = 3\frac{\dot{\varphi}^2/2}{\dot{\varphi}^2/2+V} \simeq \frac{M_{\rm Pl}}{2}\left(\frac{V'}{V}\right)^2, \qquad \eta = 2\epsilon - \frac{2\ddot{\varphi}}{H\dot{\varphi}} \simeq M_{\rm Pl}^2\frac{V''}{V},$$

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The slow-roll math

A successful inflaton model must achieve:

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So we get acceleration

So inflation can last

In this approximation:

$$3\dot{\varphi} = -\frac{1}{H}V', \qquad H^2 = \frac{8\pi G}{3}V$$

Lesson 2 a) Inflation

The slow-roll math

Example: quadratic potential (massive scalar field) $$V(\varphi) = \frac{1}{2}m^2\varphi^2$$

Lesson 2 a) Inflation

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In slow-roll regime, Friedmann equation and K-G equation read:

$$3\dot{\varphi} = -\frac{1}{H}V', \quad H^2 = \frac{8\pi G}{3}V$$

$$\dot{\varphi} = -\frac{m^2}{3q}$$

$$H = q\varphi, \quad q^2 \equiv \frac{1}{6}\frac{m^2}{M_{\rm Pl}^2}$$

Lesson 2 a) Inflation

The slow-roll math

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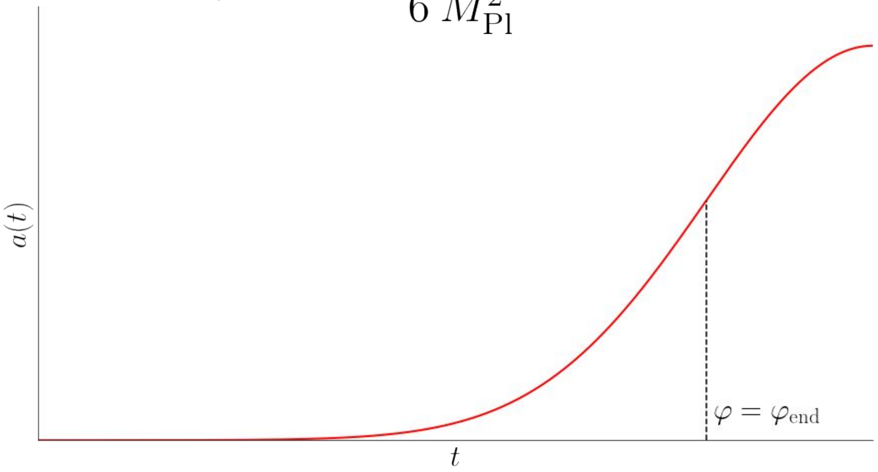
$$\dot{\varphi} = -\frac{m^2}{3q} \qquad\qquad H = q\varphi, \qquad q^2 \equiv \frac{1}{6}\frac{m^2}{M_{\rm Pl}^2}$$

These can be readily integrated:

$$\varphi(t) = \varphi_i - \frac{m^2}{3q}t,$$

$$\log\left(\frac{a(t)}{a_i}\right) = 2\pi G\varphi_i^2 - \frac{m^2}{6}\left(t - \frac{3q\varphi_i^2}{m^2}\right)^2$$



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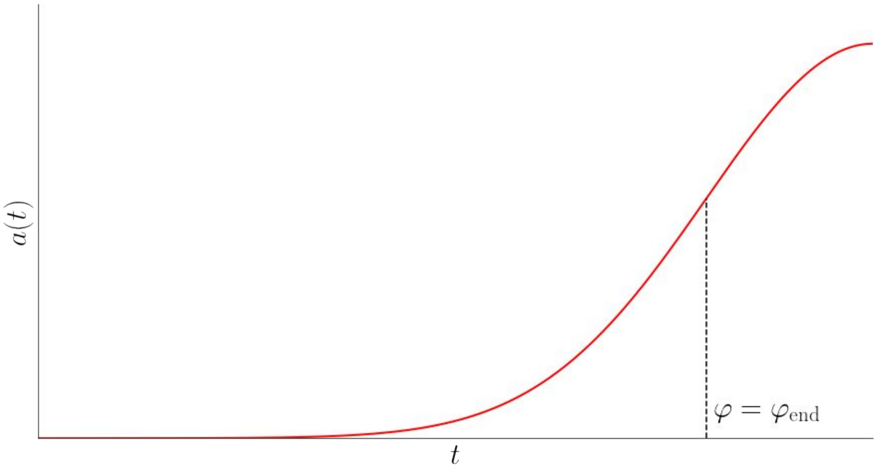
Inflation ends when $\dot{\varphi}^2/2 \sim V$

At which point:

$$\log(a_e/a_i) = \left(\frac{\varphi_i}{2M_{\rm Pl}}\right)^2$$

Since we want this to be ~60, the field must start at high, Planckian values.



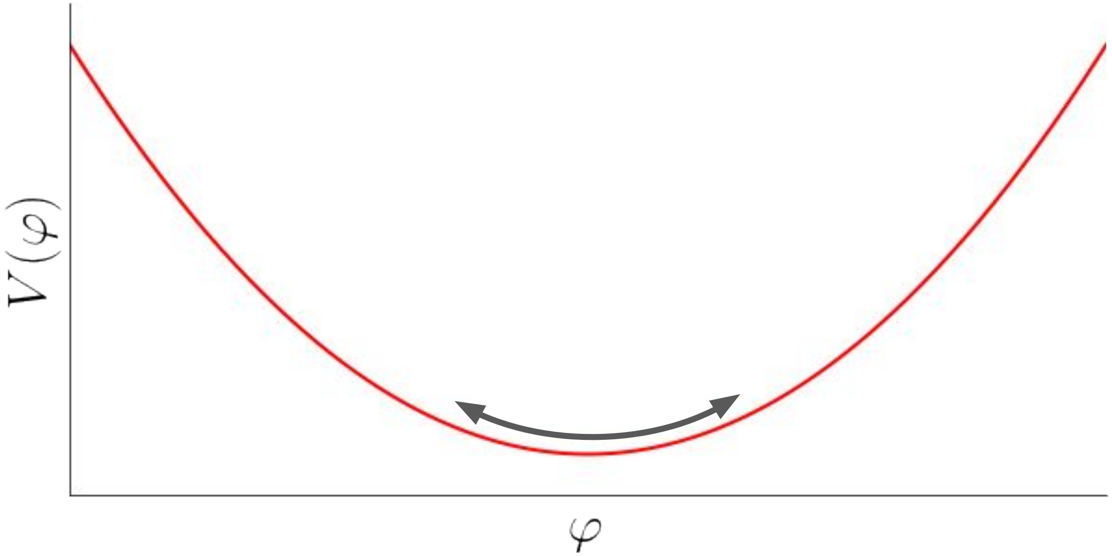
Lesson 2 a) Inflation

Graceful exit, reheating

When the field reaches the minimum:

$$\ddot{\varphi} + \cancel{3H}\dot{\varphi} + m^2\varphi = 0$$

$$\varphi \propto \cos(mt + \alpha)$$



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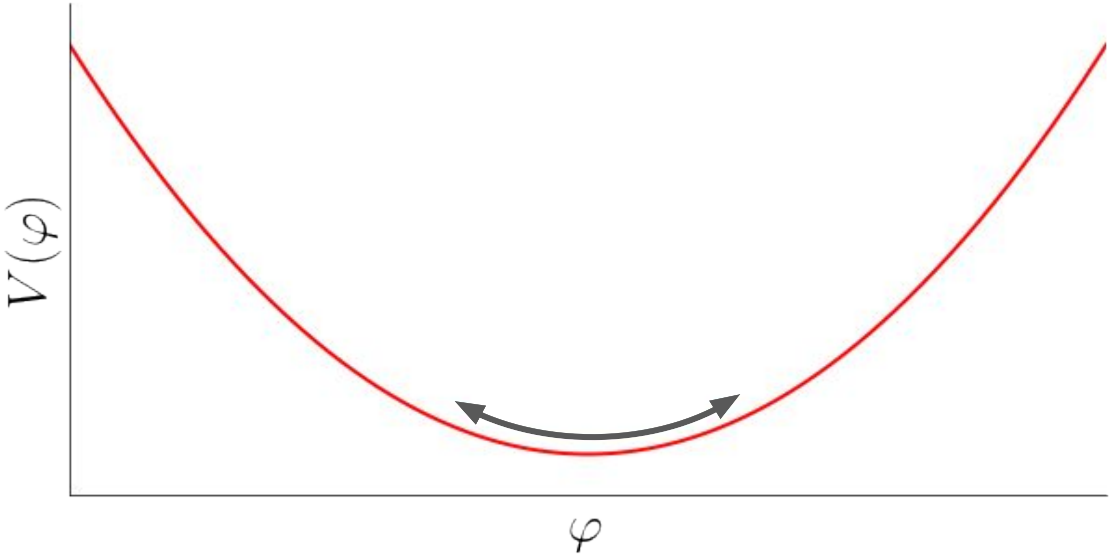

Field behaves like presureless fluid: $a \propto t^{2/3}$

Lesson 2 a) Inflation

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$$\downarrow$$

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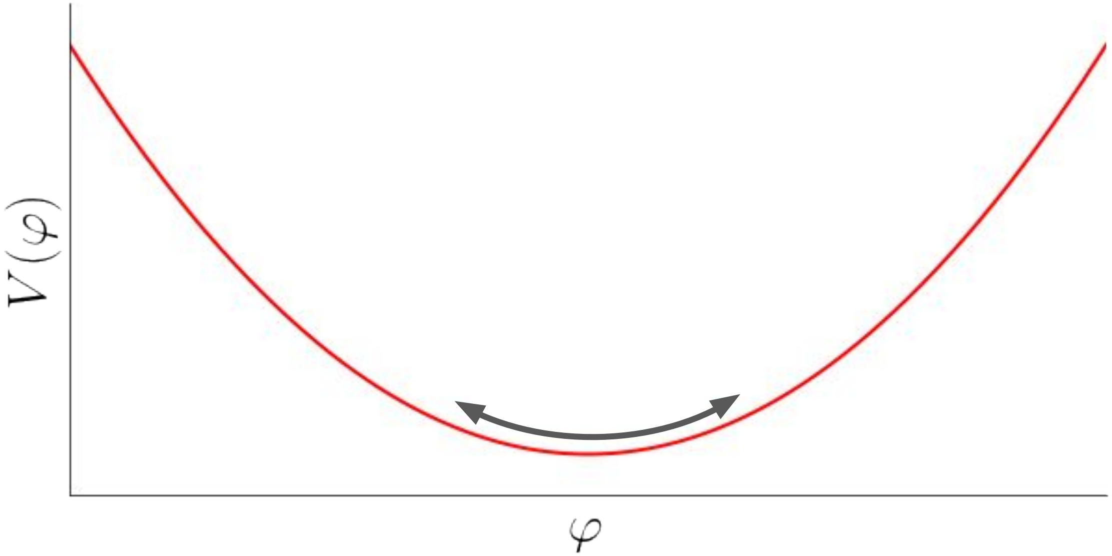

Field behaves like presureless fluid: $a \propto t^{2/3}$

Through couplings, inflaton energy transfers to other fields, eventually generating SM particles (**reheating**).

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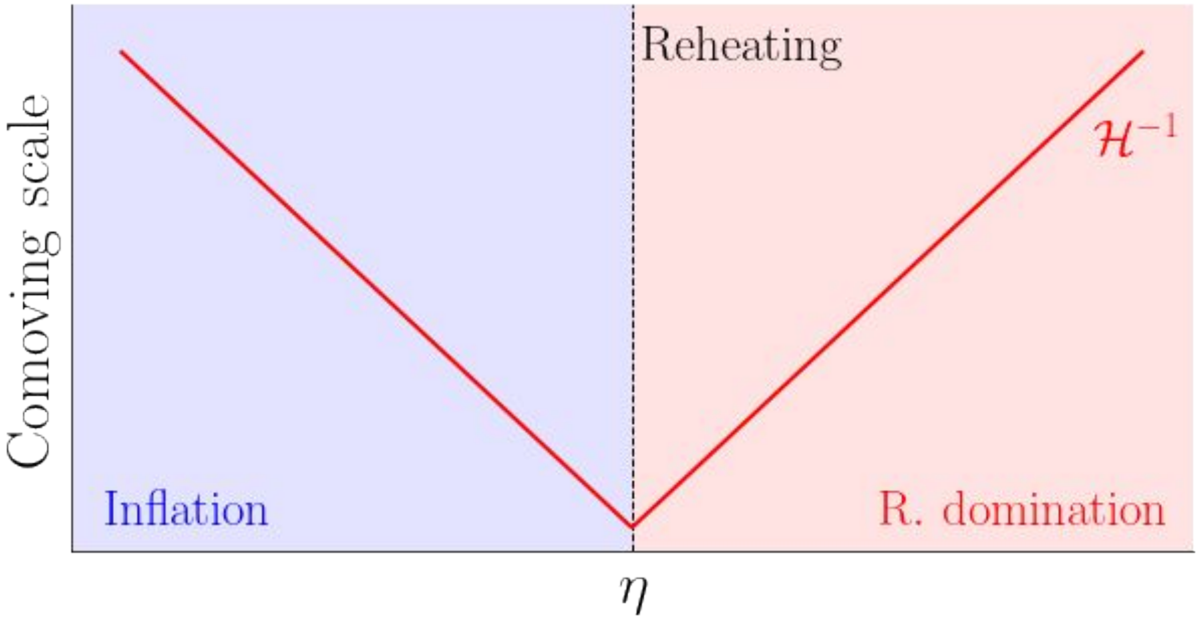
Lesson 2 b) Perturbations from inflation

Quantum fluctuations and the hubble scale

Inflation provides a natural way to generate the initial metric fluctuations.
The key fact is that the *comoving Hubble scale* $(aH)^{-1}$ shrinks dramatically during inflation:

$$(aH)^{-1} = \mathcal{H}^{-1} \propto e^{-Ht}$$

In de-Sitter phase



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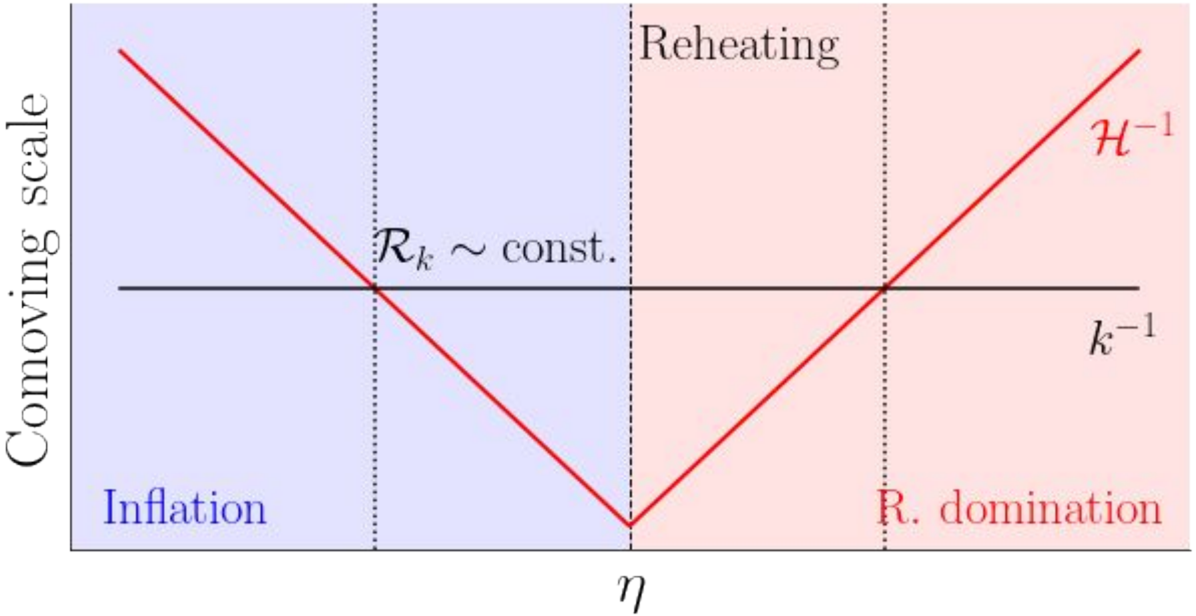
Quantum fluctuations and the hubble scale

<u>Inflation provides a natural way to generate the initial metric fluctuations.</u>
The key fact is that the *comoving Hubble scale* $(aH)^{-1}$ shrinks dramatically during inflation.
Perturbations on scales above or below $(aH)^{-1}$ behave very differently.

- They are preserved on super-horizon scales.



Lesson 2 b) Perturbations from inflation

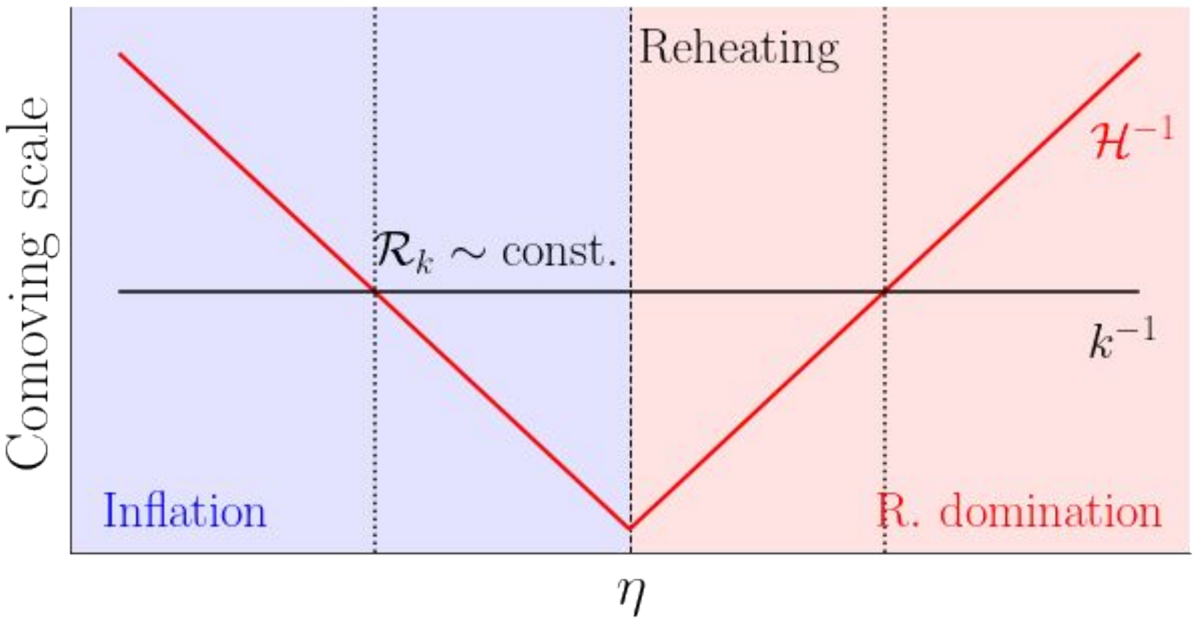
Quantum fluctuations and the hubble scale

<u>Inflation provides a natural way to generate the initial metric fluctuations.</u>

- Perturbations are preserved on super-horizon scales.

At the same time quantum mechanics prevents a field from being perfectly homogeneous.

$$\langle \varphi^2(x) \rangle > 0$$



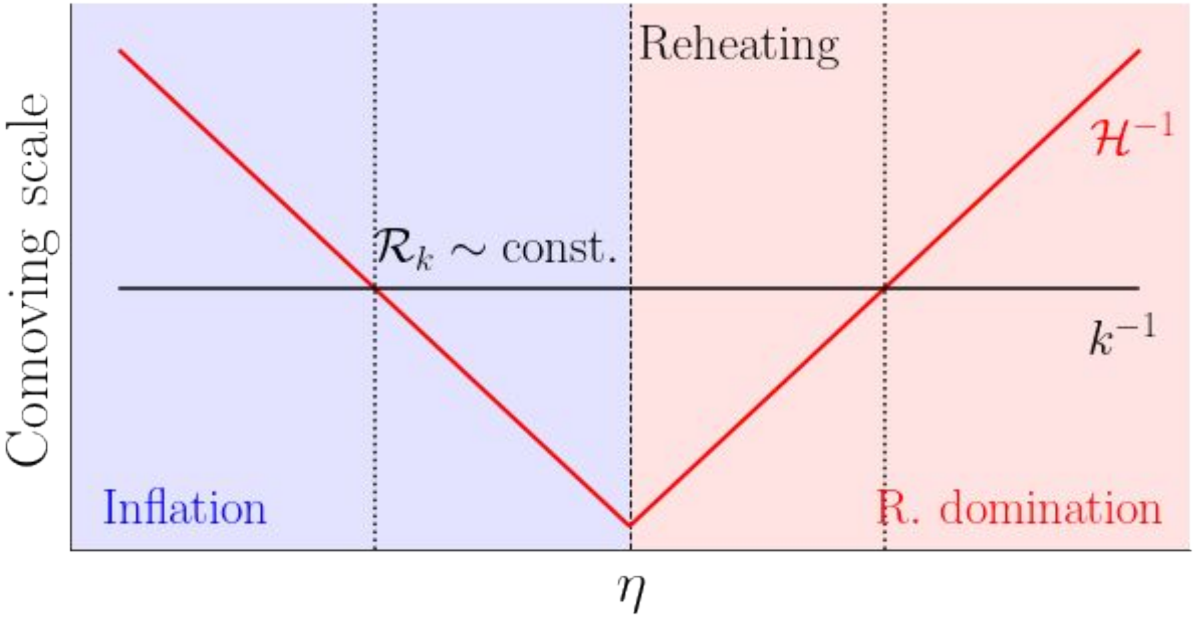
Lesson 2 b) Perturbations from inflation

Quantum fluctuations and the hubble scale

Inflation provides a natural way to generate the initial metric fluctuations.

- Perturbations are preserved on super-horizon scales.

At the same time quantum mechanics prevents a field from being perfectly homogeneous.
Even in vacuum state, quantum fluctuations are always present.



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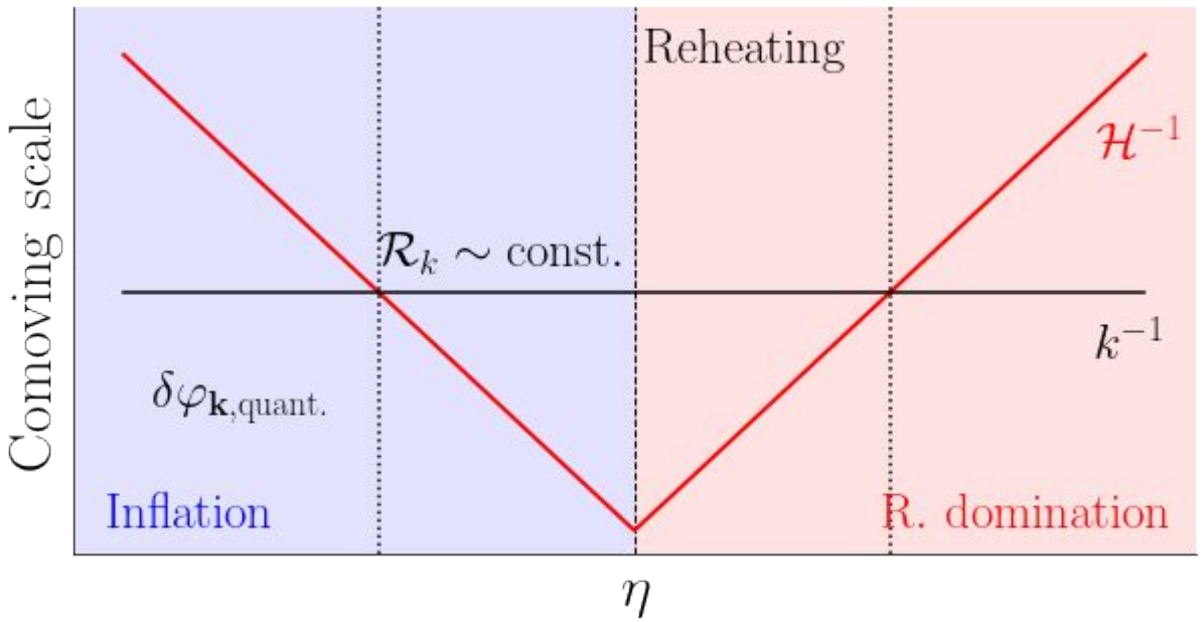

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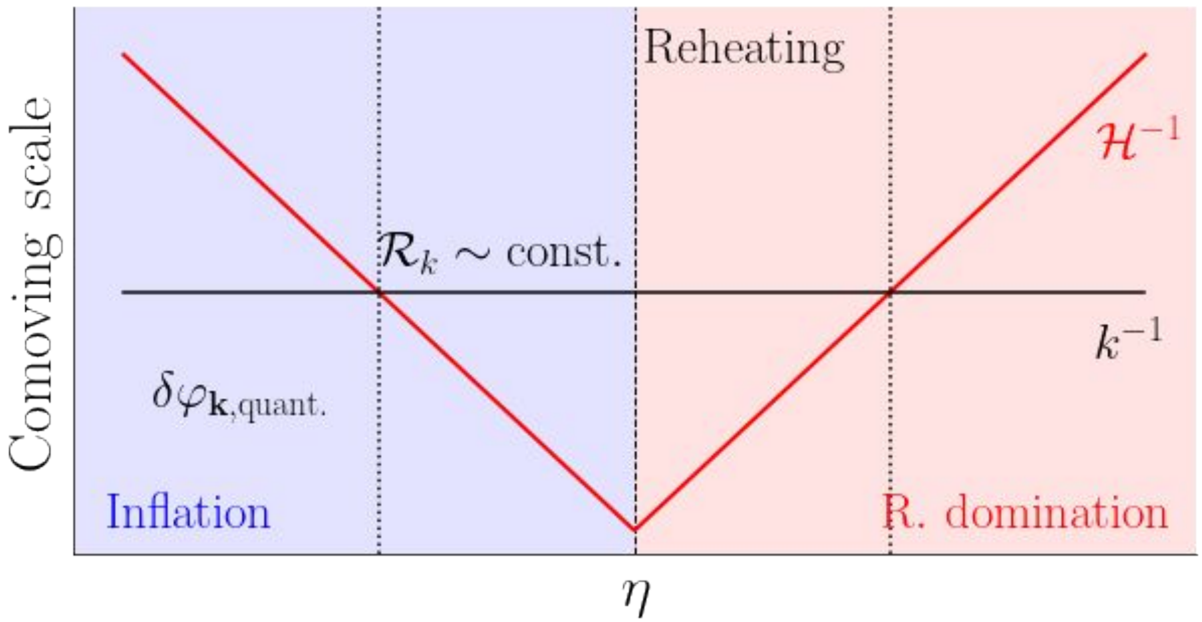

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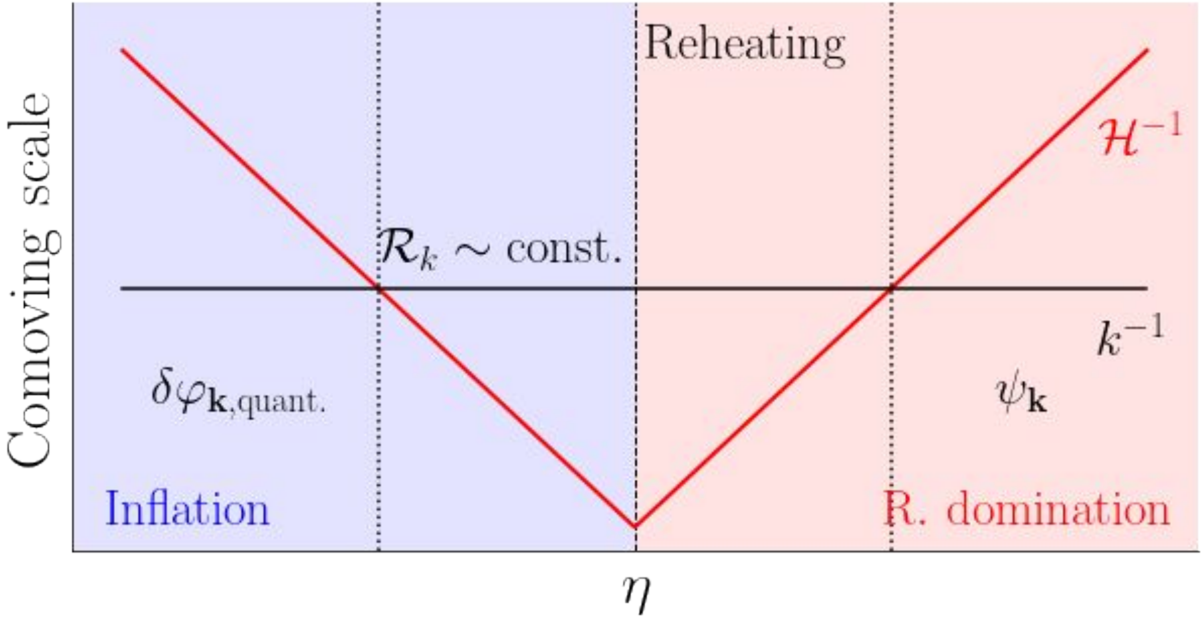

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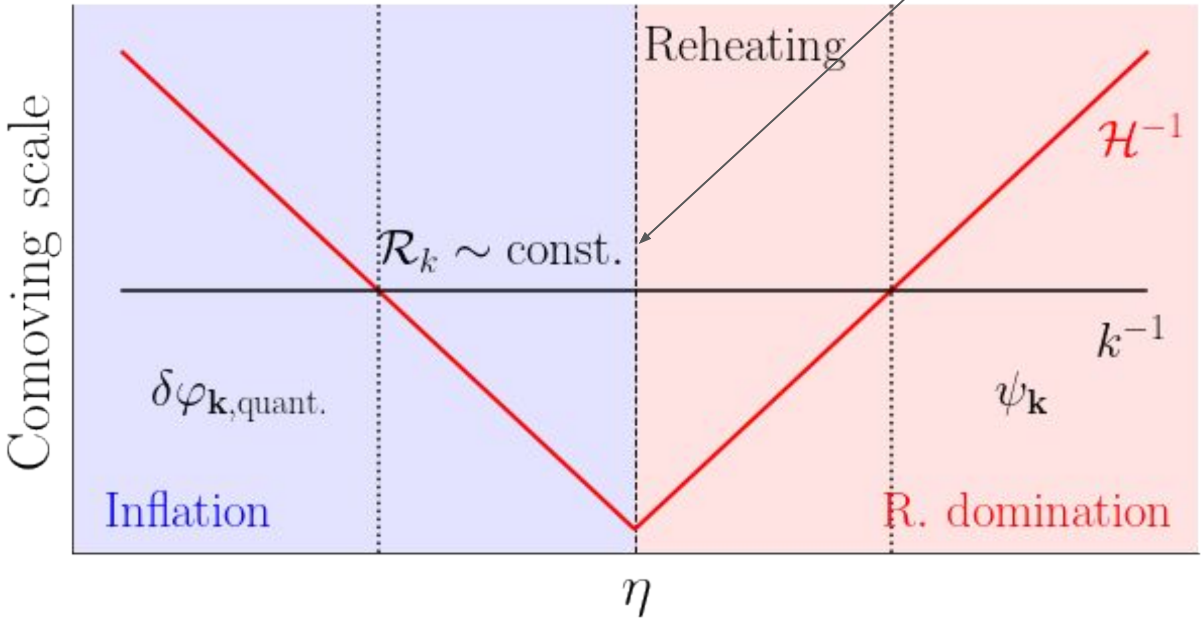
Our goal:
Predict the spectrum from quantum fluctuations after inflation



Lesson 2 b) Perturbations from inflation

1. Sub-horizon perturbations during inflation

During slow-roll, and on sub-horizon scales:

$$\varphi = \bar{\varphi}(t) + \delta\varphi(\mathbf{x}, t)$$

KG equation for the perturbation:

$$\delta\varphi''_{\mathbf{k}} + 2\mathcal{H}\delta\varphi'_{\mathbf{k}} + k^2\delta\varphi_{\mathbf{k}} \simeq 0$$

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Through a change of variables:

$$f_{\mathbf{k}} \equiv a\,\delta\varphi_{\mathbf{k}} \quad \rightarrow \quad f''_{\mathbf{k}} + \left(k^2 - \frac{a''}{a}\right) f_{\mathbf{k}} = 0$$

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On sub-horizon scales

f_k behaves like a harmonic oscillator. Let's quantize it!

Lesson 2 b) Perturbations from inflation

2. Quantize fluctuations and get vacuum statistics

Quick review of canonical quantization

1. Promote $f_{\mathbf{k}}$ to operator and split into ladder operators

$$\hat{f}_{\mathbf{k}}(\eta) = f_k(\eta)\hat{a}_{\mathbf{k}} + f_k^*(\eta)\hat{a}_{\mathbf{k}}^\dagger$$

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$$[\hat{a}_{\mathbf{k}}, \hat{a}_{\mathbf{k}'}^\dagger] = (2\pi)^3\delta^D(\mathbf{k} - \mathbf{k}')$$

$$\hat{a}_{\mathbf{k}}|0\rangle = 0$$

2. Fix normalisation by:
   a. Ensuring commutation relations of field and conjugate momentum operator.
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$$\langle \hat{f}_{\mathbf{k}} \rangle \equiv \langle 0|\hat{f}|0\rangle = 0$$

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$$\Delta_f^2(k) = \left(\frac{k}{2\pi}\right)^2$$

Power spectrum

Lesson 2 b) Perturbations from inflation

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Inflation doesn't predict the field configuration, but it does predict its degree or variability.

Lesson 2 b) Perturbations from inflation

3. Super-horizon perturbations

The behavior of $\delta\varphi_k$ changes after k crosses the Hubble scale.
Simplest strategy:

1. Relate $\delta\varphi_k$, to $\mathcal{R}_k$ at horizon crossing.
2. $\mathcal{R}_k$ is then preserved until it crosses back inside the horizon after inflation ends.
3. Relate $\mathcal{R}_k$ to all other perturbations of interest (ψ, δ, θ…) through transfer functions.

$$\mathcal{R} \equiv -\phi - \frac{\mathcal{H}(\psi' + \mathcal{H}\phi)}{4\pi G a^2(\bar{\rho} + \bar{p})}$$

Lesson 2 b) Perturbations from inflation

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Using $\Delta_f^2(k) = \left(\frac{k}{2\pi}\right)^2$, and evaluating at $k = aH$:

$$\Delta_\mathcal{R}^2(k) = \frac{1}{2M_\mathrm{Pl}^2\varepsilon}\left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH},$$

Lesson 2 b) Perturbations from inflation

The primordial power spectrum

$$\Delta^2_{\mathcal{R}}(k) = \frac{1}{2M_{\rm Pl}^2\varepsilon}\left(\frac{H}{2\pi}\right)^2\Bigg|_{k=aH},$$

During inflation, both *H* and ε vary very slowly. **Spectrum is almost scale-invariant.**

Lesson 2 b) Perturbations from inflation

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Common parametrisation:

$$\Delta^2_{\mathcal{R}}(k) = A_s\left(\frac{k}{k_*}\right)^{n_s-1}$$

A specific model does not provide a prediction for A_s, but it does for n_s:

$$n_s - 1 = -2\varepsilon - \eta$$

Lesson 2 b) Perturbations from inflation

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Latest measurement from *Planck*:

$$n_s - 1 = -0.035 \pm 0.004$$

Outline

Lesson 1: background cosmology and Newtonian perturbations

a) **Homogeneous cosmology.** The FRW metric. Distances and redshift. The Friedman Equation.
b) **Newtonian perturbations.** The perturbation equations. Linear theory. Jeans equation and growth.
c) **Relativistic perturbations.** Gauge-invariant PT. Hydrodynamic perturbations. Qualitative behavior.

Lesson 2: Inflation

a) **Inflation.** The curvature and horizon problems. Scalar fields. Slow roll.
b) **Perturbations from inflation.** Inflationary PT. Random fields. The primordial power spectrum.

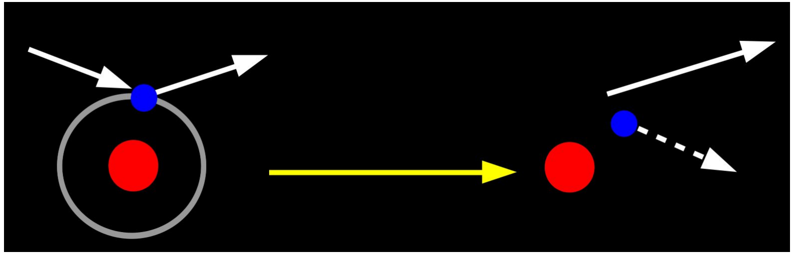
Lesson 3: cosmological probes of structure

a) **The CMB.** Recombination. Temperature anisotropies. Scattering and polarization.
b) **The matter power spectrum.** The linear power spectrum. Non-linearities
c) **Gravitational lensing.** Geodesics. Galaxy lensing. CMB lensing.

Lesson 3 a) The CMB

Recombination and last scattering

At early times, energetic photons prevent recombination



Lesson 3 a) The CMB

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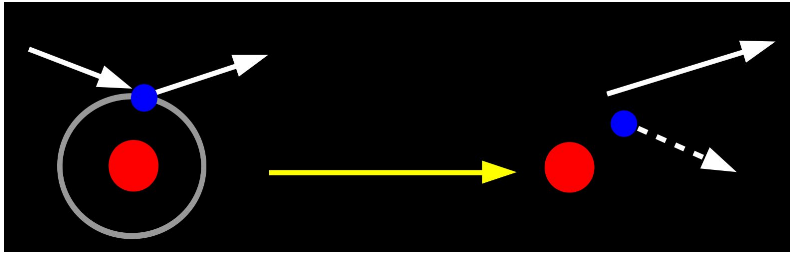
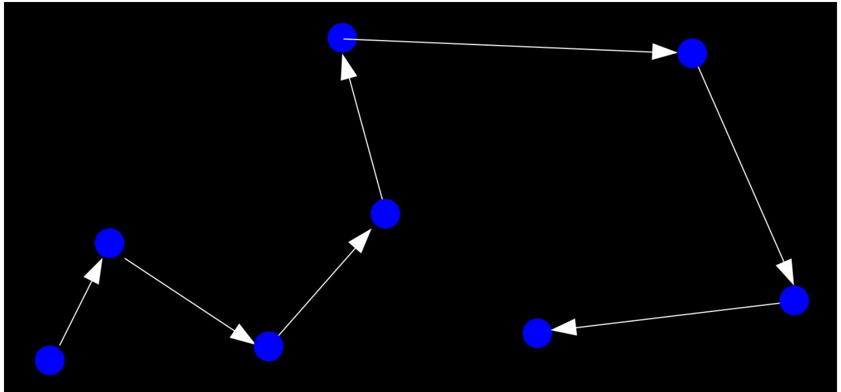


Many free electrons → short mean-free path



Lesson 3 a) The CMB

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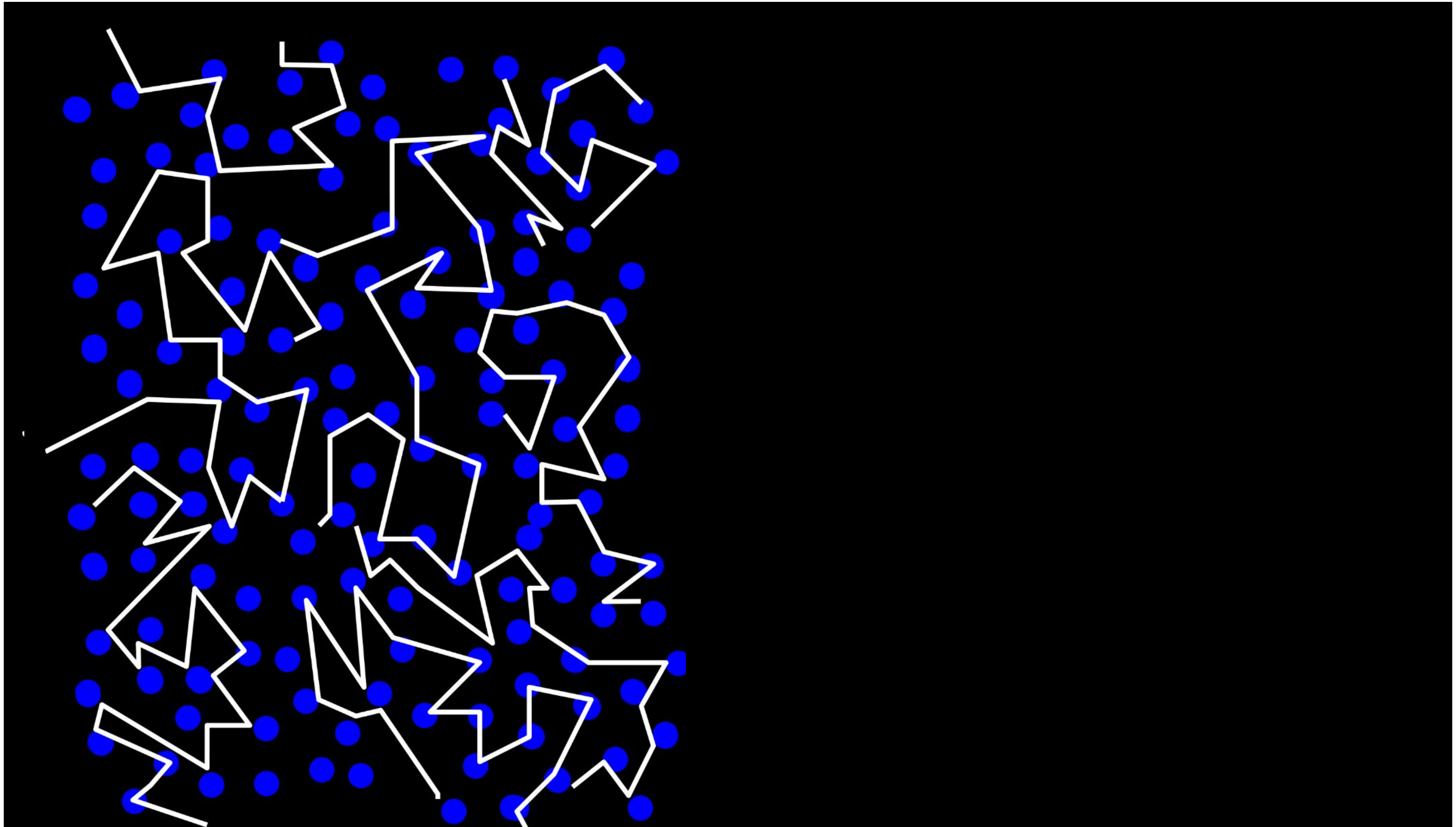

Lesson 3 a) The CMB

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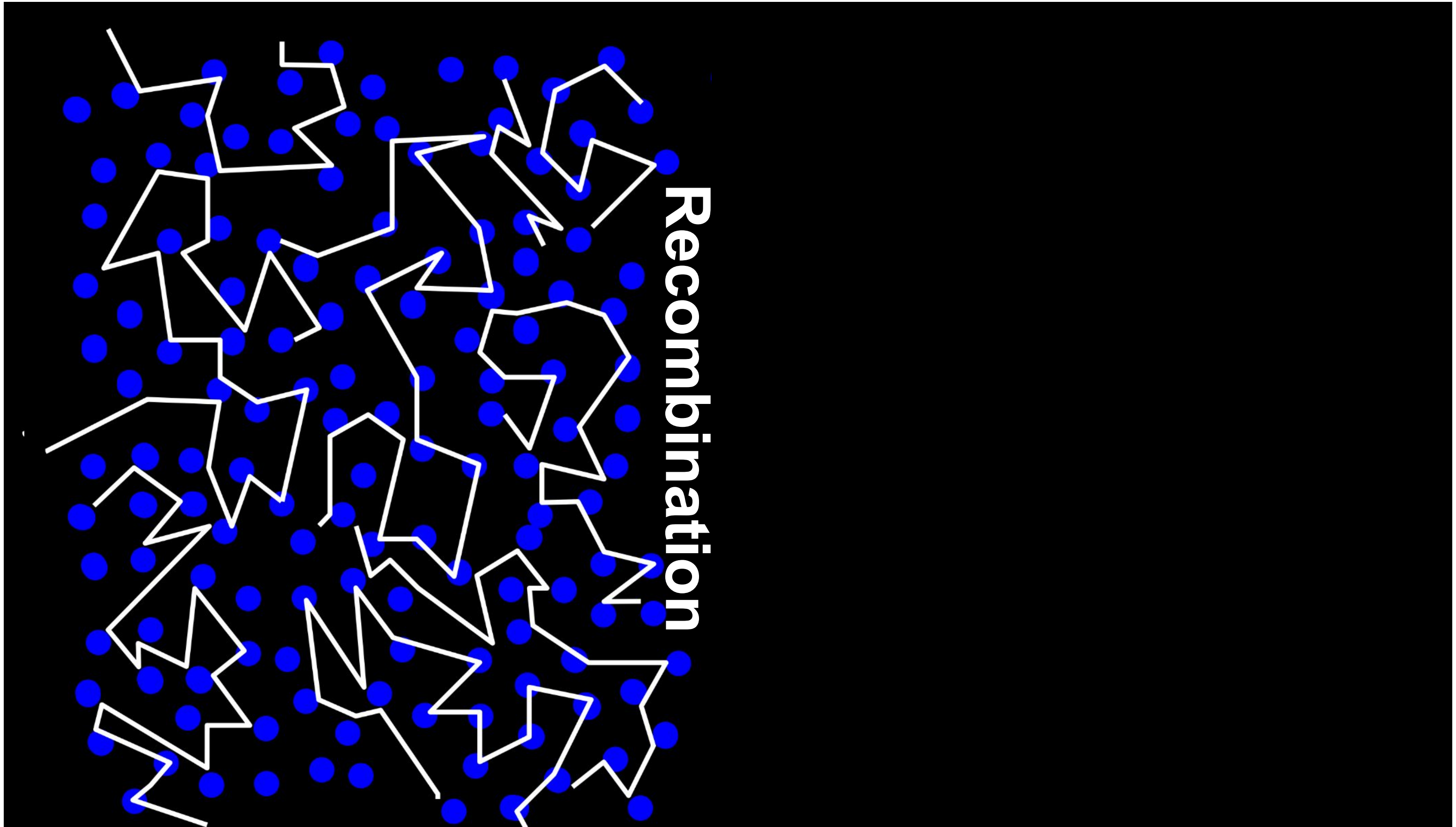

Lesson 3 a) The CMB

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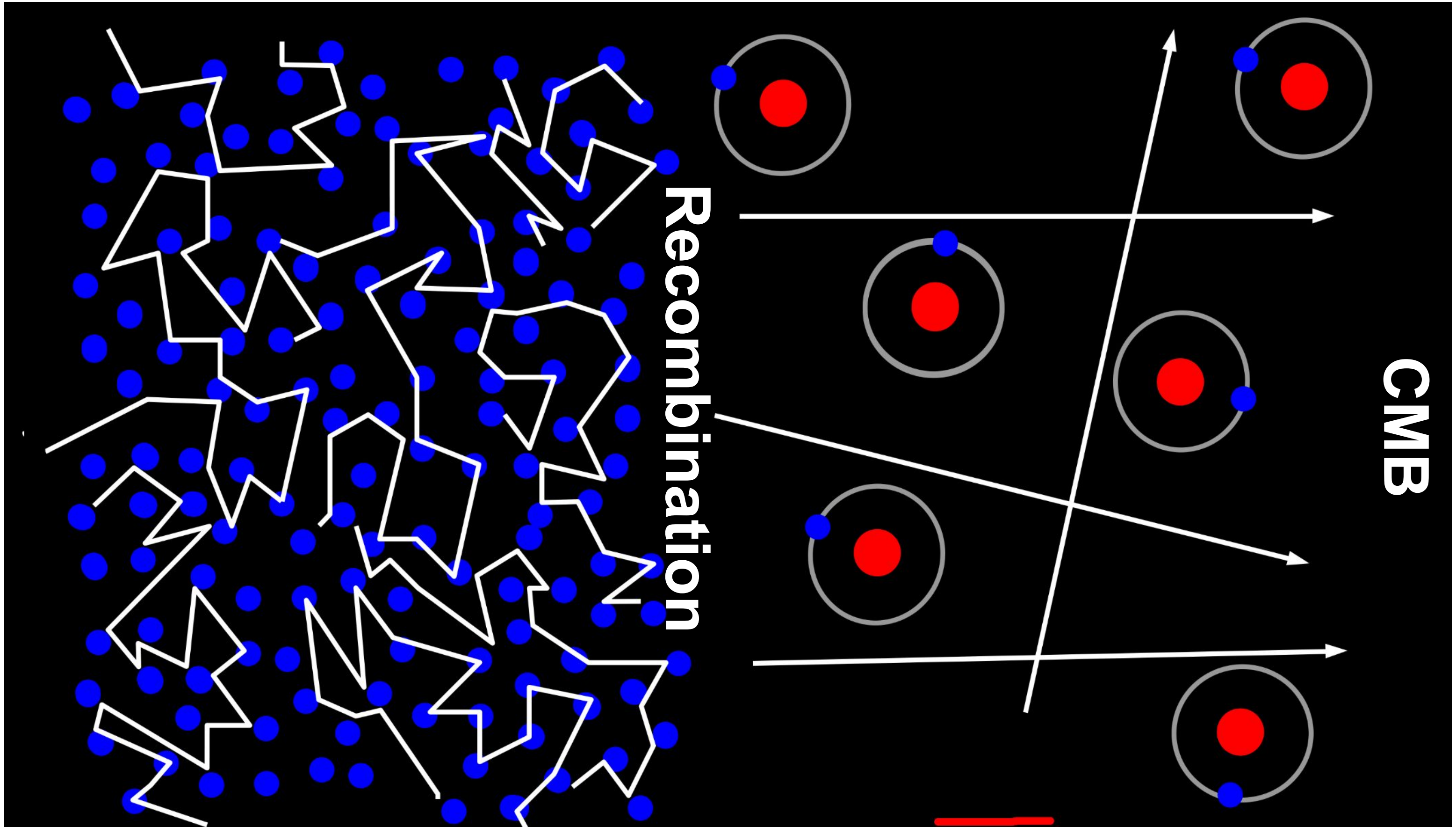

Lesson 3 a) The CMB

When does recombination happen?

1. Wild guess: hydrogen ionisation potential χ=13.6 eV.

$$1 + z_{\rm rec} = \frac{\chi}{k_B T_{\rm CMB}} \sim 5 \times 10^4$$

Rad. domination?

Lesson 3 a) The CMB

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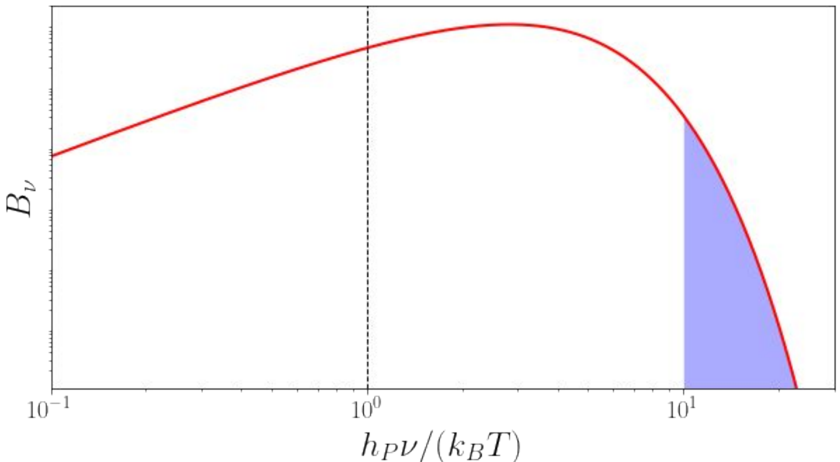
$$T_4 \equiv T/(10^4\ {\rm K}) \quad \omega_b \equiv \Omega_b h^2$$



x_e	$z_{\rm Saha}$
0.5	1370
0.1	1250
0.01	1140

Matter domination!

Lesson 3 a) The CMB

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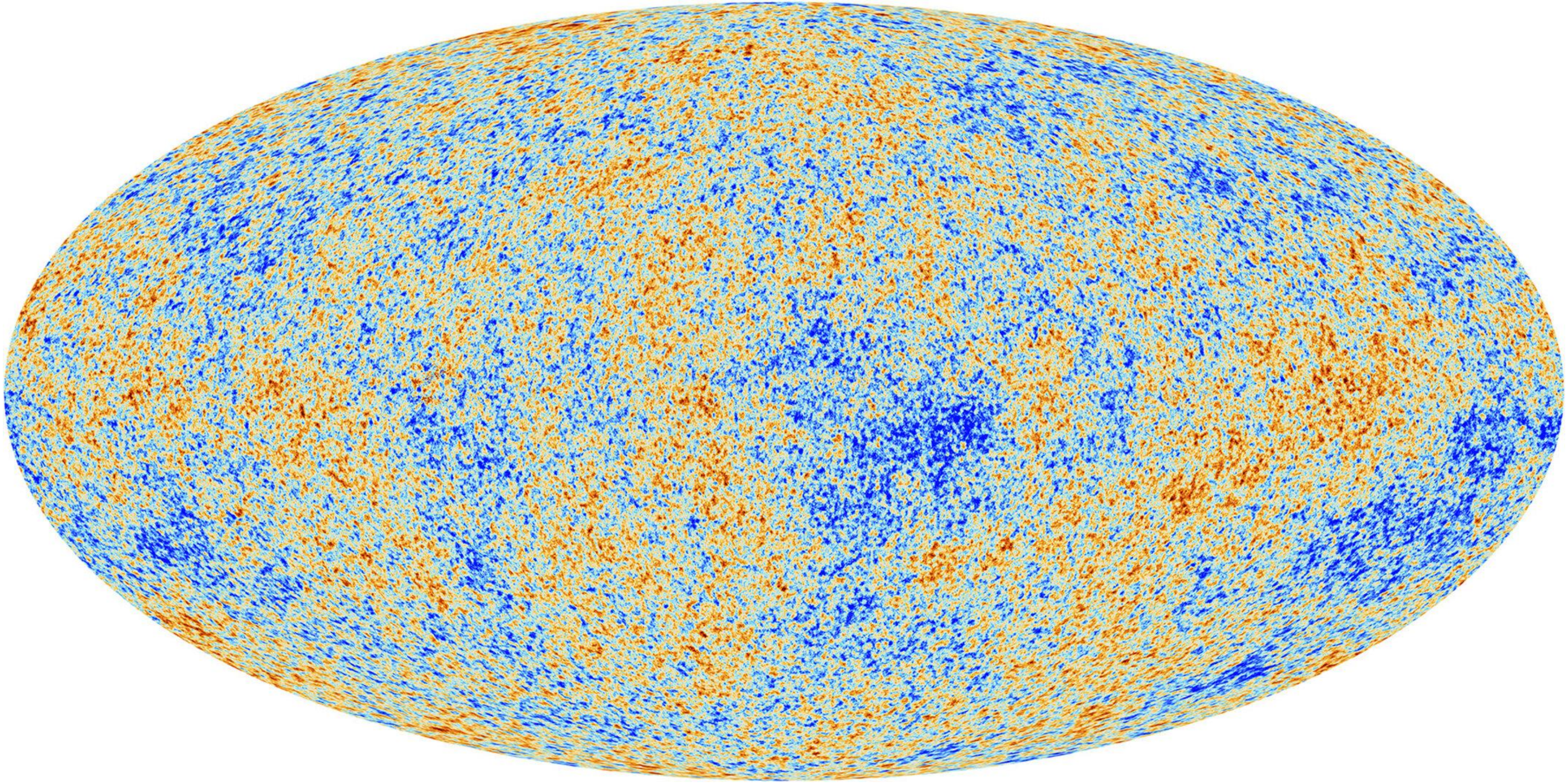
3. Non-equilibrium corrections due to:
- Re-absorption
- Lyman-α resonant scattering

x_e	$z_{\rm Saha}$	$z_{\rm exact}$
0.5	1370	1210
0.1	1250	980
0.01	1140	820

Lesson 3 a) The CMB

Perturbations during recombination



Lesson 3 a) The CMB

Perturbations during recombination

We need to solve for the evolution of 6 quantities: $\boxed{(\delta_c, \theta_c)}, \boxed{(\delta_b, \theta_b)}, \boxed{(\delta_\gamma, \theta_\gamma)}, \boxed{\psi}$

CDM baryons photons potential

Lesson 3 a) The CMB

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CDM

$$\delta_c' + \theta_c - 3\phi' = 0, \quad \theta_c' + \mathcal{H}\theta_c - k^2\psi = 0$$

DM uncoupled (except gravitationally)

Lesson 3 a) The CMB

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Baryon + photon fluid

$$\delta_b' + \theta_b - 3\phi' = 0, \quad \theta_b' + \mathcal{H}\theta_b - k^2\psi = c_{s,b}^2 k^2 \delta_b + \frac{4\bar{\rho}_\gamma}{3\bar{\rho}_b} a\, n_e\, \sigma_T (\theta_\gamma - \theta_b)$$

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$z<z_{rec}$: single, tightly-coupled, viscous fluid
$z>z_{rec}$: baryons decoupled, like DM

$c_s \sim 1/\sqrt{3} \longrightarrow$ Acoustic waves before recombination.

Lesson 3 a) The CMB

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Gravitational potential

$$k^2\psi + 3\mathcal{H}(\psi' + \mathcal{H}\psi) = -4\pi G a^2\, \bar{\rho}\, \delta,$$

Potential set mostly by DM at recombination.

$c_s \sim 1/\sqrt{3} \longrightarrow$ Acoustic waves before recombination.

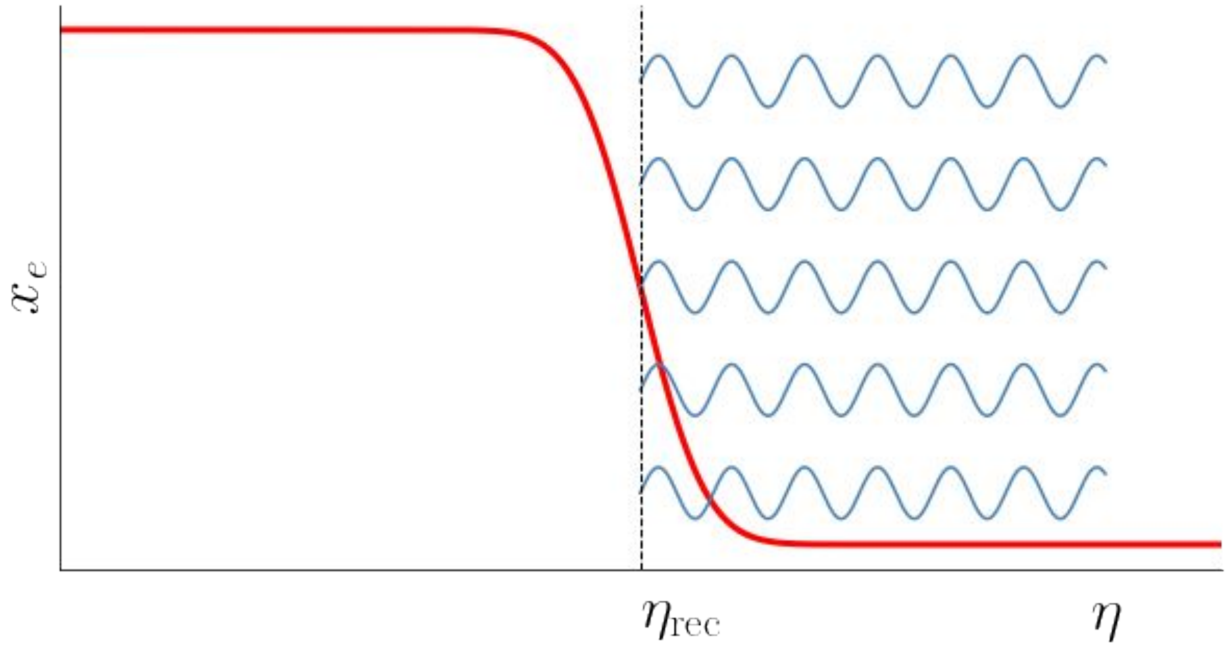
Lesson 3 a) The CMB

Temperature fluctuations

Approximations:

1. Instantaneous recombination.
2. Temperature from frequency:

$$\frac{\delta T}{\bar{T}} = \frac{\delta \nu}{\bar{\nu}}$$



Lesson 3 a) The CMB

Temperature fluctuations

Redshift in perturbed FRW (as we saw in tutorial):

$$\frac{\nu}{\nu_0} = 1 + z = \frac{1}{a}\left[1 - \psi + \psi_0 + \hat{\mathbf{n}} \cdot (\mathbf{v} - \mathbf{v}_0) + \int_{\eta_0}^{\eta} d\eta' \, (\phi' + \psi')\right]$$

Lesson 3 a) The CMB

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Therefore:

$$\left.\frac{\delta T}{T}\right|_0 = \left(\frac{\delta T}{T} + \psi - \hat{\mathbf{n}} \cdot \mathbf{v}\right)_{\rm rec} + \int_{\eta_{\rm rec}}^{\eta_0} d\eta (\phi' + \psi').$$

Lesson 3 a) The CMB

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Stefan-Boltzmann law: $\rho_\gamma \propto T^4 \quad \rightarrow \quad \frac{\delta T}{T} = \frac{1}{4}\delta_\gamma$

$$\frac{\delta T}{T}(\hat{\mathbf{n}}) = \left(\frac{\delta_\gamma}{4} + \psi - \hat{\mathbf{n}} \cdot \mathbf{v}\right)(\eta_{\mathrm{rec}}, \chi_{\mathrm{rec}}\hat{\mathbf{n}}) + \int_{\eta_{\mathrm{rec}}}^{\eta_0} d\eta \, (\phi' + \psi')(\eta, \chi\hat{\mathbf{n}})$$

Lesson 3 a) The CMB

Temperature fluctuations

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Doppler (kinematic redshift)

Sachs-Wolfe (gravitational redshift)

Intrinsic fluctuation

Lesson 3 a) The CMB

Temperature fluctuations

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Integrated Sachs-Wolfe (ISW)

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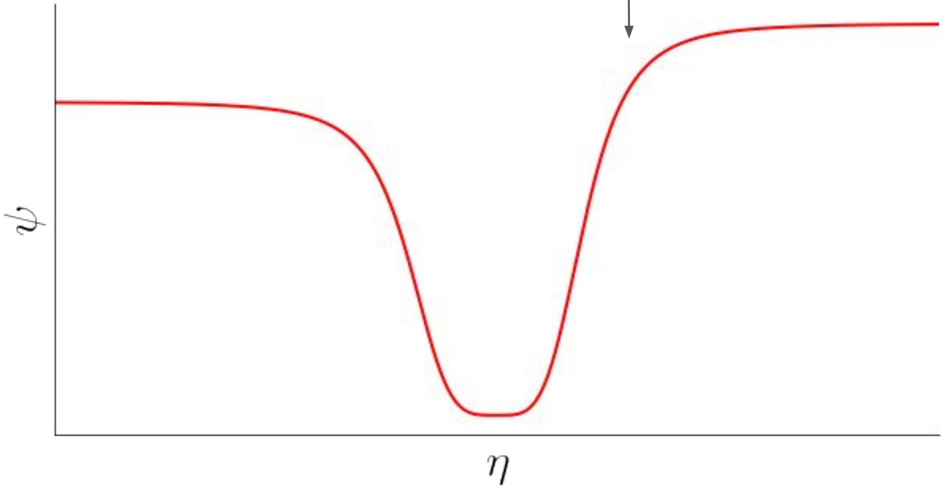
ISW:

- Caused by time-varying potentials at early (R-dom) and late (Λ-dom) times.
- Effect on CMB C_ℓ from early ISW.
- Late ISW detectable in cross-correlation with low-z probes.



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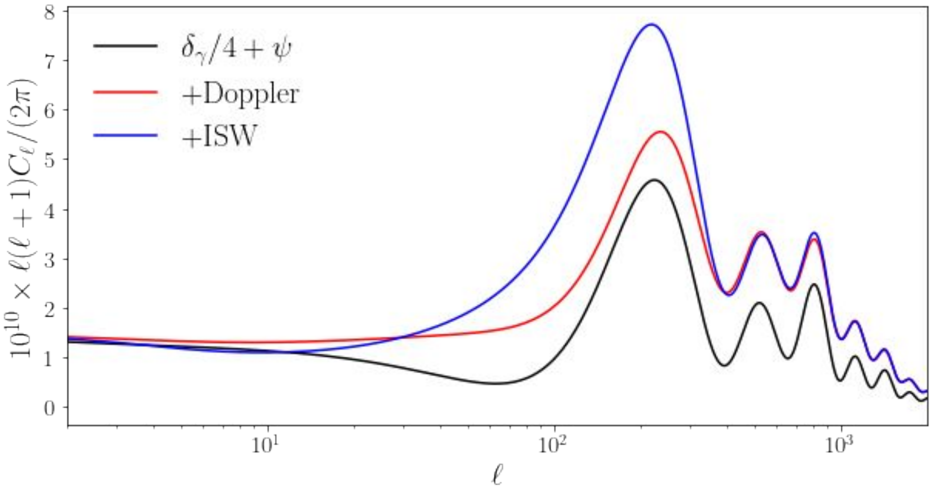

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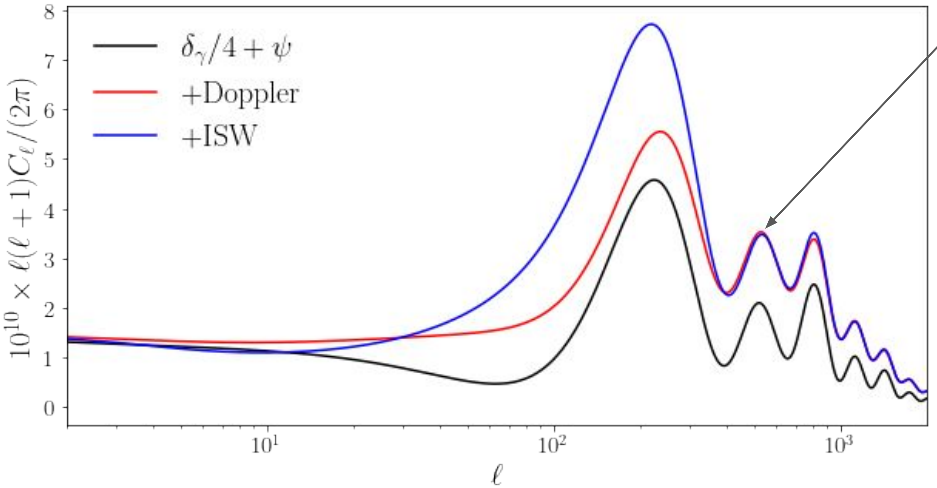

$$\sim \cos(kr_s)$$

$$r_s = \int^{\eta_{\rm rec}} d\eta\, c_s$$

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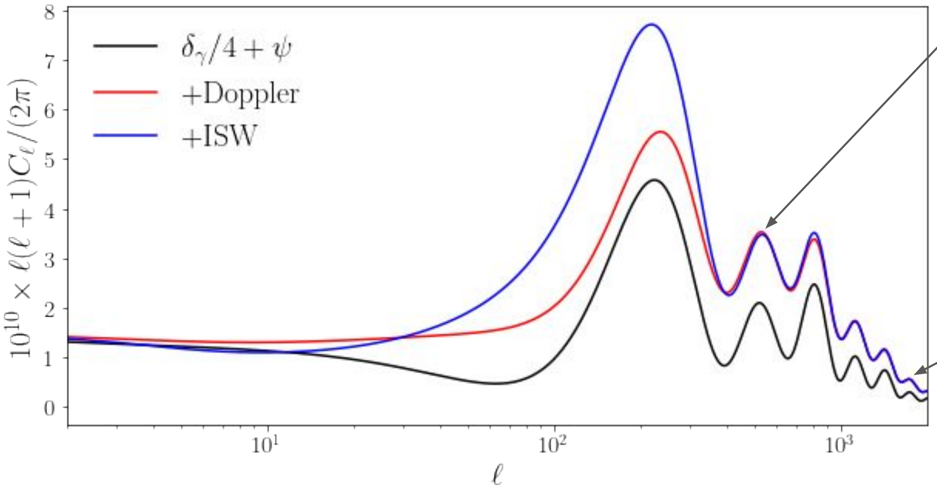


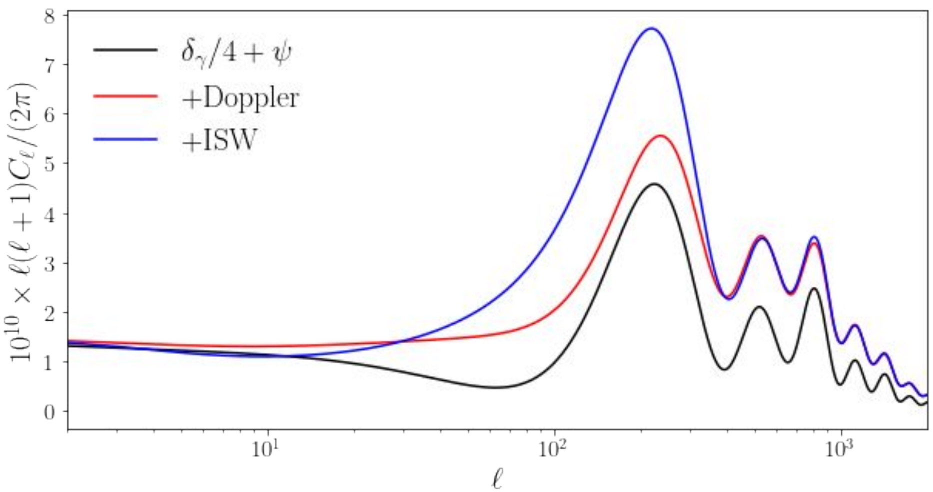
$\sim\cos(kr_s)$

$$r_s = \int^{\eta_{\rm rec}} d\eta\, c_s$$

Damping tail due to viscosity in baryon-photon fluid.

Lesson 3 a) The CMB

Peak structure (height, frequency, position) governed by r_s, d_A, relative baryon abundance f_b.



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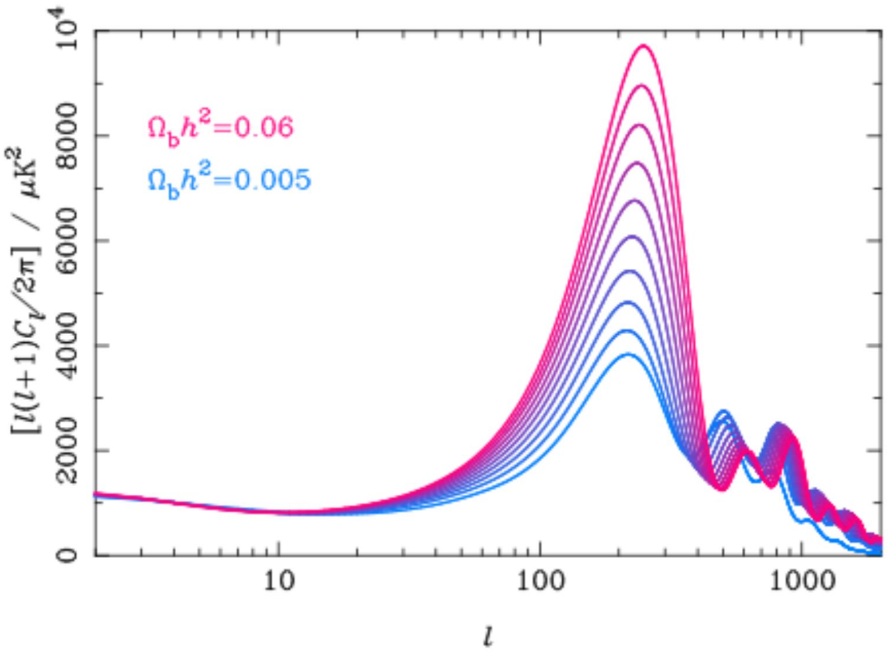

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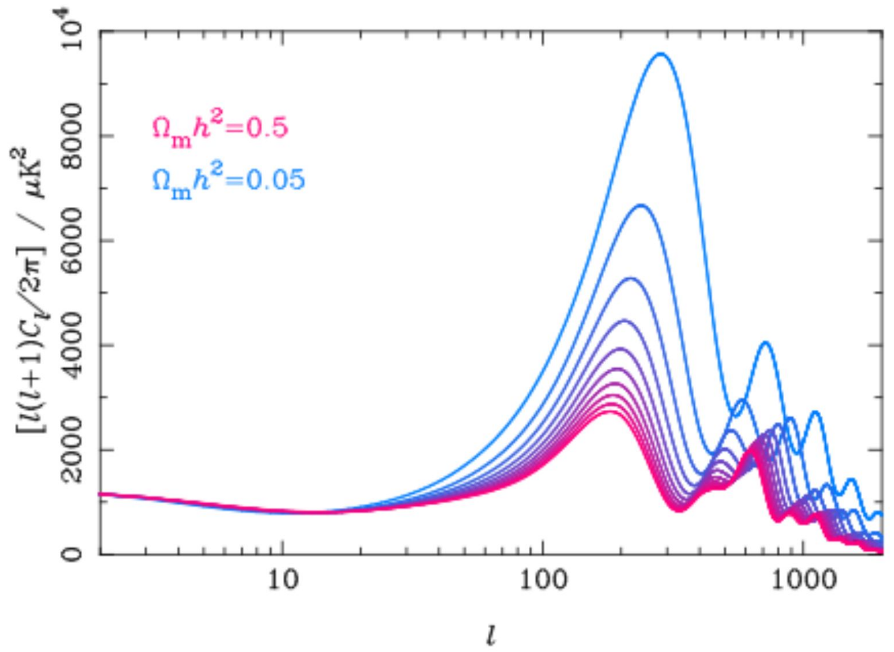

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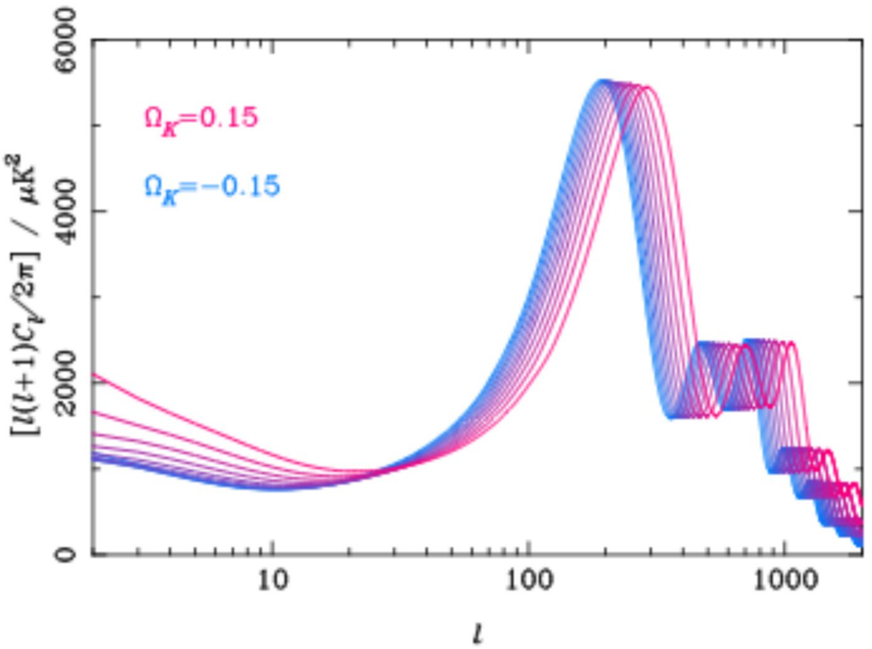
- Ω_k or Ω_Λ: peak positions (through d_A).



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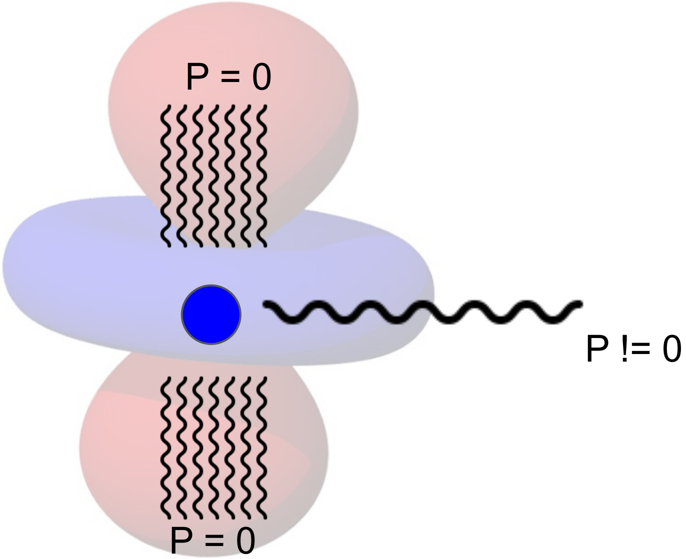

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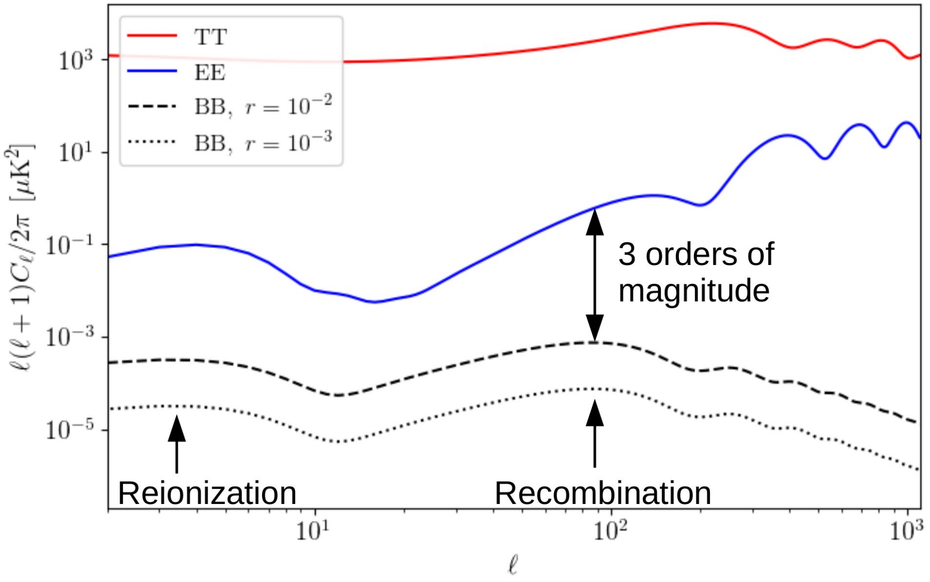

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Recommended references:

- Durrer: "*The Cosmic Microwave Background*"
- Mukhanov: "*Physical foundations of cosmology*"
- Dodelson: "*Modern cosmology*"
- Ma & Bertschinger: "*Cosmological perturbations*"

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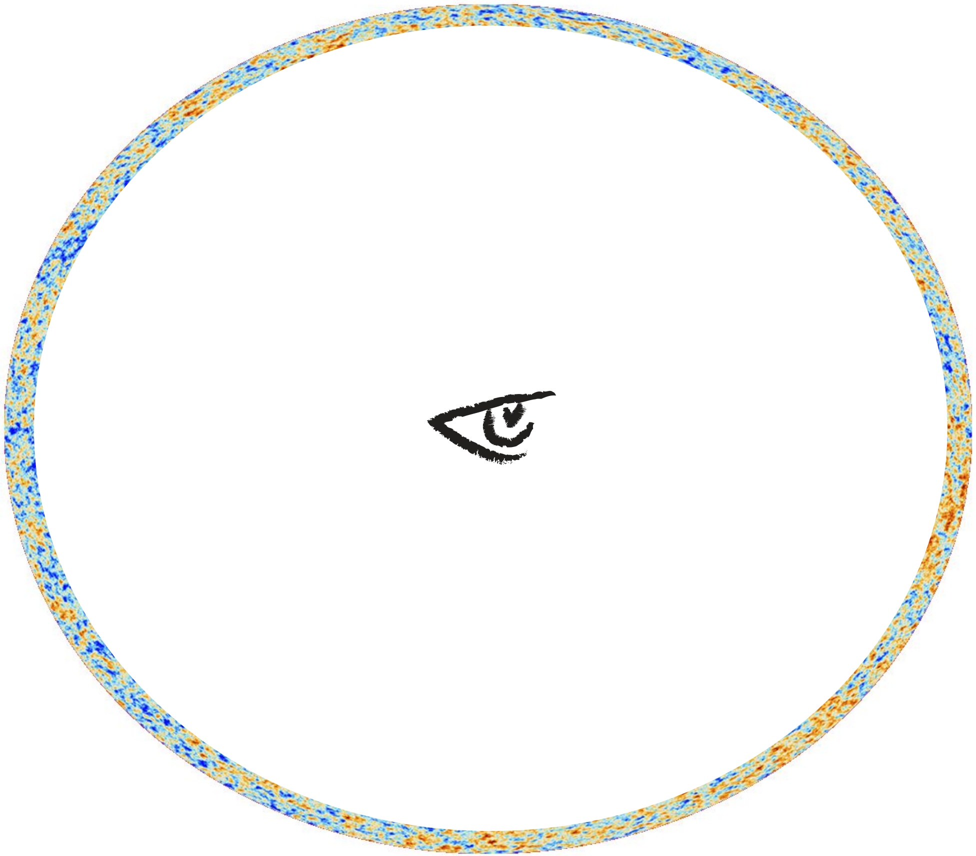
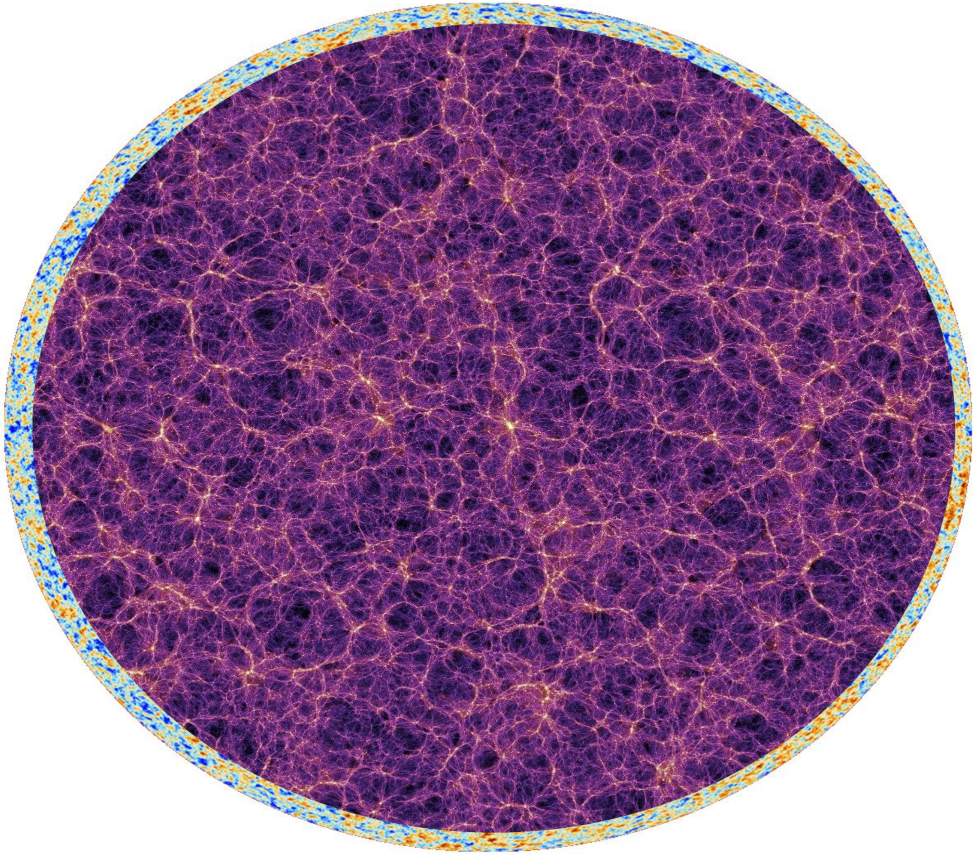
Lesson 3 b) The matter power spectrum



CMB ultimately limited by 2D nature:

$$N^{2D}_{\text{modes}} \propto \ell^2_{\max}$$

Lesson 3 b) The matter power spectrum



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Can we study 3D matter fluctuations after recombination?

$$N^{3D}_{\rm modes} \propto V k^3_{\rm max}$$

Lesson 3 b) The matter power spectrum

After recombination:

- Dark matter overdensity keeps growing
- Baryons and photons decouple
- Baryons fall into potential wells set by dark matter
- Dark matter + baryons = non-relativistic matter

What does the matter power spectrum look like?

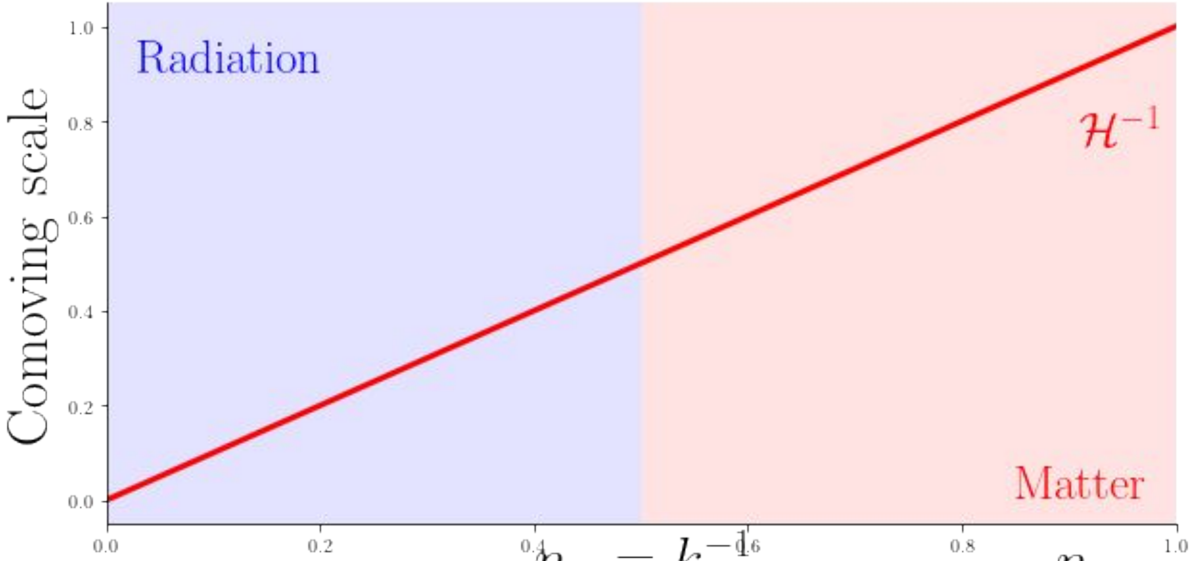
Lesson 3 b) The matter power spectrum

Evolution of perturbations depends on:

- Horizon scale
- Era (before or after R-M equality).

Key scale: horizon at equality k_{eq}

$$\mathcal{H}^{-1} \sim \eta$$



$\eta_{eq} = k_{eq}^{-1}$ $\quad \eta$

Lesson 3 b) The matter power spectrum

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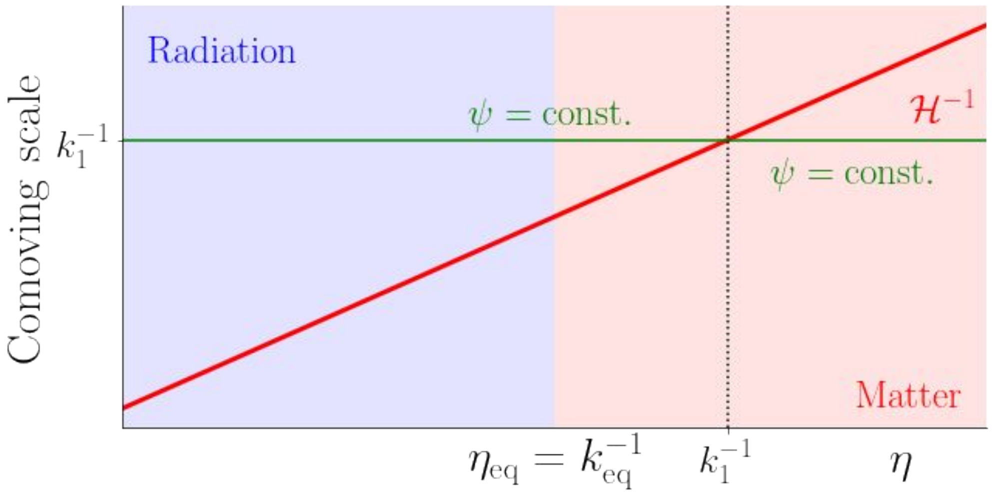

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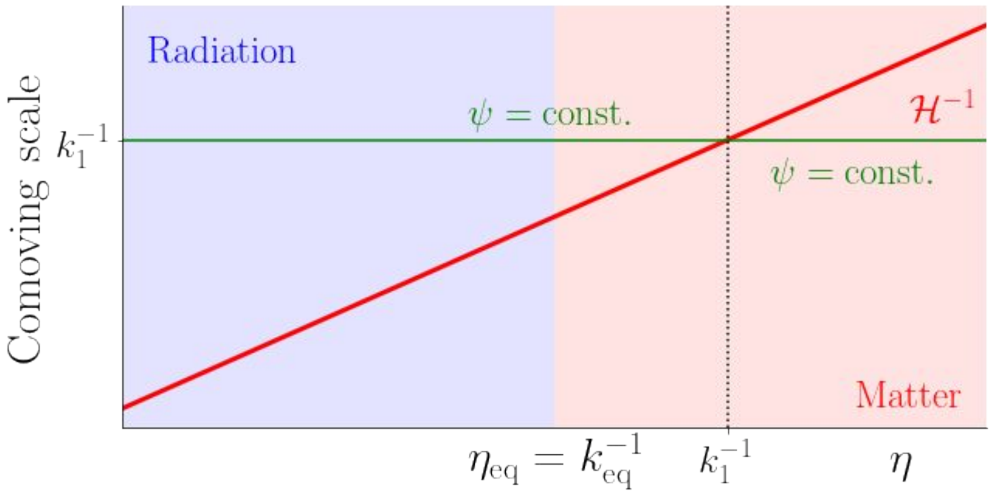

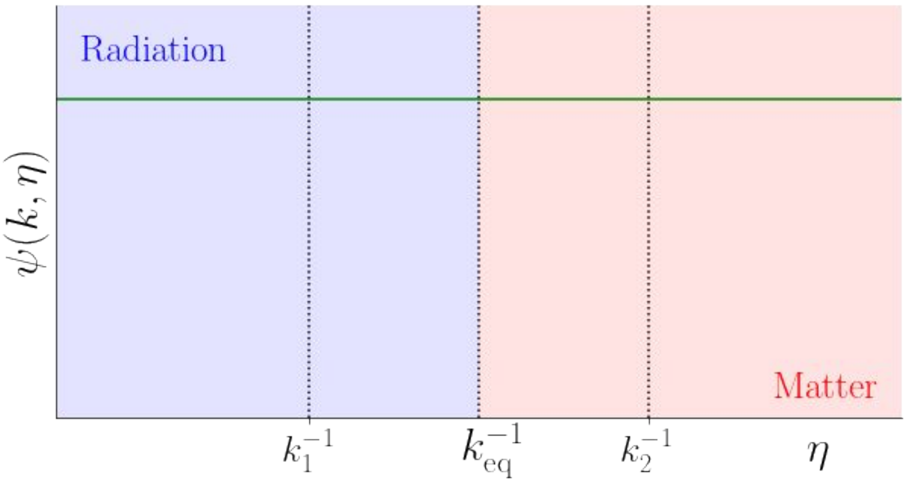

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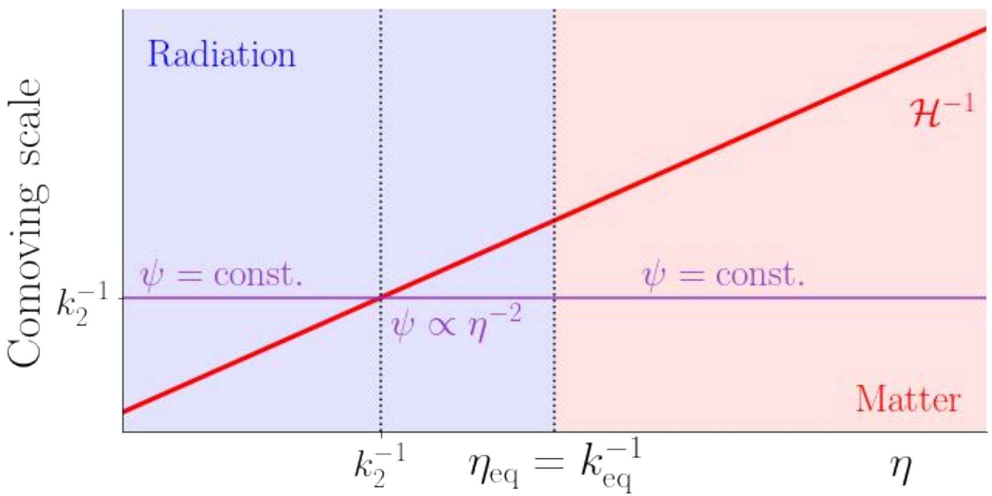

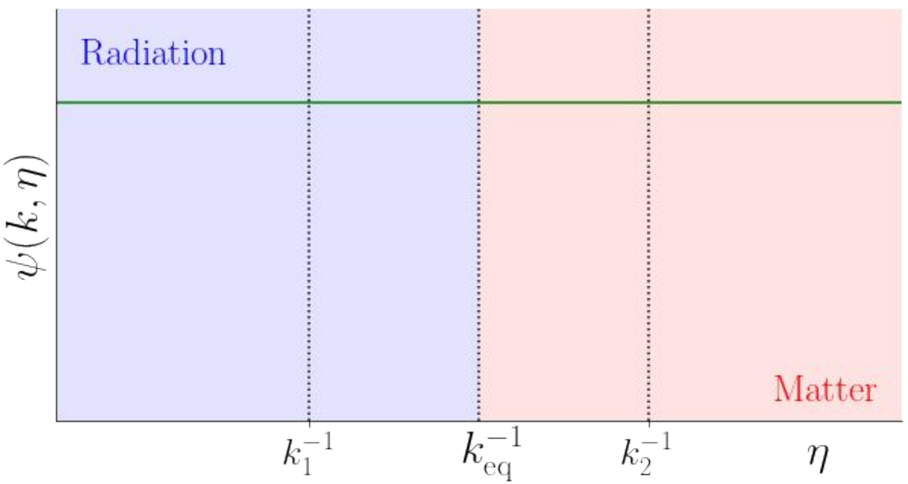

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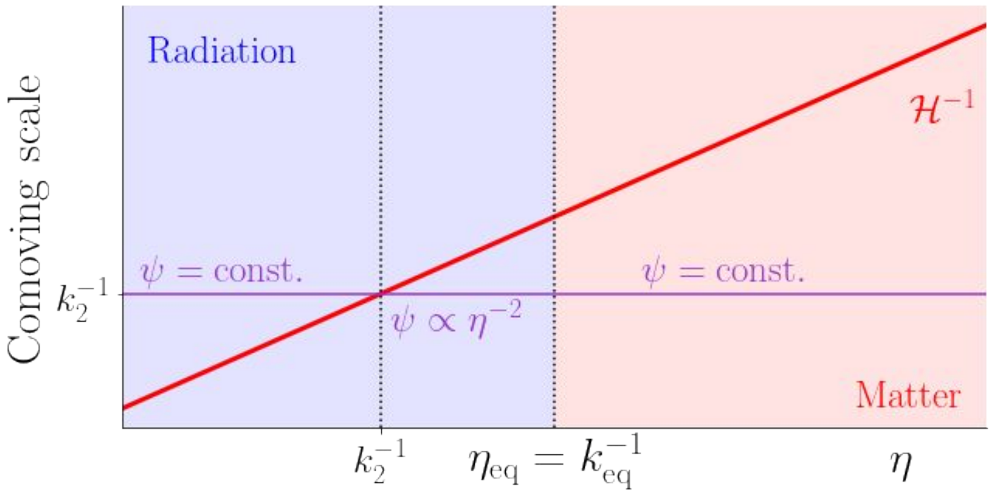

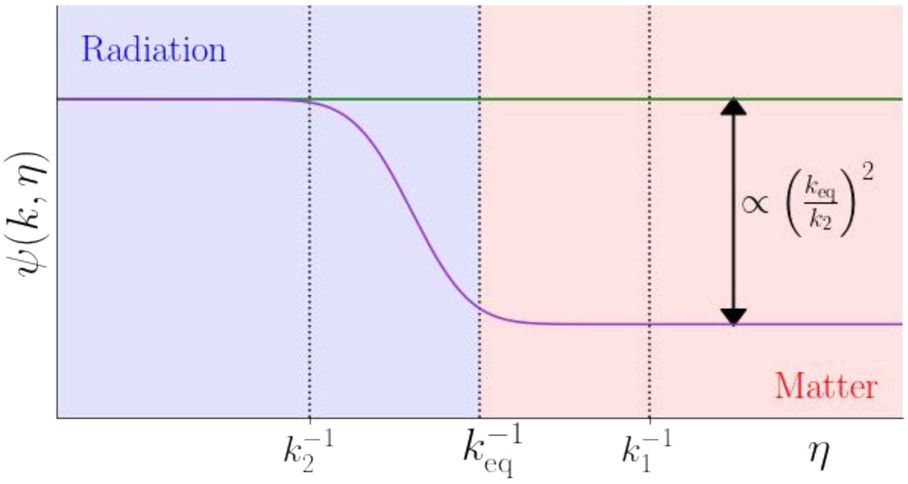

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$$\psi(k) \propto \begin{cases} \text{const.} & k < k_{\text{eq}} \\ k^{-2} & k > k_{\text{eq}} \end{cases}$$

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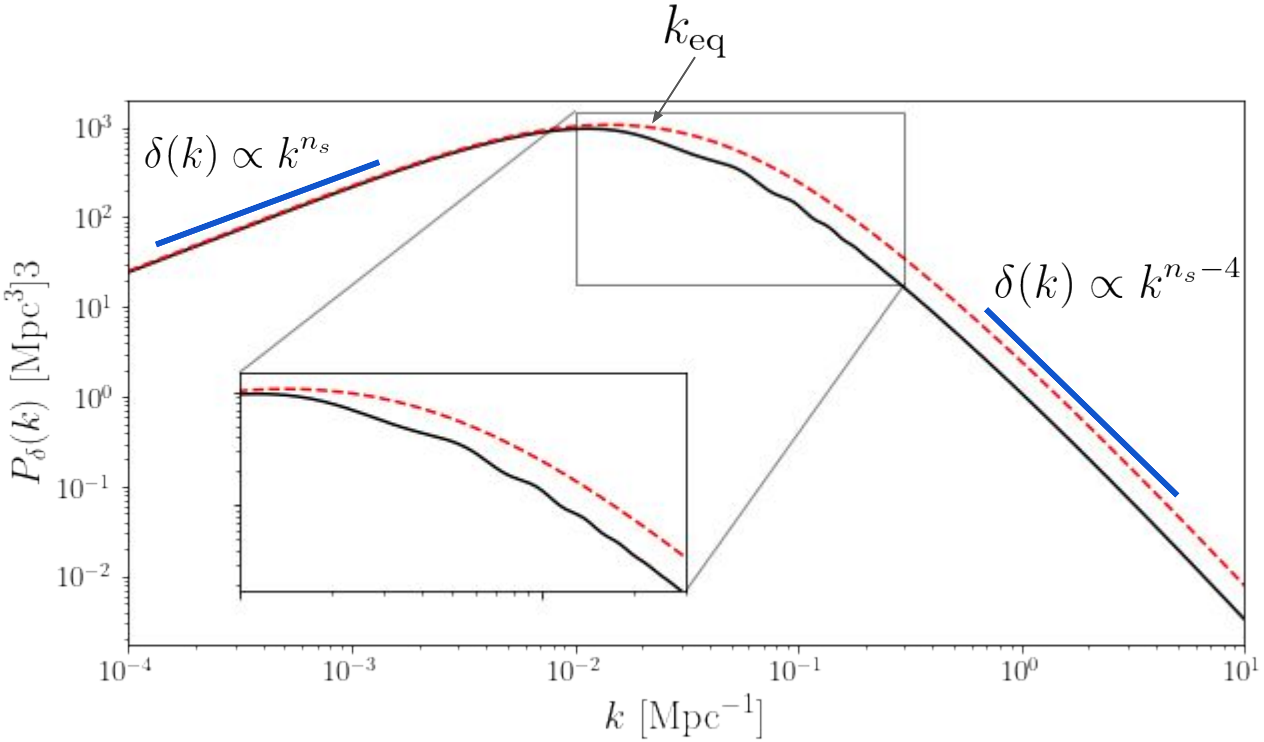
Therefore: $$P_\delta(k) \propto |\delta(k)|^2 \propto \begin{cases} k^{n_s} & k \ll k_{\rm eq} \\ k^{n_s-4} & k \gg k_{\rm eq} \end{cases}$$

Lesson 3 b) The matter power spectrum



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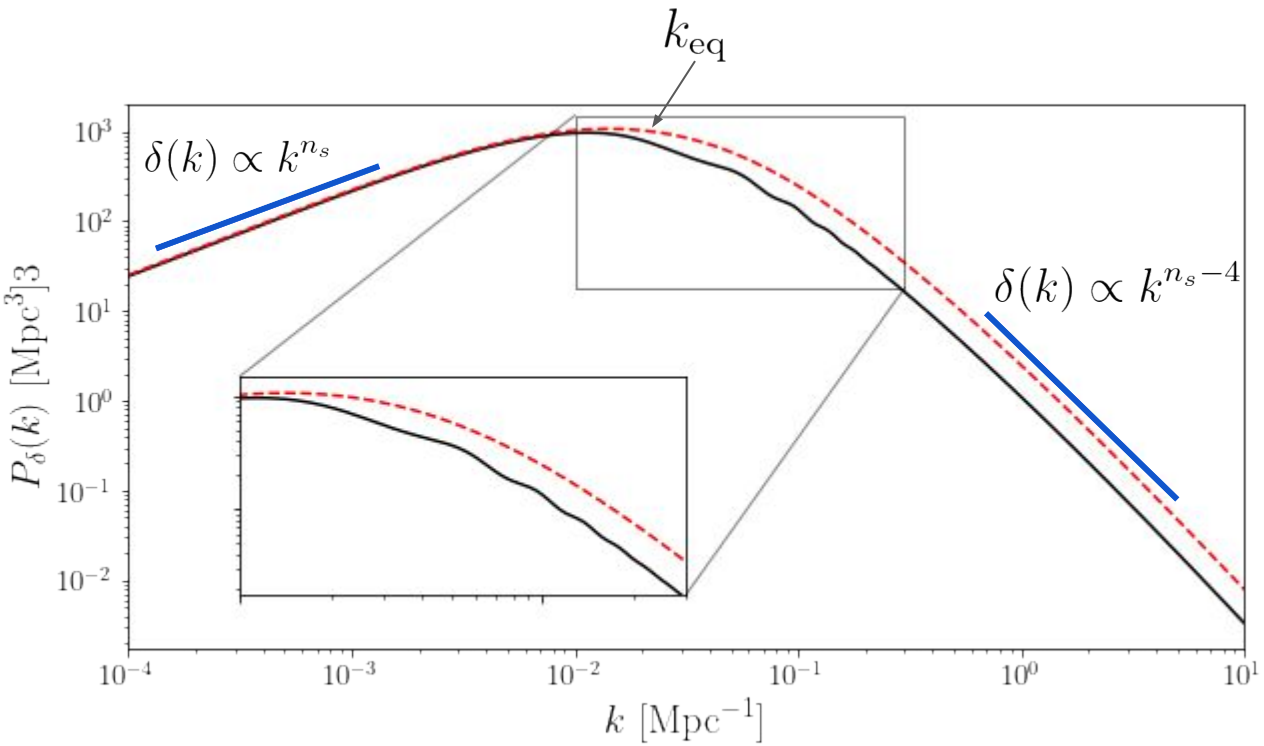


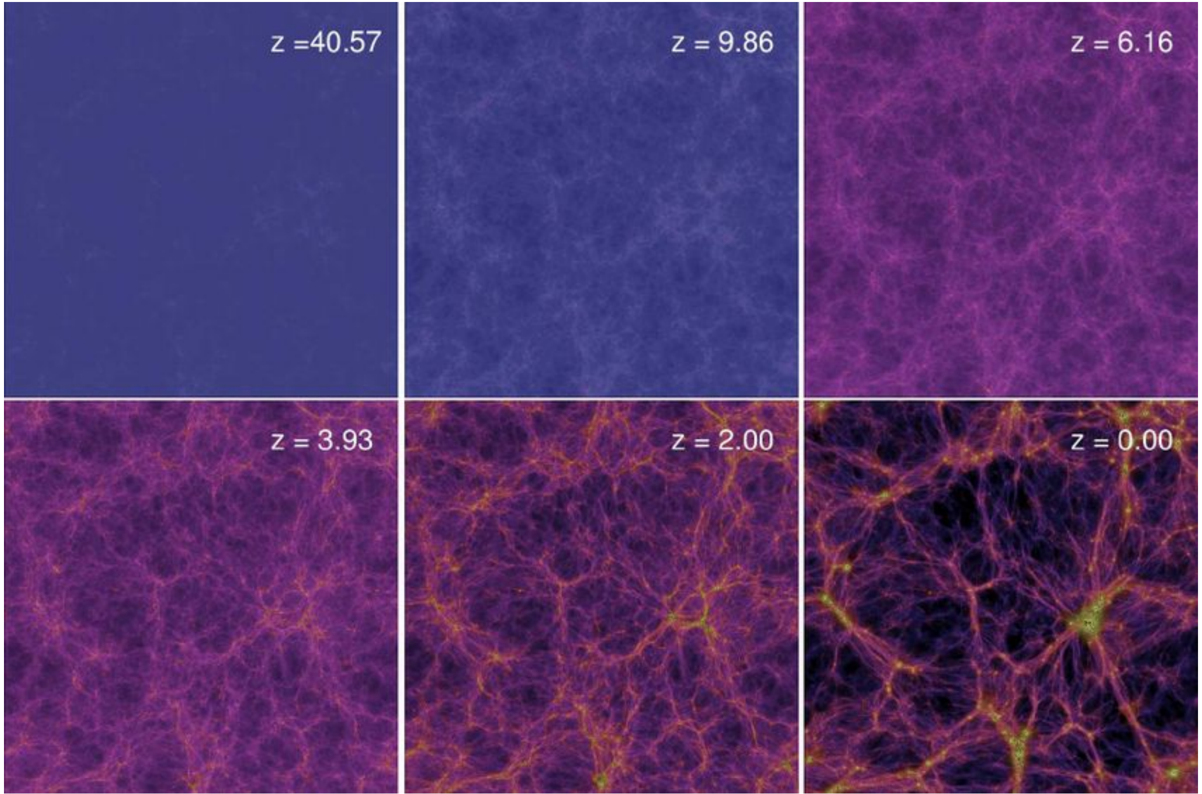
Baryon effects:

- Power decrement (baryons don't accrete before recombination).
- Baryon acoustic oscillations (standard ruler).

Lesson 3 b) The matter power spectrum

Non-linear evolution:

Eventually $\delta_{\mathbf{k}} \gtrsim 1$. PT may help a bit, but it fails fairly quickly.



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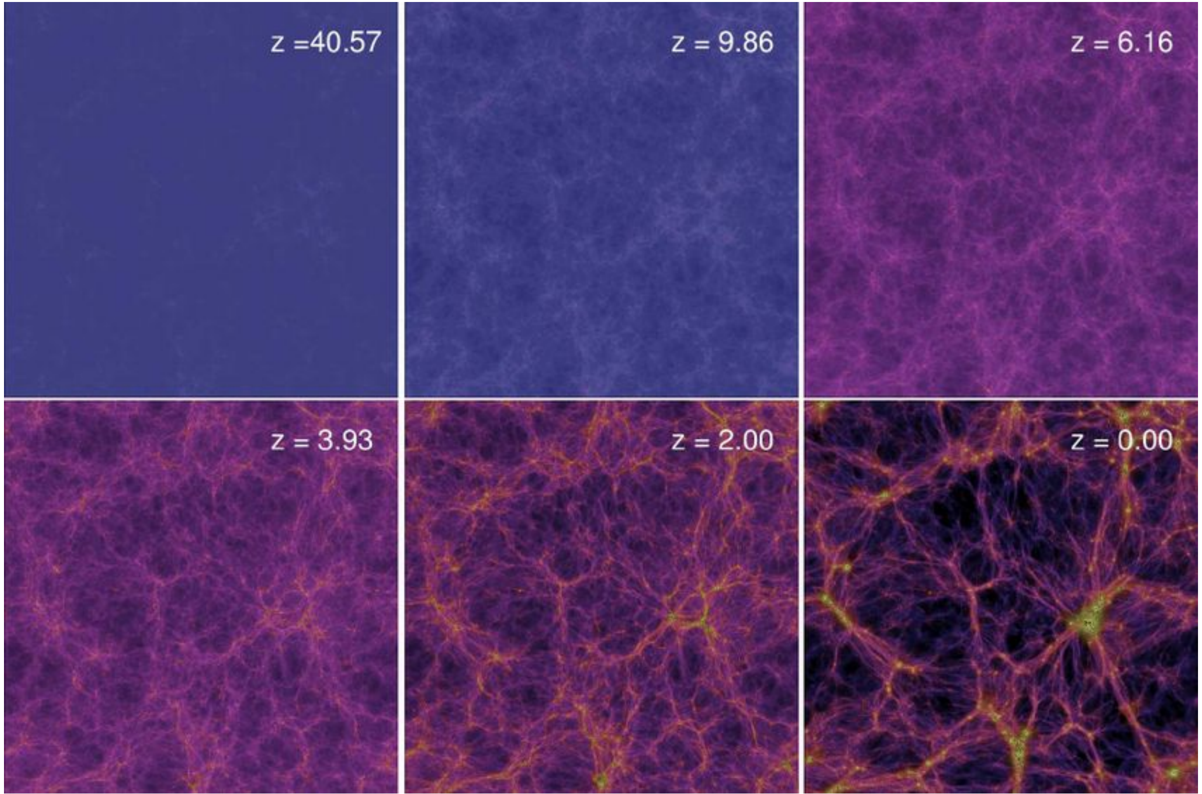
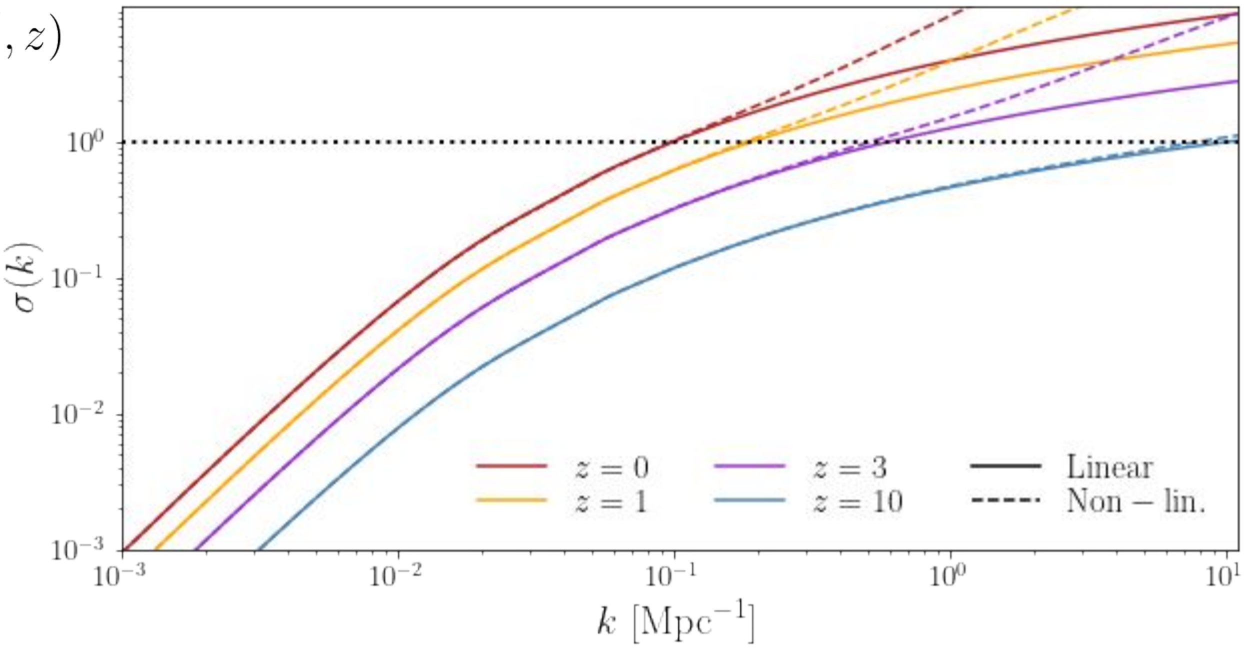


Consequences:
- Non-gaussianity: information leaks into higher-order correlators.
- Coupled evolution of Fourier modes.
- Scale-dependent growth.

Lesson 3 b) The matter power spectrum

Non-linear evolution:
Eventually $\delta_{\mathbf{k}} \gtrsim 1$. PT may help a bit, but it fails fairly quickly. When does it fail?
Useful quantity: overdensity variance

$$\sigma^2(k,z) = \int_0^k \frac{dk'\,k'^2}{2\pi^2} P(k',z)$$



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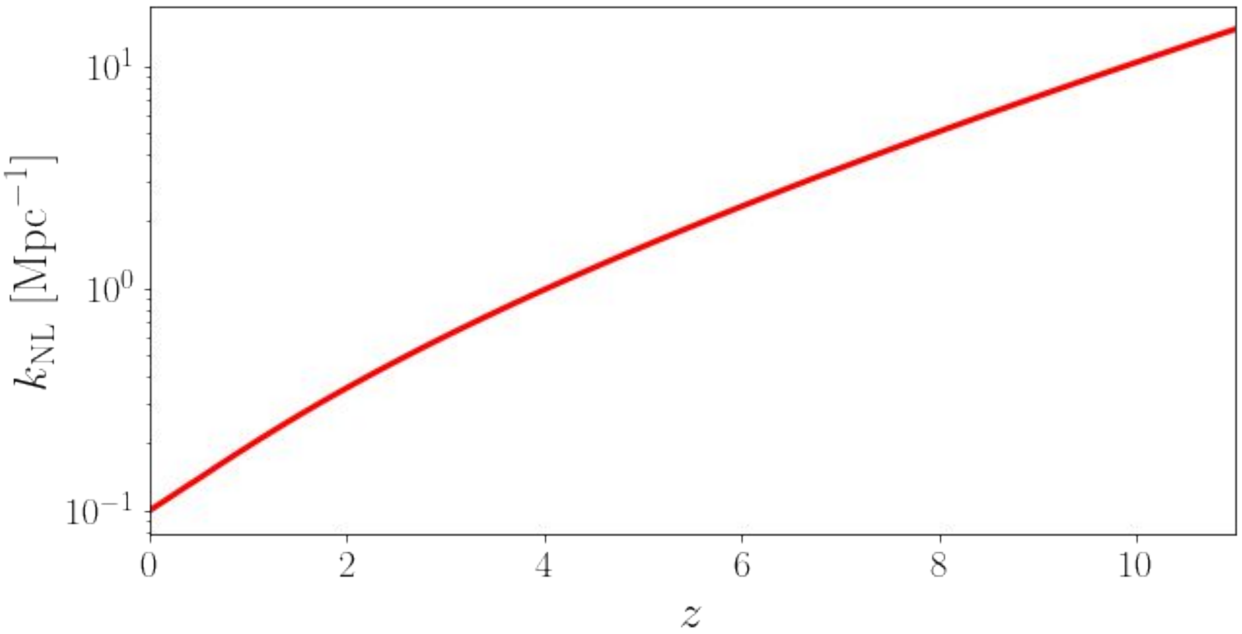
$$\sigma(k_{\rm NL}, z) \equiv 1$$

Avoiding non-linearities leads to severe limitations in constraining power, especially at z<1.

<u>We must tackle non-linearities!</u>



Lesson 3 b) The matter power spectrum

Tackling non-linearities:

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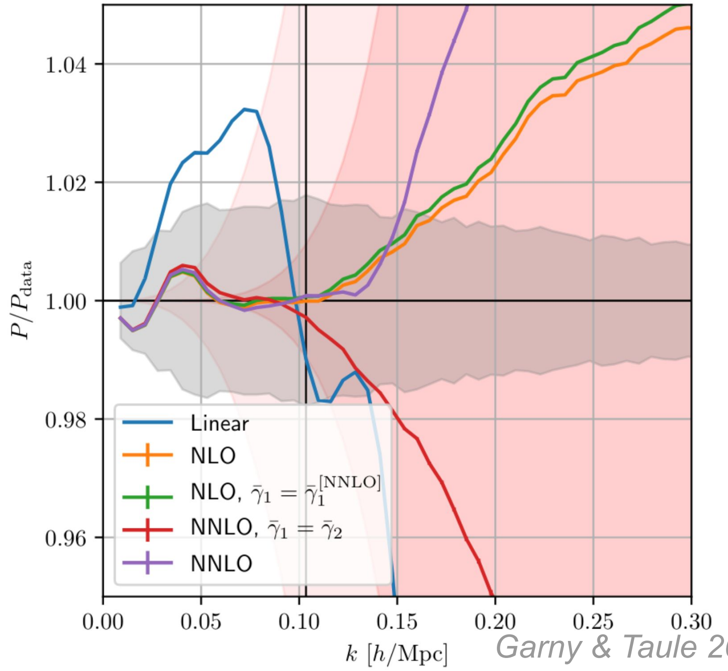

Garny & Taule 2022

Lesson 3 b) The matter power spectrum

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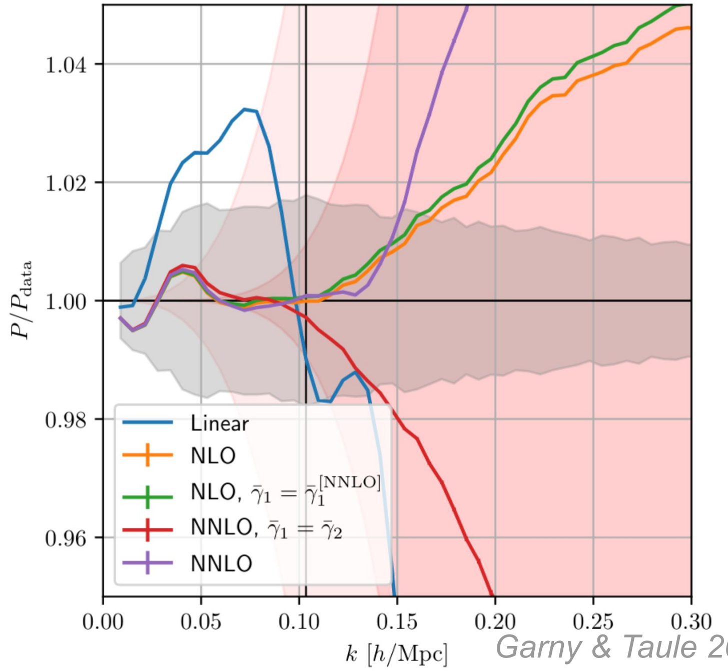

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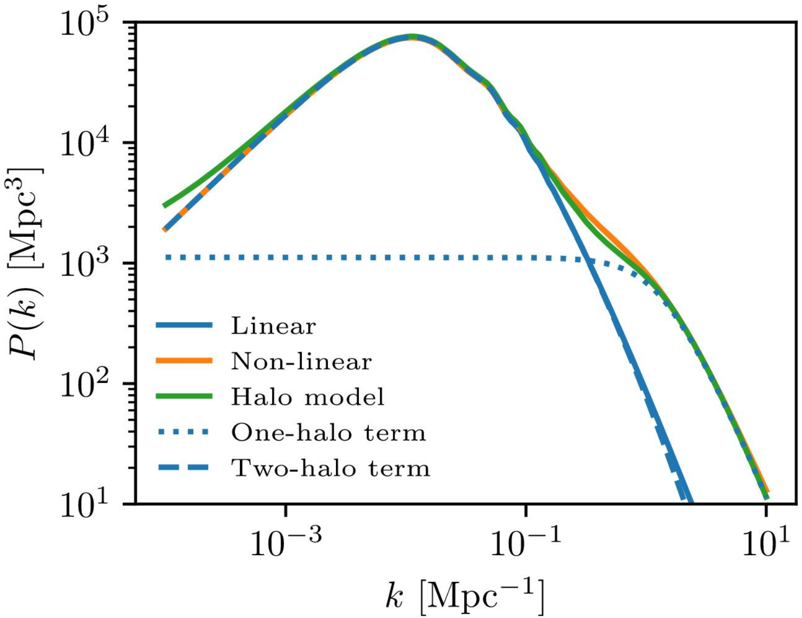

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   Potentially most robust way forward.

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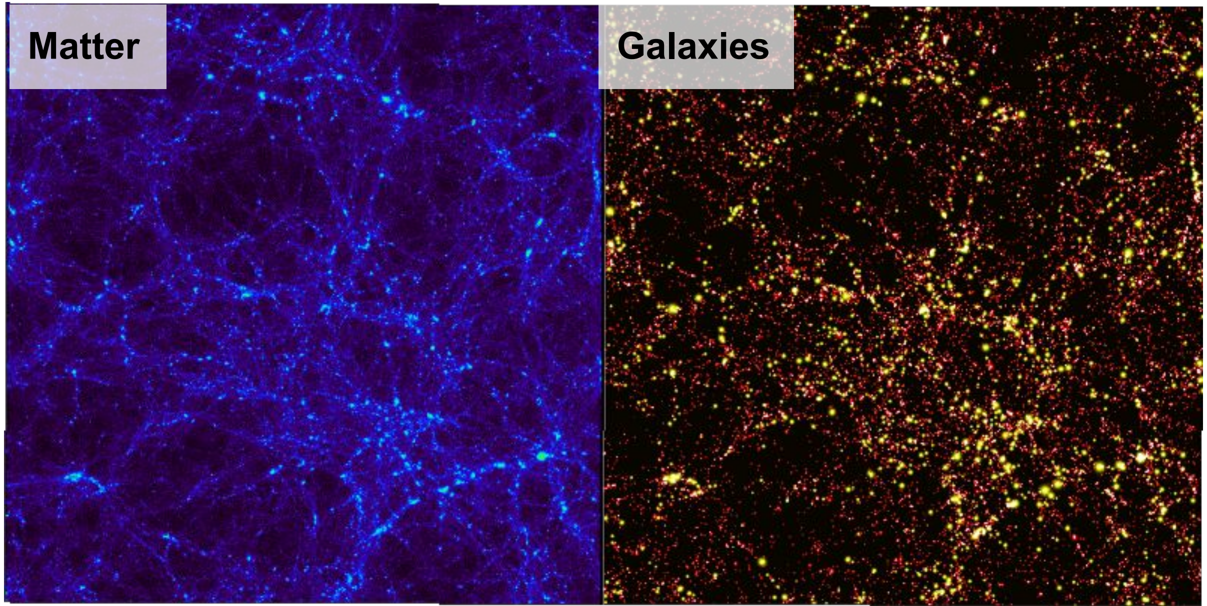
Lesson 3 c) Gravitational lensing

Probes of δ_M

There are many *indirect* probes of matter:

- Galaxy density
- Gas pressure/density (Sunyaev Zel'dovich, Lyman-α, 21cm)



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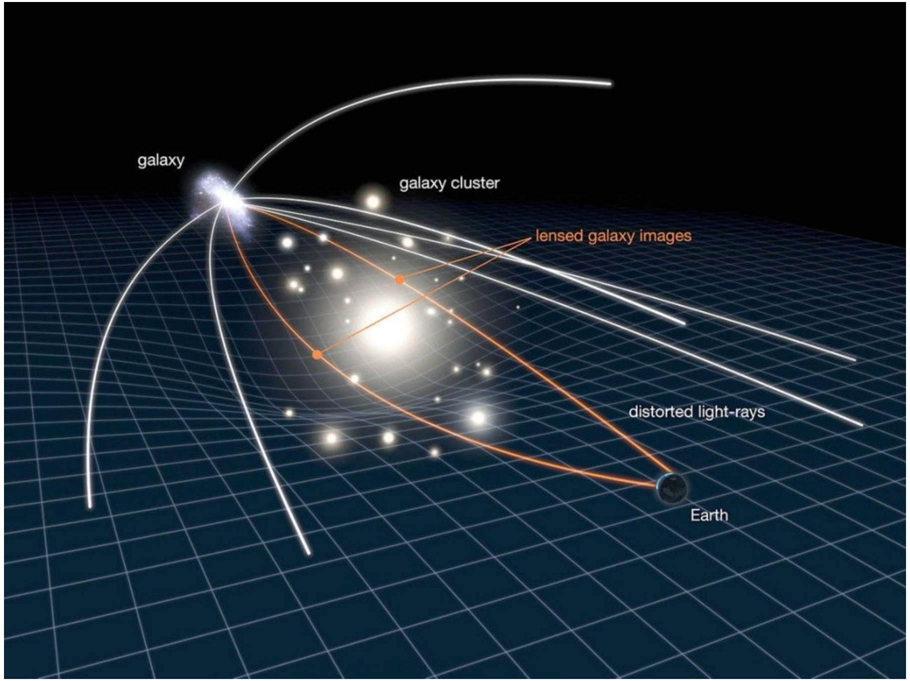

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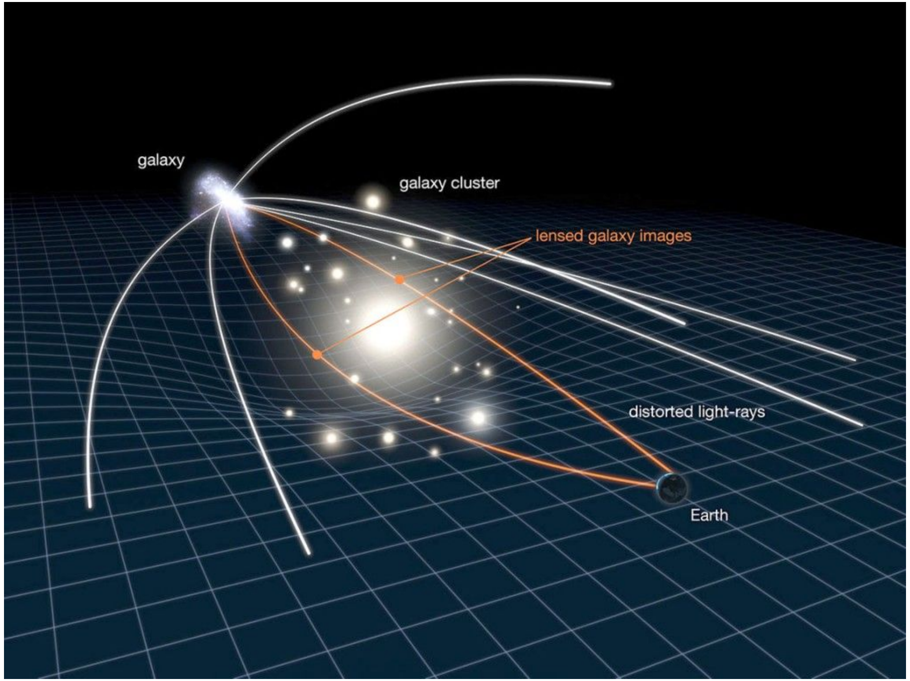
Weak lensing: gravity causes only small variations to photon path.



Lesson 3 c) Gravitational lensing

Weak lensing

<u>Starting point: geodesic equation</u>

$$\mathbf{x}(\eta) = -\hat{\mathbf{e}}_0 \int_{\eta}^{\eta_0} d\eta' \,(1+\phi+\psi) - \int_{\eta}^{\eta_0} d\eta' (\eta'-\eta)\nabla_{\perp}(\phi+\psi)$$

Lesson 3 c) Gravitational lensing

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$$\delta\vec{\boldsymbol{\theta}} = \nabla_\theta \Phi_L, \qquad \Phi_L(\hat{\mathbf{n}}, \chi_s) \equiv \int_0^{\chi_s} d\chi \, \frac{\chi_s - \chi}{\chi_s \chi} (\phi+\psi)$$

"Lensing potential"

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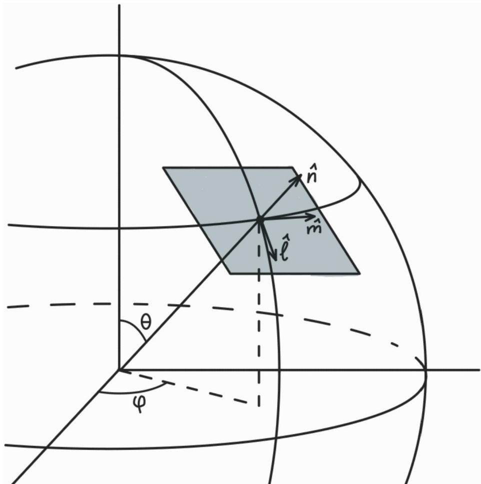
3. Flat-sky approximation (for simplicity):

$$f(\vec{\boldsymbol{\theta}}) = \int \frac{d\mathbf{l}^2}{(2\pi)^2} e^{i\mathbf{l}\cdot\vec{\boldsymbol{\theta}}}, \qquad f_{\mathbf{l}} \equiv \int d\vec{\boldsymbol{\theta}}^2 \, f(\hat{\mathbf{n}})\, e^{-i\mathbf{l}\cdot\vec{\boldsymbol{\theta}}}$$



Lesson 3 c) Gravitational lensing

Convergence and shear

Shifts described by displacement $\delta\vec{\boldsymbol{\theta}} = \nabla_\theta \Phi_L,$

Lesson 3 c) Gravitational lensing

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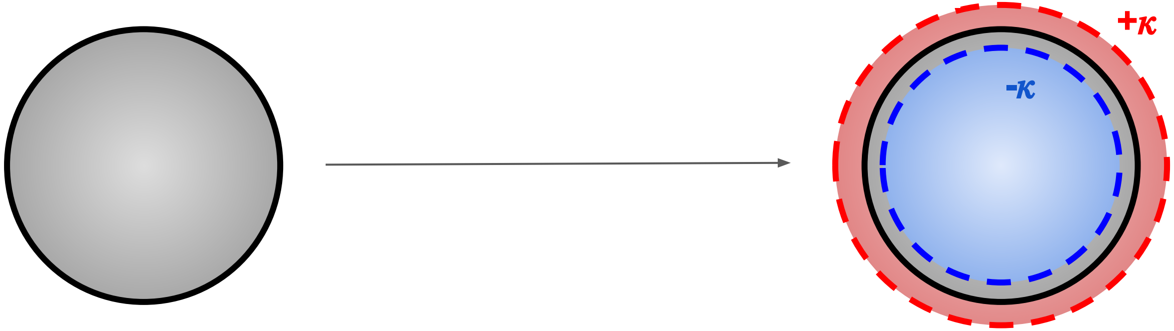
Changes to extended sources described by second derivatives: $H_{ij} \equiv \partial_{\theta_i}\partial_{\theta_j}\Phi_L$
Separate into:

- Convergence κ== trace == Laplacian: describes changes to source area.

$$\mathrm{H} = \begin{pmatrix} \kappa + \gamma_1 & \gamma_2 \\ \gamma_2 & \kappa - \gamma_1, \end{pmatrix}$$



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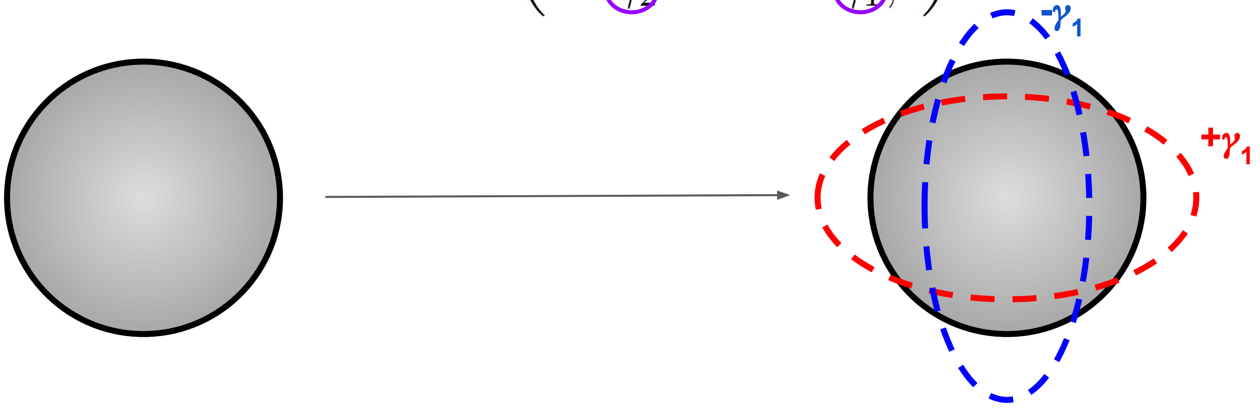

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Convergence can be related to matter overdensity:

$$\kappa(\hat{\mathbf{n}}, \chi_s) = \frac{3}{2}H_0^2\Omega_m \int_0^{\chi_s} d\chi\, \frac{\chi}{a(\chi)}\,\frac{\chi_s - \chi}{\chi_s}\,\delta(\chi\hat{\mathbf{n}}, \eta)$$

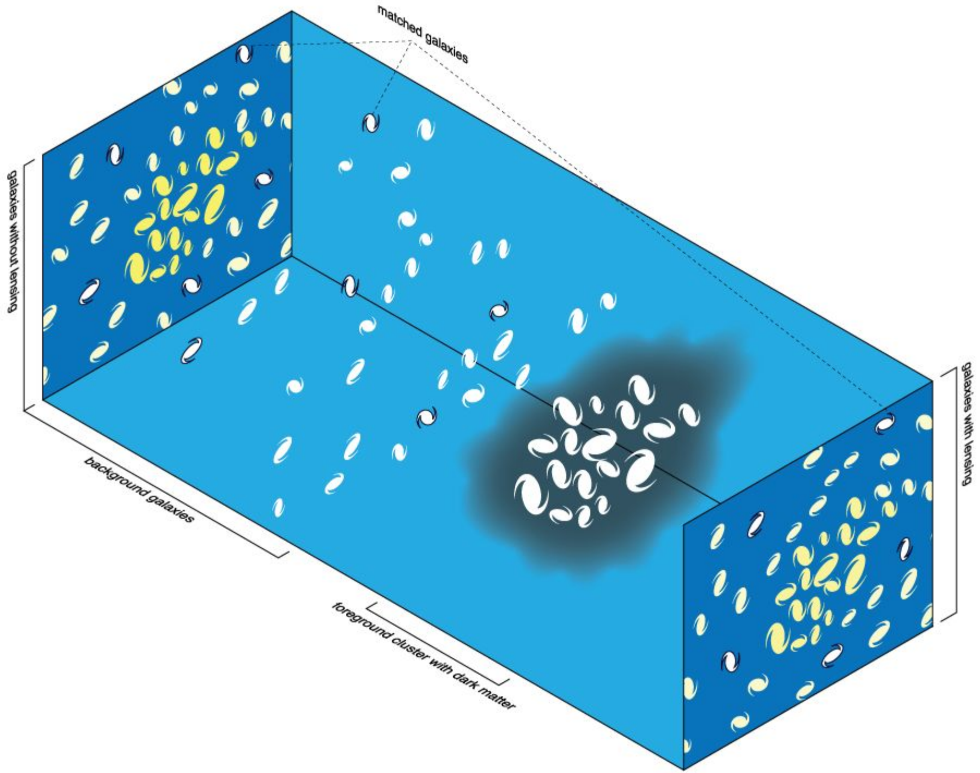
Lesson 3 c) Gravitational lensing

Probes: 1. Galaxy lensing

Lensing modifies galaxy ellipticity in a correlated manner

$$(\varepsilon_1, \varepsilon_2) = 2(\gamma_1, \gamma_2)$$

Map mean galaxy ellipticity == map shear.



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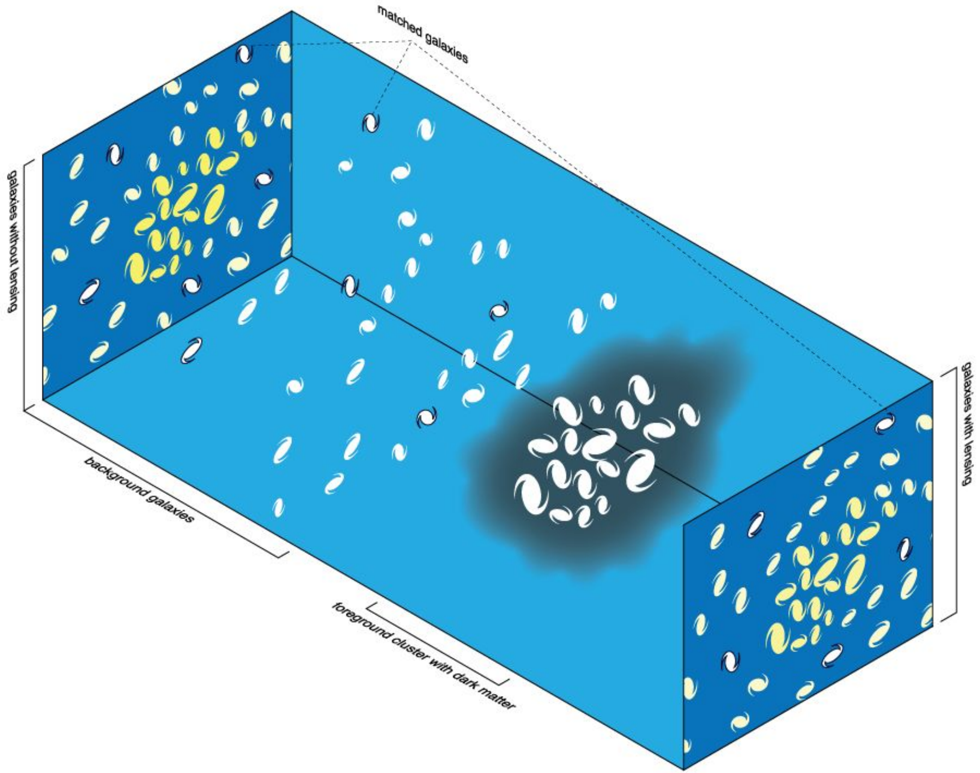
- High number density
- Deep and wide surveys
- Excellent imaging resolution for shape measurement.



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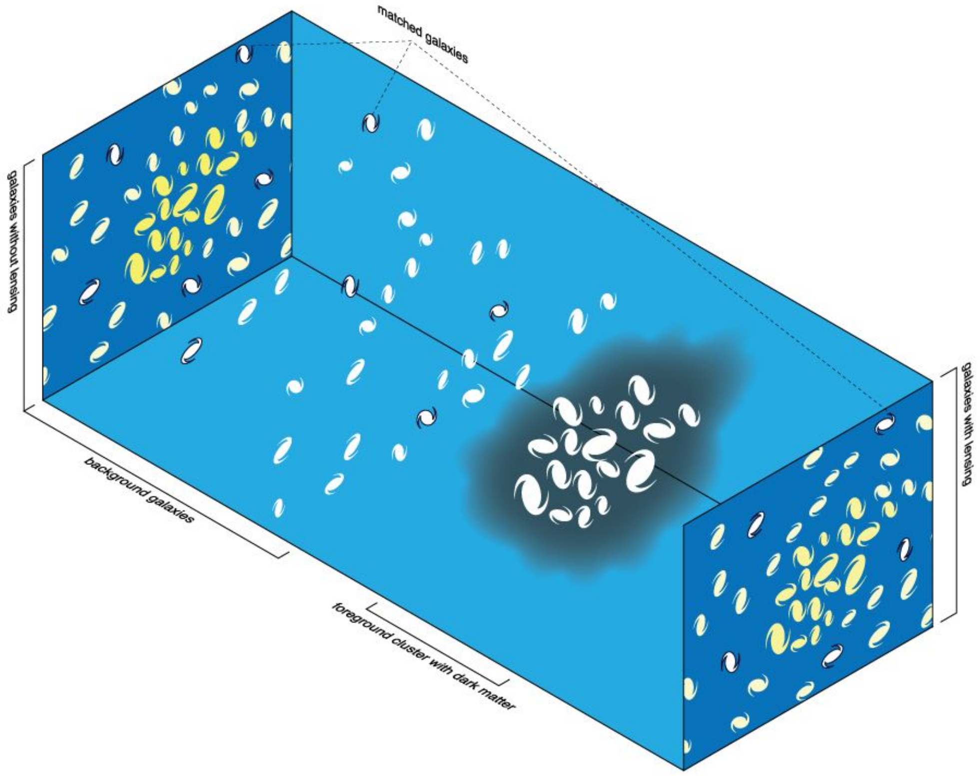

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Lensing also changes galaxy positions and fluxes. Modification to galaxy overdensity:

$$\delta_g^{\mu} = (5s - 2)\kappa$$

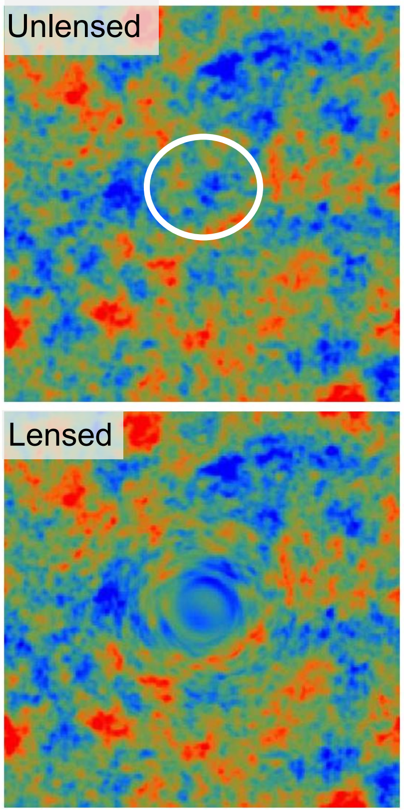
"Magnification bias"

Lesson 3 c) Gravitational lensing

Probes: 2. CMB lensing

Lensing modifies the trajectories of CMB photons.
The effect is second-order, but detectable at high significance.

$$\delta T(\vec{\boldsymbol{\theta}}) = \delta T_u(\vec{\boldsymbol{\theta}} - \delta\vec{\boldsymbol{\theta}}) \simeq \delta T_u(\vec{\boldsymbol{\theta}}) - \nabla_\theta \Phi_L(\vec{\boldsymbol{\theta}}) \cdot \nabla_\theta \delta T_u(\vec{\boldsymbol{\theta}})$$



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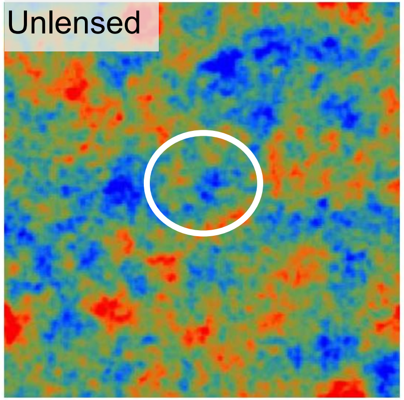
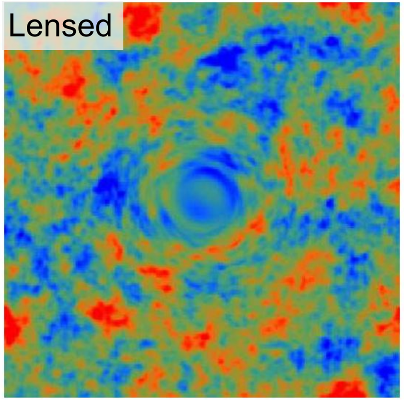
Lensing can be thought of as breaking statistical isotropy.
Causes unequal-mode correlations:

$$\langle \delta T_{\mathbf{l}} \delta T^*_{\mathbf{l}'} \rangle_{\Phi_L} = \Phi_{L,\mathbf{l}-\mathbf{l}'} \, (\mathbf{l} - \mathbf{l}') \cdot (\mathbf{l}\, C_\ell^T - \mathbf{l}' C_{\ell'}^T)$$





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Idea: reconstruct Φ_L from pairs of Fourier modes (quadratic estimator):

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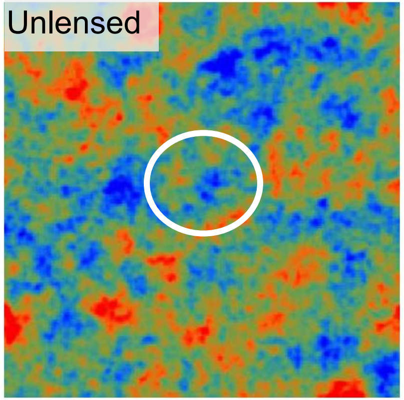

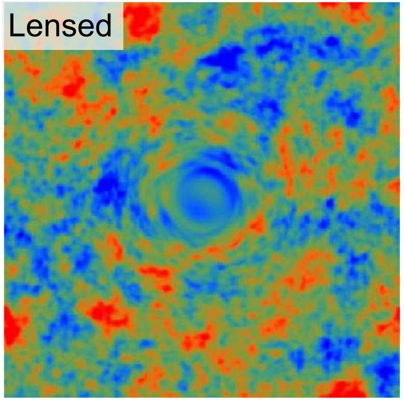

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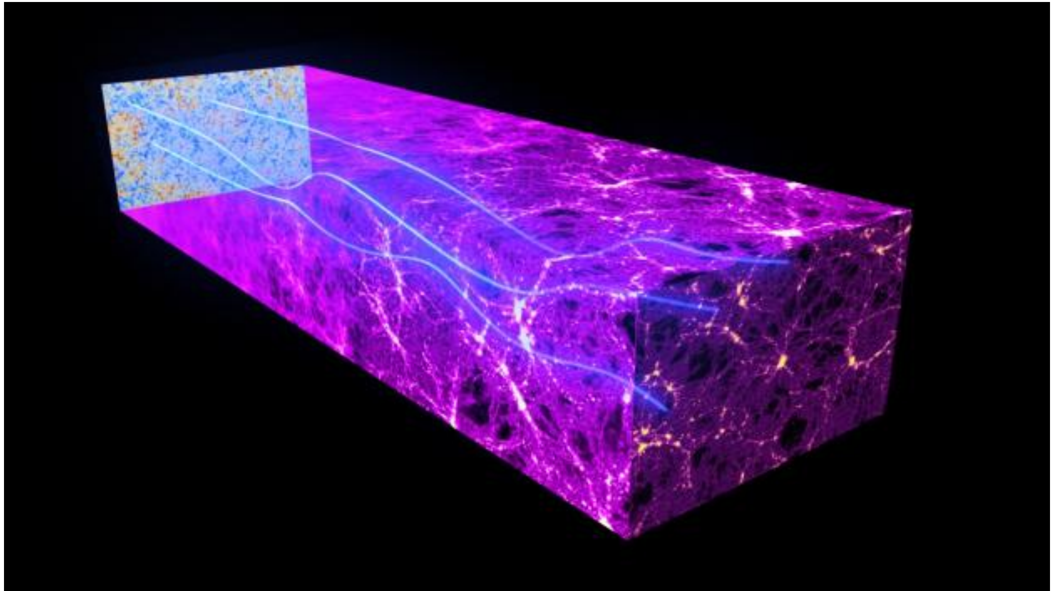
Resulting map contains information about structure growth since recombination!



Coda: opportunities and challenges

Fundamental physics from cosmology

1. CMB: primordial gravitational waves from B-mode polarization.

2. Large-scale structure: dark energy, massive neutrinos, primordial non-Gaussianity

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   <u>Problem:</u> Galactic foregrounds dominate the B-mode signal.
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   Problems:
   - Lensing: intrinsic galaxy alignments, AGN feedback
   - Galaxy clustering: how do galaxies relate to matter?
   - 21cm: Galactic foregrounds dominate by many orders of magnitude.

   How will we believe a detection? Precise understanding of galaxy formation and evolution.

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We must become astrophysicists before we can use cosmology reliably!