Probing mixed-state, symmetry-resolved, entanglement with randomized measurements

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Motivation: quantum technologies

Quantum simulators

Fermi-Hubbard simulation (MPQ)

Understand quantum matter (superconductivity, topology, High energy physics,..)

Quantum computers

Google Sycamore chip

Quantum algorithms
Optimization problems (Annealing)

Key challenge: probe quantum properties of these many-body systems
Two subsystems A and B are **bipartite entangled** iff

\[ |\Psi\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle \quad \rho \neq \sum_j p_j \rho_j^{(A)} \otimes \rho_j^{(B)} \]

**Reduced density matrix**

\[ \rho_A = \text{Tr}_B(\rho) \]

**Entanglement condition (Horodecki 1996)**

\[ \text{Tr} \left[ \rho_A^2 \right], \text{Tr} \left[ \rho_B^2 \right] < \text{Tr} \left[ \rho^2 \right] \]

**Quantifying entanglement for pure states → Entanglement entropies**

\[ S_A = -\text{Tr}_A [\rho_A \log \rho_A] \]

\[ S_A^{(n)} = \frac{1}{1-n} \log \text{Tr}_A [\rho_A^n] \quad \leq S_A \]

**purity**

\[ S_A^{(2)} = -\log(\text{Tr}_A(\rho_A^2)) \]

**von-Neumann**

**Nth Rényi**

**2nd Rényi**
Measuring entanglement entropies is fundamental for Quantum Simulation.

Many-body ground states | Quantum Phase transitions | Topological order

Area law: \( S_A^{(2)} \propto L_A^{D-1} \)

\[ S_A^{(2)} \approx \left( \frac{c}{4} \right) \log(L_A) \]

central charge

Topological entanglement Entropy

\[ S_A^{(n)} \approx \alpha_n L_A - \gamma \]

Quantum Thermalization

P. Calabrese and J. Cardy, PRL 2006
Badarson et al, PRL 2012

Amico et al., Rev.Mod.Phys, 80, 517 (2008)
Eisert et al., Rev. Mod. Phys. 82, 277 (2010)
Humeniuk, Roscilde PRB (2012)

Kitaev, Preskill, PRL 2006
Levin, Wen, PRL 2006
Jian et al, NP 2012
Measuring the entanglement “power” of quantum computers

“How to measure entanglement in such many-body quantum systems?”

A new tool: randomized measurements

“Checks”
- Purity checks
- Entanglement checks

Universal behaviors

Google Sycamore chip

A standard measurement protocol

Simple ‘classical’ initial state

Quantum simulation/computation

Controlled time evolution

- Spins interact

Quantum state $\rho$

Measurement - eg. local spin direction

Repeat a finite number of times $N_M$

Probability to see a specific configuration

$P_{\rho}(s) + \frac{\epsilon}{\sqrt{N_M}}$

Quantum Expectation value

‘Projection’ noise

Limited to `observables’, correlation functions, etc

Not applicable to Entanglement-related quantities, nonlinear functions w.r.t the density matrix

Example: $\text{tr}(\rho^2)$
A new tool: randomized measurement protocols

**Randomized measurement**

- Controlled time evolution
- Spins interact

**Simple ‘classical’ initial state**

**Quantum Simulation**

**Measurement**
- Controlled randomness
  - $U$
  - random unitary

- Measurement
  - eg. local spin direction

- Repeat

- Measurement $P_{\rho, U}(s)$

- Ensemble average over random unitaries

**Correlations of probabilities**

$$P_{\rho, U}(s_1)P_{\rho, U}(s_2)$$

**Entanglement entropies**

**Fidelities**

- $F_{\max}(\rho, \rho_E)$

- $i(t_{\text{ms}})$

- $N$ (ms)

**Scrambling**

- $\alpha(t)$

- $t - \frac{r}{\sqrt{2}}$ (ms)

**Mixed-state entanglement**

- Zhou et al, PRL 2020
- Elben, .., BV PRL 2020
- See also works by Knips, Ketterer

**Topology**

- Elben,..,BV, Sci. Adv. 6 (2020)
- ZP Cian, BV, et al, PRL 2021

- BV et al., PRX 2019
- Joshi, BV, et al  PRL 2020

**Van Enk, Beenakker PRL 2012**
- PRL 2018, PRA 2018, PRA 2019
- Brydges,..,BV,.., Science 2019
- Huang et al Nature Physics 2020
- Elben et al, arXiv:2101.07814
Part I: Mixed-State Entanglement from Local Randomized Measurements


A. Elben (Innsbruck) R. Kueng (Caltech → Linz), R. Huang (Caltech), R. van Bijnen (Innsbruck) C. Kokail (Innsbruck), M. Dalmonte (Trieste), P. Calabrese (Trieste), B. Kraus, (Innsbruck) John Preskill (Caltech), Peter Zoller (Innsbruck), and BV

Part II: Symmetry-resolved dynamical purification in synthetic quantum matter


V. Vitale, A. Elben, R. Kueng, A. Neven, J. Carrasco, B. Kraus, P. Zoller, P. Calabrese, B. Vermersch, M. Dalmonte
Mixed-state entanglement

What kind of entanglement detection?

Purity test:
\[
\text{Tr} \left[ \rho_A^2 \right], \text{Tr} \left[ \rho_B^2 \right] < \text{Tr} \left[ \rho^2 \right]
\]
Not very powerful for highly mixed states (Brydges 2019)

Entanglement witness:
\[
\text{Tr} (O \rho_{AB}) < 0
\]
The relevant operator is state-dependent (ex: CHSH inequalities..)
Not a quantifier of mixed-state entanglement

PPT condition
\[
\rho_{AB} \overset{T_A}{\rightarrow} \text{is not positive semi-definite}
\]
Powerful (ex: sufficient for two qubits)
Basis-independent
Entanglement monotone: negativity
Relevant in quantum field theories
Mixed-state entanglement

Positive-Partial-Transpose (PPT) Condition for mixed state entanglement

If the state is separable \( \rho_{AB} = \sum_k c_k \rho_A^{(k)} \otimes \rho_B^{(k)} \)

Partial transposition

Then \( \rho_{AB}^{T_A} = \sum_k c_k \left[ (\rho_A^{(k)})^T \otimes \rho_B^{(k)} \right], \) is positive semi-definite, i.e only has positive eigenvalues

Conversely, if \( \rho_{AB}^{T_A} \) is not positive semi-definite \[\rightarrow\] The state is entangled

Example: Bell state \( \rho_{AB} = \left| \text{Bell} \right> \left< \text{Bell} \right| \)

\[= \begin{bmatrix} 0. & 0. & 0. & 0. \\ 0. & 0.5 & 0.5 & 0. \\ 0. & 0.5 & 0.5 & 0. \\ 0. & 0. & 0. & 0. \end{bmatrix} \]

\[\rho_{AB}^{T_A} = \begin{bmatrix} 0. & 0. & 0. & 0.5 \\ 0. & 0.5 & 0. & 0. \\ 0. & 0.5 & 0. & 0. \\ 0.5 & 0. & 0. & 0. \end{bmatrix} \]

Spec = \( (0.5, 0.5, 0.5, -0.5) \)

How to detect entanglement via the PPT condition in multi qubit systems??
Mixed-state entanglement

**PT moments**

\[ p_n = \text{Tr}[(\rho_{AB}^{T_A})^n] \quad \text{for } n = 1, 2, 3, \ldots \]

→ **Quantify mixed-entanglement in quantum-field theories:**
Works by P. Calabrese, etc

Chung, PRB 2014

→ **A measurable powerful entanglement condition**
Elben et al, PRL 2020

**p_3** PPT condition

\[ p_3 < p_2^2 \] implies PPT violation (implies entanglement)

Hint for the proof: large negative eigenvalues make \( p_3 \) small, and \( p_2 \) large
Measuring PT moments via local randomized measurements

**Protocol**

![Diagram of a protocol](image)

Randomized measurements are tomographically complete

Measured bit strings

\[
\hat{\rho}_{AB}^{(r)} = \bigotimes_{i \in AB} \left[3 (u_i^{(r)})^\dagger |k_i^{(r)}\rangle \langle k_i^{(r)}| - \mathbb{I}_2\right]
\]

\[\mathbb{E}[\hat{\rho}_{AB}^{(r)}] = \rho_{AB}\]

Polynomials of the density matrix can be estimated via U-statistics
See also Huang et al, Nature Physics 2020 for the purity

\[
p_3 = \mathbb{E} \left[ \text{Tr} \left( (\rho_{AB}^{(r_1)})^{T_A} (\rho_{AB}^{(r_2)})^{T_A} (\rho_{AB}^{(r_3)})^{T_A} \right) \right]
\]

→ Multi-linear post-processing of the data (no tomography)
→ Measurement budget \( \sim 2^{N_{[AB]}} \)
First experimental measurements of PT moments

**State:** Quench of a Neel state with long range XY model

**Data:** Brydges, Science 2019 (reanalyzed)

Entanglement spreading
Quantum-field theory predictions: P. Calabrese et al

**Part I:** Mixed-State Entanglement from Local Randomized Measurements


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**Part II:** Symmetry-resolved dynamical purification in synthetic quantum matter


V. Vitale, A. Elben, R. Kueng, A. Neven, J. Carrasco, B. Kraus, P. Zoller, P. Calabrese, B. Vermersch, M. Dalmonte
Motivation:

- **quantum simulators and noisy-intermediate scale quantum devices**

The total number of excitations (atoms, spin up states) is conserved [U(1) symmetry]

**Local interactions**

**Independent** sources of dissipation [Spontaneous emission, particle loss, etc]

**Dynamics from a product state** (ex Neel State 01010101)

Can we observe universal short-time entanglement signatures of the competition between unitary versus decoherence dynamics?
Part II: Symmetry-resolved dynamical purification in synthetic quantum matter

Our model:

\[ \frac{\partial \rho}{\partial t} = -\frac{i}{\hbar} [H, \rho] + \sum_k [L_k \rho L_k^\dagger - \frac{1}{2} \{L_k^\dagger L_k, \rho\}] \]

Hard-core bosons dynamics

\[ H = J \sum_{<i,j>} (b_i^\dagger b_j + \text{h.c.}) \]

Single particle loss

\[ L_j = \sqrt{\gamma} b_j \]
Part II: Symmetry-resolved dynamical purification in synthetic quantum matter

**Consequence of U(1) Symmetry:** Reduced density matrices are block-diagonal

\[ \rho_A = \bigoplus_q p(q) \rho_A(q) = \]

\[ \rho_A(q) = \frac{\Pi_q \rho_A \Pi_q}{\text{Tr} \rho_A \Pi_q}, \quad \text{Tr} \rho_A(q) = 1 \]

Description in terms of *m* symmetry-resolved reduced (and normalized) density matrices

See early works in gauge theories + more recent by Calabrese, Goldstein, Laflorencie, Sela.
Consequence of U(1) Symmetry: Reduced density matrices are block-diagonal

\[ \rho_A = \bigoplus_q p(q) \rho_A(q) = \]

\[ \rho_A(q) = \frac{\prod_q \rho_A \prod_q}{\text{Tr} \rho_A \prod_q}, \quad \text{Tr} \rho_A(q) = 1 \]

How to quantify SR dynamics?

SR-Renyi entropies

\[ S_A^{(n)}(q) \equiv \frac{1}{1 - n} \log \text{Tr} \rho_A(q)^n \]

SR-purity

\[ P_A(q) \equiv \text{Tr} \rho_A(q)^2 \]

(\sim\text{ inverse of the number of dominant eigenstates in each block})

And SR-PPT entanglement conditions...
Symmetry-resolved dynamical purification at short times

**Tool:** Second-Order Perturbation Theory at short times (w.r.t Lindblad rates)

In the $q=-1$ sector (I lost one particle)

Loss terms always win at very short times!

Coherent terms will progressively beat the loss terms
Universal symmetry-resolved dynamical purification at short times

\[ |\Psi(t = 0)\rangle \quad \in E_0(-1) \]

\[ \in E_0(-1) \]

\[ \in E_1(-1) \]

\[ \mathcal{P}_A(-1) \propto \begin{cases} 
1/V_A & t_1 \gg t \geq 0 \text{ (short time)} \\
1/(\partial V_A) & t > t > t_1 \text{ (int. time)}, \\
1/2V_A & t \gg t_j \text{ (long time)}. 
\end{cases} \]

"Log-volume" "Log-area" Volume

Dissipative processes dominant

Coherent ‘boundary’ terms dominant

Thermalization

Dynamical purification reveals the locality of interactions ‘on top’ of a dissipative environment
How to measure SR entropies in an experiment via randomized measurements?

\[
\hat{\rho}_A^{(r)} = \bigotimes_{i \in A} \left[ 3 (u_i^{(r)})^\dagger k_i^{(r)} \langle k_i^{(r)} | u_i^{(r)} - \mathbb{I}_2 \right]
\]

Random single qubit rotation (Haar distributed)

Measure bitstrings

Symmetry projection

\[
\mathcal{P}_A(q)^{(r,r')} = \frac{1}{2} \text{tr}[(\hat{\rho}_A^{(r)} \Pi_q)(\hat{\rho}_A^{(r')} \Pi_q)] + \frac{1}{2} \text{tr}[(\hat{\rho}_A^{(r')} \Pi_q)(\hat{\rho}_A^{(r)} \Pi_q)]
\]

Experimental observation of dynamical purification

**State:** Quench of a 10 qubit Neel state via a long range XY model

**Data:** Brydges , Science 2019 (reanalyzed)

SR purity increases! Then decreases

The total purity decays
Part II: Symmetry-resolved dynamical purification in synthetic quantum matter

1) What about entanglement???

- In one sector, the partition A *purifies*
- However, A is entangled with B
- Proof: SR PPT condition

2) Symmetry-resolved Partial transpose

\[ \rho^{T_A}(\tilde{q}) \equiv \frac{\Pi_{\tilde{q}} \rho^{T_A} \Pi_{\tilde{q}}}{\text{Tr} \rho^{T_A} \Pi_{\tilde{q}}} \]

Cornfeld, Goldstein and Sela, PRA 2018

3) Symmetry-resolved p3 PPT condition

\[ p_3(\tilde{q}) < p_2(\tilde{q})^2 \]

SR negative eigenvalue \rightarrow entanglement

Perturbation theory: SR p3 ppt detects entanglement at arbitrary short times

*Neven, Carrasco et al, in preparation*
Part II: Symmetry-resolved dynamical purification in synthetic quantum matter

Observation of symmetry-resolved entanglement

*Neven, Carrasco et al, in preparation*

**State:** Quench of a 10 qubit Neel state via a long range XY model

**Data:** Brydges, Science 2019 (reanalyzed)
**Conclusion**

**Randomized measurements:** a versatile toolbox to probe many-body physics in quantum experiments

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**Optimized protocols**

A. Rath, A. Elben, R. van Bijnen, P. Zoller

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**Current efforts**

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**Random Time-of-flight Microscopy**

P. Naldesi, A. Elben, P. Zoller, A. Minguzzi

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**Measuring Spectral Form Factors**

L. Joshi, A. Elben, P. Zoller
Thank you!

Funding available for PhD

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