Dirac, Weyl and Majorana Fermions

Recall the Dirac equation for a four-component (Dirac) fermion:

$$(\not{p} - m)\Psi = 0 \qquad \text{where} \qquad \not{p} = p_\mu \gamma^\mu \,. \tag{1}$$

Further recall (from Standard Model tutorial 1) that the action of charge congugation can be represented as a matrix acting on Ψ:

$$\Psi^c = C\,\overline{\Psi}^T \qquad\qquad C = -i\,\gamma^2\gamma^0 \tag{2}$$

If we define

$$\Psi \equiv \begin{pmatrix} \xi \\ \overline{\eta} \end{pmatrix} \equiv \begin{pmatrix} \psi_L \\ \psi_R \end{pmatrix} \,, \tag{3}$$

then ξ and η are left- and right-handed[1] two-component (Weyl) spinors respectively, and the equation of motion (1) becomes two coupled differential equations:

$$(\overline{\sigma}_\mu p^\mu)\,\xi = m\,\overline{\eta} \tag{5a}$$
$$(\sigma_\mu p^\mu)\,\overline{\eta} = m\,\xi \tag{5b}$$

Remember that in the chiral basis,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \overline{\sigma}^\mu & 0 \end{pmatrix} \qquad \text{where} \qquad \sigma^\mu = (\mathbb{1}_2, \vec{\sigma}) \,, \quad \overline{\sigma}^\mu = (\mathbb{1}_2, -\vec{\sigma}) \,. \tag{6}$$

Note that the two equations (5) decouple when $m = 0$.

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[1]We can project onto the left- and right-handed components with

$$P_L = \frac{1}{2}(\mathbb{1} - \gamma^5) \qquad\qquad P_R = \frac{1}{2}(\mathbb{1} + \gamma^5) \,. \tag{4}$$

Note: $P_R + P_L = \mathbb{1}$ and $P_R P_L = P_L P_R = 0$.

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a) A *Majorana spinor* is one which is equal to its charge congugate. In 4-component form, this condition reads

$$\Psi^c = \Psi \tag{7}$$

One can think of this as a reality condition for the spinor, just as real numbers satisfy $z^* = z$. Write the Majorana condition (7) in Weyl language.

b) Is this condition preserved under charge conjugation?

c) Translate the following Dirac bilinears into Weyl notation:

$$\overline{\Psi}_1 \Psi_2 \,, \qquad \overline{\Psi}_1 P_L \Psi_2 \,, \qquad \overline{\Psi}_1 P_R \Psi_2 \,, \qquad \overline{\Psi}_1 \gamma_\mu \Psi_2 \,. \tag{8}$$

d) Re-write the two-component expressions you got for (8) assuming that Ψ_1 and Ψ_2 are Majorana fields.

There are two different types of mass terms that one can write for fermions:

$$\text{Dirac} \qquad M_0 \, \overline{\Psi} \Psi \tag{9a}$$

$$\text{Majorana} \qquad m_L \left(\overline{(\Psi^c)} \, P_L \, \Psi + \text{h.c.} \right) + m_R \left(\overline{(\Psi^c)} \, P_R \, \Psi + \text{h.c.} \right) \tag{9b}$$

e) Write the mass terms (9) in the language of Weyl spinors, combining all the terms and expressing the masses in the form of a matrix in $(\xi\,,\eta)$-space.

f) Show how M_D, m_L and m_R transform under the action of charge conjugation.

g) Show that a fermion with a Dirac mass term is equivalent to two degenerate Majorana fermions.