Gauge Coupling Unification and Split Supersymmetry

1 Unification

There are various arguments as to why a Supersymmetric extension of the Standard Model may be of interest for understanding TeV scale physics such as we will probed at the Large Hadron Collider. One motivation people often give is that SUSY 'predicts a unification of gauge couplings'. In this question, we'll see what this means...

We write the renormalisation group equation for the gauge couplings g_3, g_2, g_1 of the Standard Model group $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$ as

$$\mu \frac{dg_i}{d\mu} = \frac{\beta_i}{16\pi^2}\, g_i^3 \qquad \text{(no sum on } i\text{)} \tag{1}$$

where μ here is the renormalisation scale, and β_i are the one-loop beta-function coefficients (real constants).

For $\mathrm{SU}(N)$ gauge groups, we calculated the β_i coefficients in the Standard Model course:

$$\beta_i = -\frac{11N}{3} + \frac{2}{3}\sum_f T_R(f) + \frac{1}{3}\sum_s T_R(s)\ , \tag{2}$$

where f denotes a 2-component Weyl fermion and s a complex scalar. T_R is the Dynkin Index of the appropriate representation of $\mathrm{SU}(N)$ corresponding to the field f or s; explicitly, this is 1/2 for the fundamental rep[1] and N for the adjoint rep.

For U(1) we have

$$\beta_1 = \frac{2}{3}\sum_f Y_f^2 + \frac{1}{3}\sum_s Y_s^2 \tag{3}$$

where $Y_{f,s}$ is the hypercharge of a (2-component) fermion or complex scalar respectively.

[1]This choice is just a convention — once fixed, all the other T_R values follow.

In tutorial 3 we saw that for the Standard Model, at one-loop order,

$$\beta_1 = \frac{41}{6} \qquad\qquad \beta_2 = -\frac{19}{6} \qquad\qquad \beta_3 = -7\ ,$$

whereas for the MSSM

$$\beta_1 = 11 \qquad\qquad \beta_2 = 1 \qquad\qquad \beta_3 = -3\ . \tag{4}$$

a) Defining $\alpha_i(\mu) \equiv \frac{g_i^2(\mu)}{4\pi}$, solve the RG equation (1) to find a relationship between $\alpha_i(M_Z)$ and $\alpha_i(\mu_0)$ for a general scale μ_0.
Hint: Equation (1) takes the simplest form when written in terms of α^{-1}.

b) Grand Unified Theories predict that at some scale $\mu_0 = M_{GUT}$,

$$\frac{5}{3}\alpha_1(M_{GUT}) = \alpha_2(M_{GUT}) = \alpha_3(M_{GUT}). \tag{5}$$

Assuming this, derive

$$\alpha_3(M_Z)^{-1} = \alpha_2(M_Z)^{-1} + \frac{\beta_3 - \beta_2}{3\beta_1/5 - \beta_2}\left[\frac{3}{5}\alpha_1^{-1}(M_Z) - \alpha_2^{-1}(M_Z)\right]. \tag{6}$$

c) Taking the (rough) experimental values $g_1(M_Z) = 0.357$ and $g_2(M_Z) = 0.652$, and assuming all the Standard Model couplings unify at M_{GUT}, what value of $g_3(M_Z)$ do we predict from equation (6)? Does the MSSM do any better, if we assume that SUSY is broken just above the electroweak scale?

d) Show that if we introduce the fine-structure constant $\alpha = \frac{e^2}{4\pi}$, with $e = g_2 \sin\theta_W$ and $\tan\theta_W = \frac{g_1}{g_2}$, then equation (6) can be recast as

$$\alpha_3(M_Z)^{-1} = \alpha^{-1}\left[\sin^2\theta_W + \frac{3 - 8\sin^2\theta_W}{5}\,\frac{b_3 - b_2}{b_1 - b_2}\right], \tag{7}$$

where $b_3 = \beta_3$, $b_2 = \beta_2$ and $b_1 = \frac{3}{5}\beta_1$. Furthermore, show that the unification scale is given by

$$\log\left(\frac{M_{GUT}}{M_Z}\right) = \frac{2\pi\,(3 - 8\sin^2\theta_W)}{5\alpha\,(b_1 - b_2)}, \tag{8}$$

and that at the unification scale, the value of the coupling is

$$\alpha_{GUT} = \frac{5\alpha\,(b_1 - b_2)}{5\sin^2\theta_W\, b_1 - 3\cos^2\theta_W\, b_2}. \tag{9}$$

e) What is the Unification scale and value of the coupling at M_{GUT} predicted by:

(i) the Standard Model?

(ii) the MSSM?

2 Split Supersymmetry

The idea of Split Supersymmetry is to forget using SUSY as a solution to the hierarchy problem, but to still require that it leads the unification of gauge couplings and provides a dark matter candidate. We'll look at this idea, following reference [1]; their starting point was to note that the beta-function coefficients, b_i, can be written as

$$b_3 = \frac{1}{3}\left(4N_g - 33 + N_3\right) \tag{10a}$$

$$b_2 = \frac{1}{3}\left(4N_g - 22 + \frac{n_H}{2} + N_2\right) \tag{10b}$$

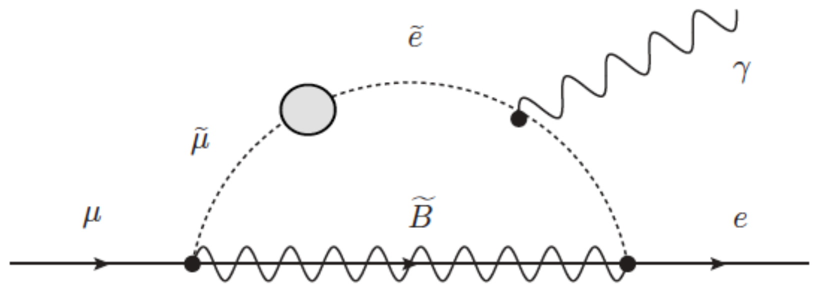
$$b_1 = \frac{1}{3}\left(4N_g + \frac{3n_H}{10} + N_1\right) \tag{10c}$$

where N_g counts the contribution to the β-functions from complete SU(5) irreps, and it is normalized such that the 3 families of SM quarks and leptons give $N_g = 3$.[2] For the MSSM one can easily show that $N_g = \frac{9}{2}$. The number of Higgs doublets is n_H, and N_i $(i = 1, 2, 3)$ give the contributions from matter in incomplete GUT multiplets (for example, in the MSSM, this includes contributions from the gauginos and higgsinos).

The important observation is that N_g actually cancels out in the equations (7) and (8), and so doesn't enter into the predictions for α_s or M_{GUT}. Split SUSY makes use of this fact: All *scalars* in the MSSM can be very heavy, except one Higgs, and unification can still take place.[3] We still need the gauginos ($\widetilde{g}$, $\widetilde{W}$ and $\widetilde{B}$) and Higgsinos $\widetilde{h}_{u,d}$ to have masses of order the TeV scale in order to retain the nice features of unification, and to also have interesting dark matter candidates.

a) If we send the scale of ~~SUSY~~ to the GUT scale, what are the natural values for the squark and slepton masses? What about the fermionic superpartners (gauginos and higgsinos)?

b) Another interesting feature of split SUSY is that pushing the scalar masses to high scales alleviates the most pressing bounds from flavor-changing neutral currents (FCNCs), CP violation, proton decay and so on. The reason is that all those dangerous bounds are based on calculating a diagram that is suppressed by a factor of the scalar masses. For example, let's look at the M_{scalar} dependence of the $\mu \to e\,\gamma$ bound: the SUSY particles typically contribute to this process through a diagram of the type:



where the mass insertion (grey blob) comes from a flavor-violating, soft SUSY-breaking term of the form $-m^2_{\tilde{e}\tilde{\mu}}\,\tilde{e}\,\tilde{\mu}$. One can use naïve dimensional analysis (NDA) to estimate the size of this contribution to the branching ratio to be

$$\mathrm{BR}(\mu \to e\,\gamma) \approx \frac{g'^2\,e^2}{16\pi^2}\left(\frac{m^2_{\tilde{e}\tilde{\mu}}}{m^2_{\tilde{\ell}}}\right)^2 \frac{v^2}{m^2_{\tilde{\ell}}}\,\frac{v^2}{m^2_{\widetilde{B}}} \tag{11}$$

where $m_{\tilde{\ell}}$ is the slepton mass, and we have used the fact that μ decays are dominated by $\mu \to e\,\nu_\mu \overline{\nu}_e$, which goes as G_F^2. Is this formula dimensionally correct?

c) Assume $m^2_{\tilde{e}\tilde{\mu}} \approx m^2_{\tilde{\ell}}$ (no flavor hierarchy) and $m_{\widetilde{B}} \approx v$. Find the experimental constraint on the $\mathrm{BR}(\mu \to e\,\gamma)$ and use it to derive a lower bound on $m_{\tilde{\ell}}$.

d) In split SUSY, gluinos (gluini?!) are lighter than squarks, so it is interesting to think about how gluinos decay. Use NDA to estimate the decay width $\Gamma_{\tilde{g}}$, and hence the decay length, $c\tau$, of the gluino as a function of $m_{\tilde{g}}$ and $m_{\tilde{q}}$ (assuming that $m_{\tilde{g}} \gg m_{\mathrm{LSP}}$, so there *are* SUSY particles for $\tilde{g}$ to decay into).

Long lived gluinos are a ‘smoking gun’ feature of split SUSY. The LHC is looking for them by keeping the detectors on when there are no collisions; as gluinos carry color charge, if they hang around long enough they end up getting bound up into R-hadrons (hadrons with non-trivial charge under R-parity) that can potentially be brought to rest by all the material in the detector. If the beams are colliding, the detector is too busy detecting other things to notice the intermittent decays of these R-hadrons, but when there are no collisions, one would only expect to register cosmic rays, and *possibly* the decay of interesting stuff trapped in the detector.

References

[1] G. F. Giudice and A. Romanino, “Split supersymmetry,” Nucl. Phys. B **699** (2004) 65 [Erratum-ibid. B **706** (2005) 65] [arXiv:hep-ph/0406088].

1 Goldstone Bosons

According to Goldstone's theorem, whenever a global symmetry group G is spontaneously broken down to a smaller one H, it gives rise to $\dim(G) - \dim(H)$ massless bosons known as *Goldstone bosons.*

Today we're going to look at what happens when we spontaneously break a global symmetry:

$$\mathrm{SU}(N) \longrightarrow \mathrm{SU}(N-1)\,. \tag{1}$$

a) How many Goldstone bosons (GBs) are generated by this breaking?

There are many ways to parameterise the GB fields, but we will try to be smart and choose a representation which clearly shows how all the fields transform under $\mathrm{SU}(N)$ and $\mathrm{SU}(N-1)$.

b) Explain how the $N \times N$ matrix

$$U_{N-1} \equiv \begin{pmatrix} \hat{U}_{N-1} & 0 \\ 0 & 1 \end{pmatrix} \qquad \text{with} \quad \hat{U}_{N-1} \ \text{an} \ (N-1)\times(N-1) \ \text{matrix} \tag{2}$$

provides a represention of the unbroken symmetry transformations.

Let's represent the GBs by introducing an $N \times N$ matrix Π in the following way

$$\phi(x) = e^{i\Pi(x)/f}\,\phi_0(x) \tag{3}$$

where

$$\Pi(x) = \begin{pmatrix} 0_{(N-1)\times(N-1)} & \vec{\pi}(x) \\ \vec{\pi}^\dagger(x) & 0 \end{pmatrix} \qquad \vec{\pi}(x) = \begin{pmatrix} \pi_1(x) \\ \vdots \\ \pi_{N-1}(x) \end{pmatrix} \in \mathbb{C}^{N-1} \tag{4}$$

$$\phi_0(x) = \frac{f}{\sqrt{2}} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \pi_0(x) \end{pmatrix} \qquad \pi_0(x) \in \mathbb{R} \tag{5}$$

c) How does ϕ transform under the unbroken symmetries?

d) Does ϕ contain the right number of degrees of freedom?

e) We would like to see how ϕ transforms under the *broken* symmetries. We will first represent the broken symmetries by the transformation:

$$U_{\text{broken}} = \exp\left\{ i \begin{pmatrix} 0 & \vec{\alpha} \\ \vec{\alpha}^\dagger & 0 \end{pmatrix} \right\} \qquad \vec{\alpha} \in \mathbb{C}^{N-1} \tag{6}$$

Show that ϕ transforms as

$$\phi \to U_{\text{broken}}\, e^{i\Pi/f} \phi_0 = e^{i\Pi'/f} \phi_0 \tag{7}$$

to first order in $\vec{\alpha}$, where

(i) The $\vec{\pi}$ field shifts linearly:

$$\vec{\pi}' = \vec{\pi} + f\,\vec{\alpha}. \tag{8}$$

(ii) The field ϕ_0 is invariant under $\mathrm{SU}(N-1)$ transformations.

f) Although one says that the $\mathrm{SU}(N)$ symmetry has been spontaneously broken down to $\mathrm{SU}(N-1)$ what really happens is that the broken part of the symmetry is realized in a way that is different from the unbroken parts. To see this more clearly compare how the fields transform under a broken symmetry vs. how they transform under the unbroken ones. For the broken generators one says that the symmetries are "non-linearly" realized. Thus for infinitesimal tranformations involving the broken generators one requires that the shifts in (8) are symmetries. Show that this statement is consistent with the statement of Goldstone's theorem that the GBs are massless.

g) This shift symmetry also implies that no potential is generated (no quartic coupling, no term made up of powers of the field) and only derivative interactions are allowed. To see this explicitly, expand the GB kinetic term

$$\partial_\mu \phi^\dagger\, \partial^\mu \phi \tag{9}$$

up to quartic order in the fields.