Detecting Topological Phase Transitions through Symmetric Tensor Networks

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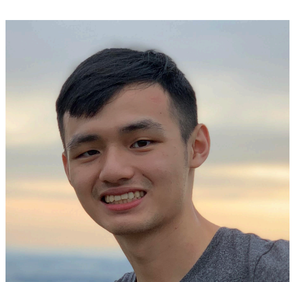
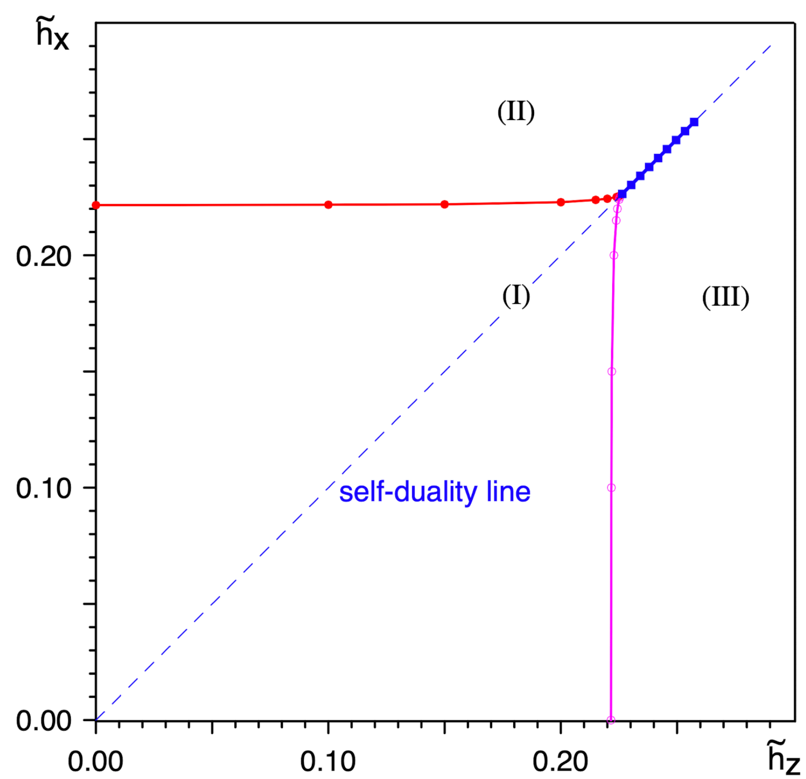
arXiv: 2102.10980




Topological Phase Transitions

Toric code in magnetic field

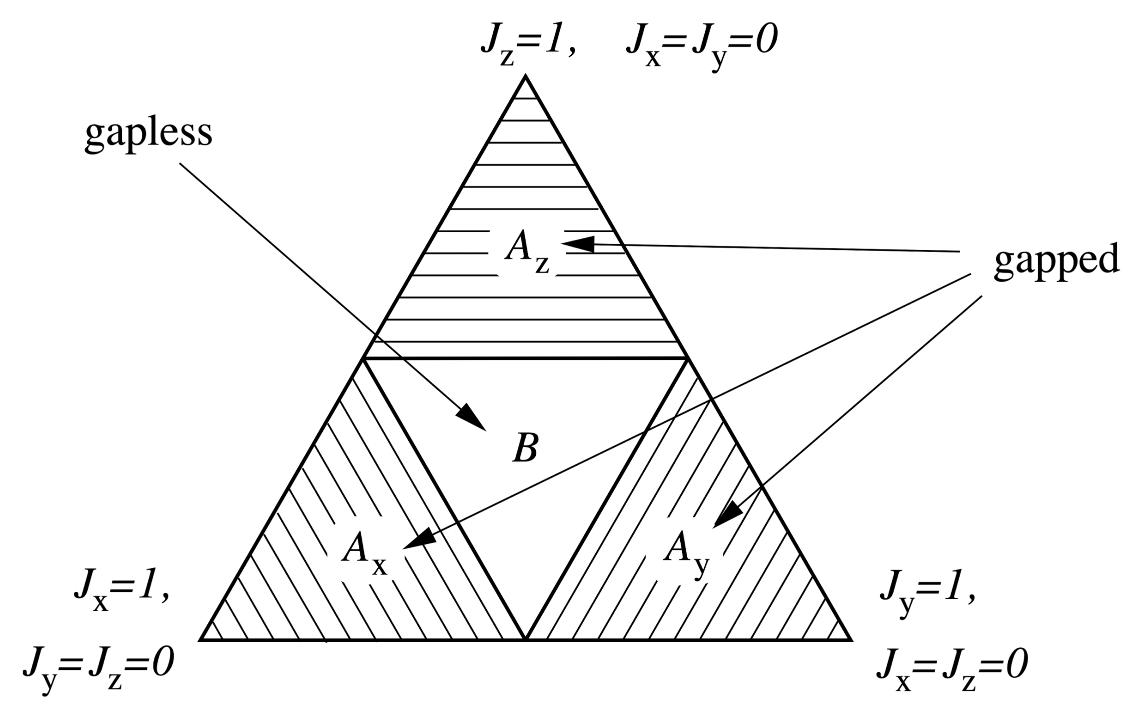
(I) $\mathbb{Z}_2$ Topological Orders (TO)
(II) Flux Condensed (FC) Phase
(III) Charge Condensed (CC) Phase



I. S. Tupitsyn et al: Phys. Rev. B **82**, 085114 (2010)

Kitaev's honeycomb model

A: $\mathbb{Z}_2$ Topological Orders (Abelian)
B: Ising Anyons (non-Abelian)



A. Kitaev, Annals of Physics **321**, 2–111 (2006)

Can we understand both transitions out of the $\mathbb{Z}_2$-TO using the same tensor network formalism?

Outline

Topological Order in Tensor Network

- $\mathbb{Z}_2$-injective Projected Entangled Pair States
- Using Minimally Entangled States to Detect Topological Transitions
- Warmup: Toric Code with Finite String Tension

Transitions from Abelian to non-Abelian Topological Phase

- Kitaev Model on the Star Lattice
- Loop Gas and String Gas Ansatzs as $\mathbb{Z}_2$-injective PEPS
- Results

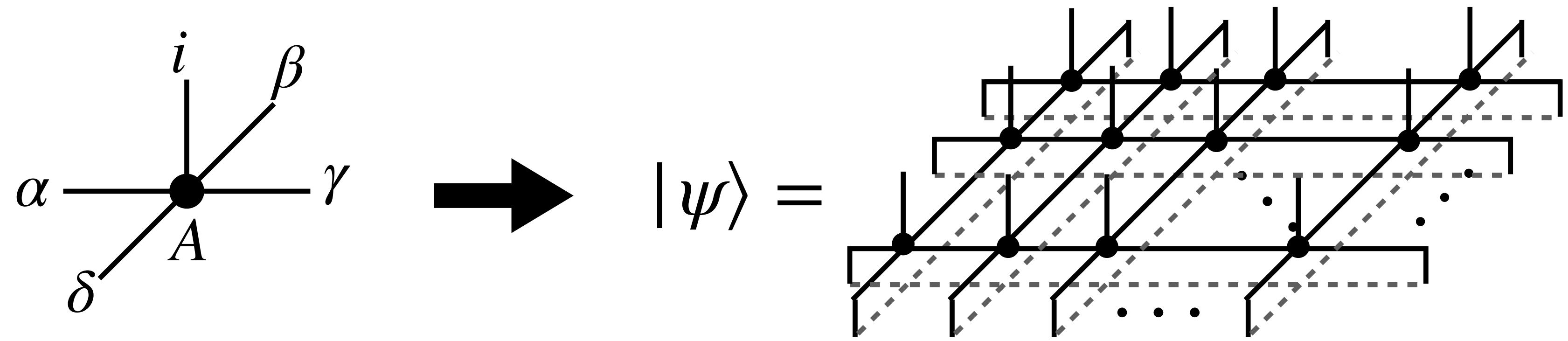
Conclusion

Projected Entangled Pair States (PEPS)

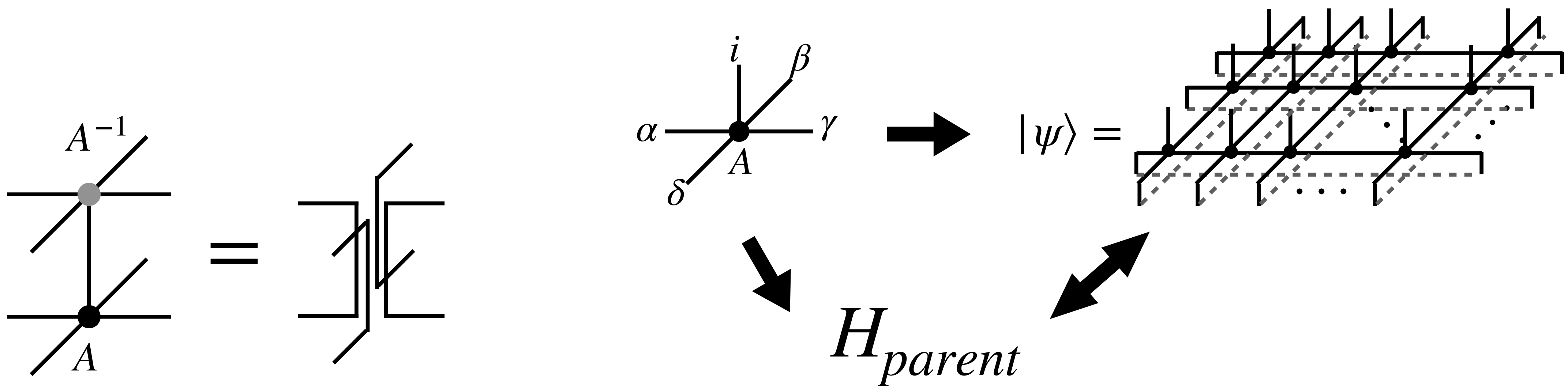
PEPS:

i: physical index

$\alpha, \beta, \gamma, \delta$: virtual index



Injective PEPS:

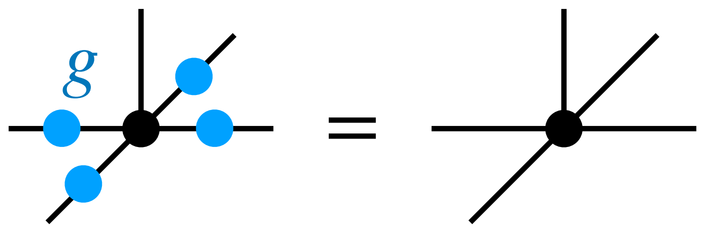


Injective PEPSs give rise to parent Hamiltonians, to which they are unique ground states

N. Schuch, I. Cirac, and D. Pérez-García, Annals of Physics 325, 2153 (2010)

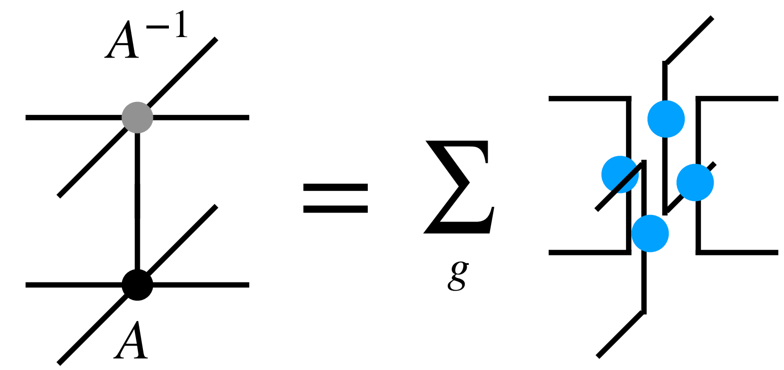
$\mathbb{Z}_2$-Injective PEPS: Natural Framework for $\mathbb{Z}_2$ Topological Order

$\mathbb{Z}_2$-injective PEPS:



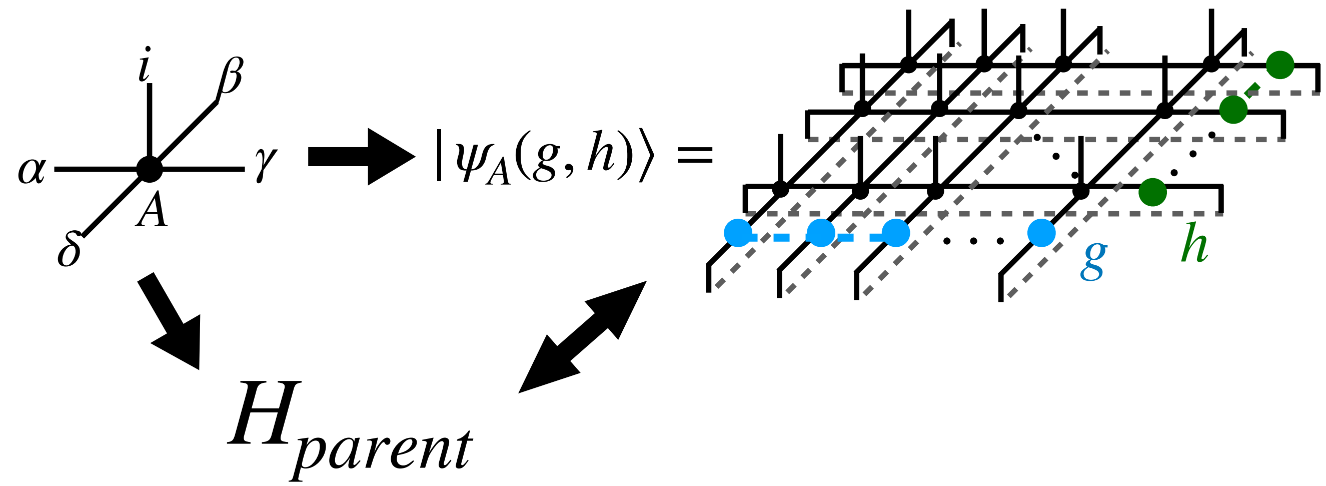
$g \in \{I, Z\}$

and

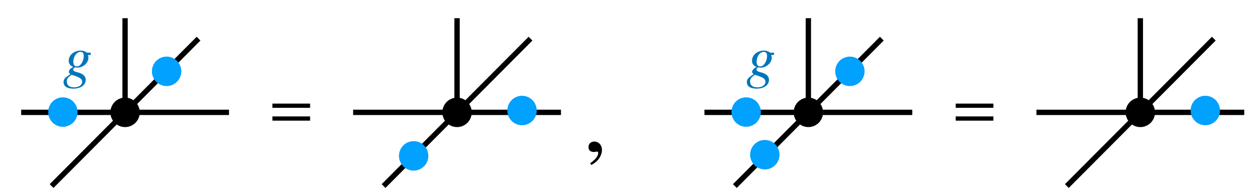


- The ground subspace of the parent Hamiltonian of a $\mathbb{Z}_2$-injective PEPS is spanned by two non-contractible loops $(u_g^{\otimes L_x}, u_h^{\otimes L_y})$.

- These ground states are locally indistinguishable owing to the pulling through conditions. $\rightarrow$ topological degeneracy.



Pulling through conditions:



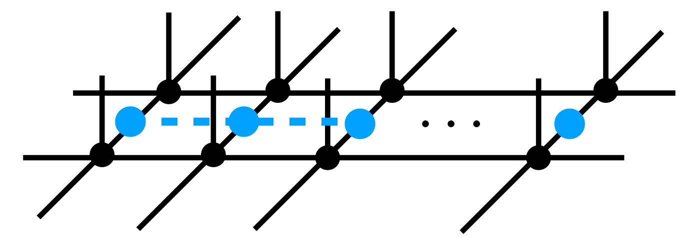
N. Schuch, I. Cirac, and D. Pérez-García, Annals of Physics 325, 2153 (2010)

Anyons and Minimally Entangled States

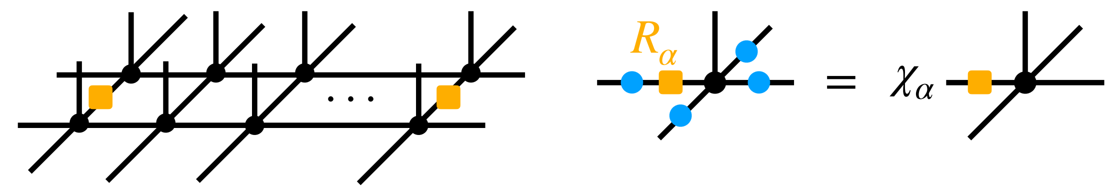
Anyons ($\mathbb{Z}_2$ TC)

- Flux: Characterized by group action g.
- Charge: Characterized by irreducible representation α.
- Fermion: Combination of charge and flux.

Flux:



Charge:

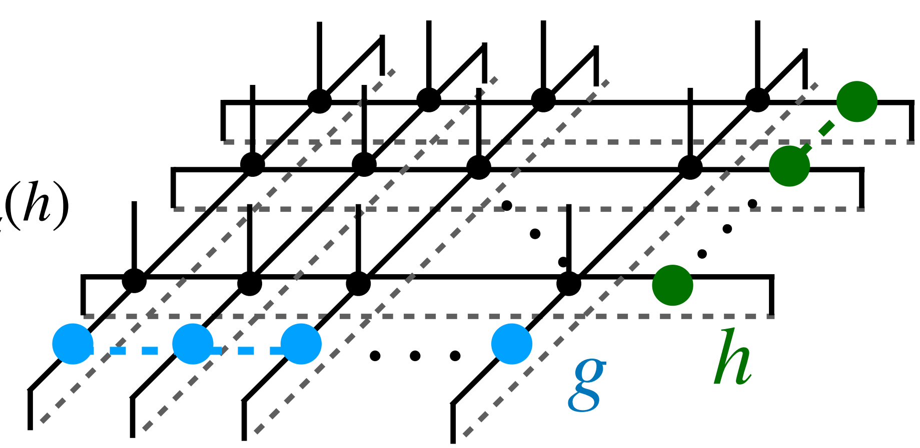


Minimally Entangled States (MES)

- A special basis that spans the ground state with the minimum entanglement entropy.
- Reflects the anyonic excitations of topological phases:

$$|\psi(I,e)\rangle = |I\rangle \text{ (vacuum)}$$
$$|\psi(I,o)\rangle = |e\rangle \text{ (charge)}$$
$$|\psi(Z,e)\rangle = |m\rangle \text{ (flux)}$$
$$|\psi(Z,o)\rangle = |\epsilon\rangle \text{ (fermion)}$$

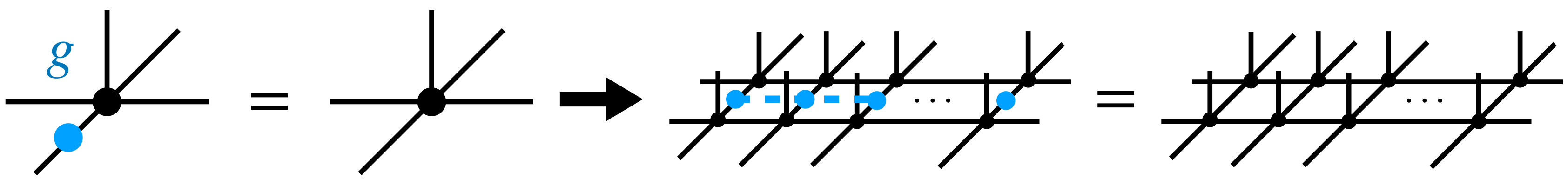
$$|\psi(g,\alpha)\rangle = \sum_{h\in\mathbb{Z}_2} \chi_\alpha(h)$$



N. Schuch, I. Cirac, and D. Pérez-García, Annals of Physics 325, 2153 (2010)
Y. Zhang, T. Grover, A. Turner, M. Oshikawa, and A. Vishwanath, Phys Rev B 85, 235151 (2012)

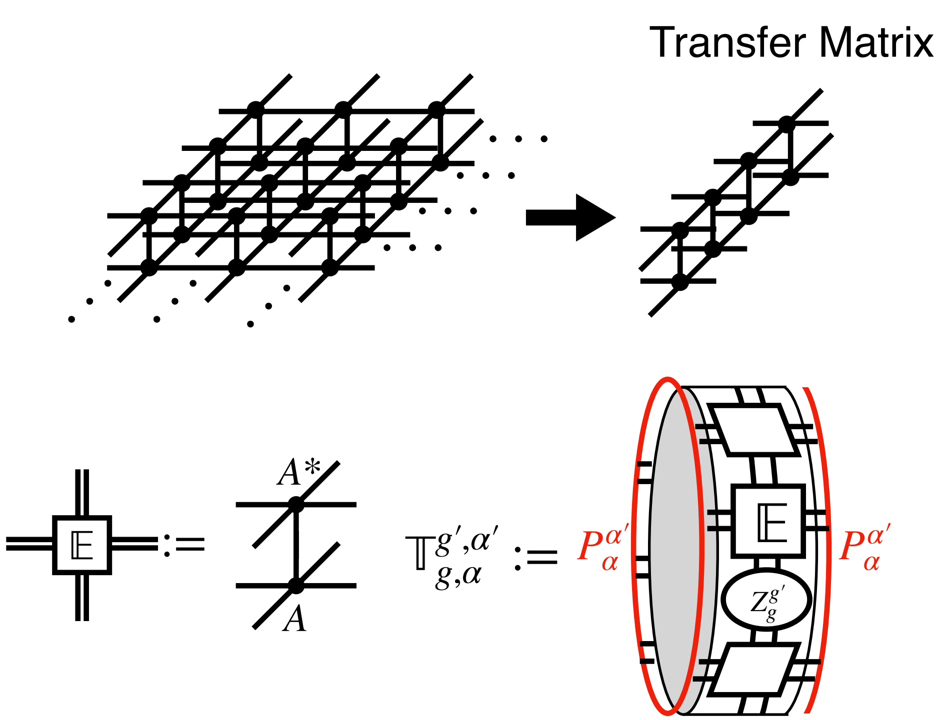
Using MES to Detect Anyon Condensation

A $\mathbb{Z}_2$-injective PEPS does not guarantee topological order



Flux is identical to GS
→ Flux condensation

To detect anyon condensation, we compute the overlap of MESs, which is encoded in the dominant eigenvalue (λ) of the transfer matrix (TM).



	II	IZ	ZI	ZZ
ee	$\langle I \| I \rangle$	$\langle I \| m \rangle$	$\langle m \| I \rangle$	$\langle m \| m \rangle$
eo	$\langle I \| e \rangle$	$\langle I \| \epsilon \rangle$	$\langle m \| e \rangle$	$\langle m \| \epsilon \rangle$
oe	$\langle e \| I \rangle$	$\langle e \| m \rangle$	$\langle \epsilon \| I \rangle$	$\langle \epsilon \| m \rangle$
oo	$\langle e \| e \rangle$	$\langle e \| \epsilon \rangle$	$\langle \epsilon \| e \rangle$	$\langle \epsilon \| \epsilon \rangle$

Different projector correspond to different MES overlap.

- Red: Regular TM.
- Blue: TM measuring charge difference.
- Yellow: TM measuring flux difference.
- Green: TM measuring flux and charge (fermion) difference.

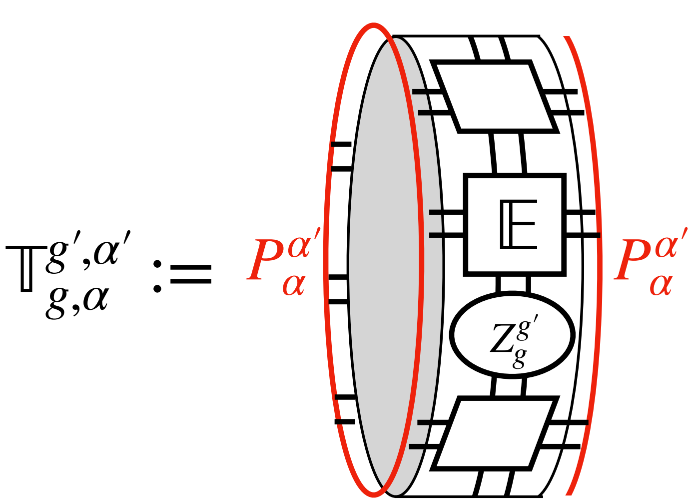
N. Schuch, D. Poilblanc, J. I. Cirac, and D. Pérez-García, Phys Rev Lett 111, 090501 (2013).
J. Haegeman, V. Zauner, N. Schuch, and F. Verstraete, Nat Commun 6, 8284 (2015).

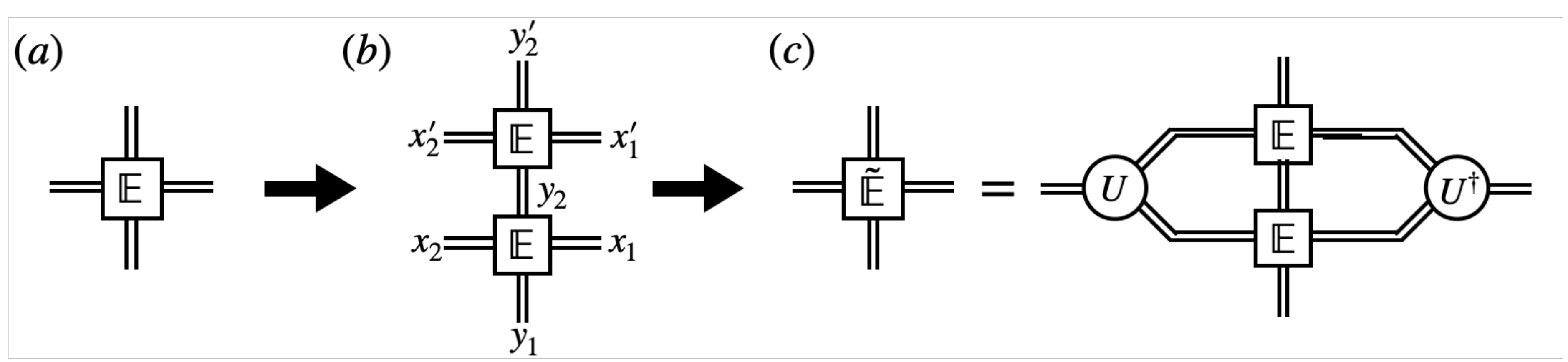
Method: Gauge-Symmetry Preserved HOTRG

Goal: Calculate TM on a long (infinite) cylinder with appropriate boundary conditions

(a) Start from a one-site double tensor.
(b) Merge two double tensors to form a new rank-6 tensor.
(c) Apply appropriate isometries U which truncates the bond dimension.

Preserve symmetries at each step





Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

Warmup: Toric Code with String Tension

$$|\Psi(\beta_x, \beta_z)\rangle = \prod_e Q_e(\beta_x, \beta_z) \times |\Omega_{TC}\rangle$$

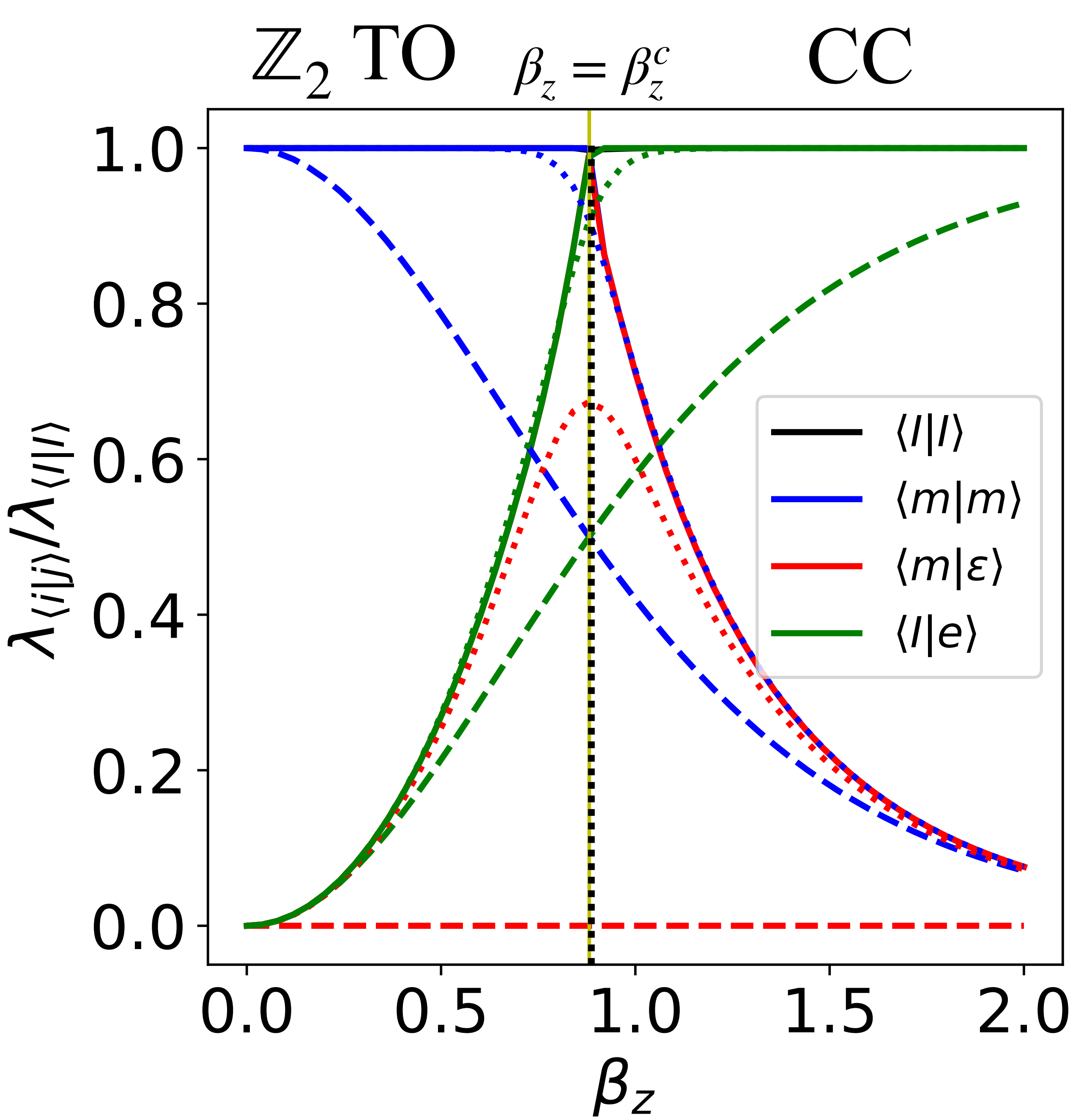
with $Q_e(\beta_x, \beta_z) = \exp\left(\frac{\beta_x \sigma_e^x + \beta_z \sigma_e^z}{4}\right)$

Along $\beta_x = 0$ axis, varying β_z
Only regular and the charge difference TMs are nonzero.

$\beta_z < \beta_z^c$ ($\mathbb{Z}_2$ TO phase):

- $\lambda_{\langle I|I\rangle} = \lambda_{\langle m|m\rangle} = 1$: $|I\rangle$ and $|m\rangle$ are normalizable ground states.
- $\lambda_{\langle I|e\rangle} = \lambda_{\langle m|\epsilon\rangle} < 1$: Four MESs are orthogonal.

	II	IZ	ZI	ZZ
ee	$\langle I\|I\rangle$	$\langle I\|m\rangle$	$\langle m\|I\rangle$	$\langle m\|m\rangle$
eo	$\langle I\|e\rangle$	$\langle I\|\epsilon\rangle$	$\langle m\|e\rangle$	$\langle m\|\epsilon\rangle$
oe	$\langle e\|I\rangle$	$\langle e\|m\rangle$	$\langle \epsilon\|I\rangle$	$\langle \epsilon\|m\rangle$
oo	$\langle e\|e\rangle$	$\langle e\|\epsilon\rangle$	$\langle \epsilon\|e\rangle$	$\langle \epsilon\|\epsilon\rangle$



dashed: L = 1, dotted: L = 16, solid: L = 256

Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

Warmup: Toric Code with String Tension

$$|\Psi(\beta_x, \beta_z)\rangle = \prod_e Q_e(\beta_x, \beta_z) \times |\Omega_{TC}\rangle$$

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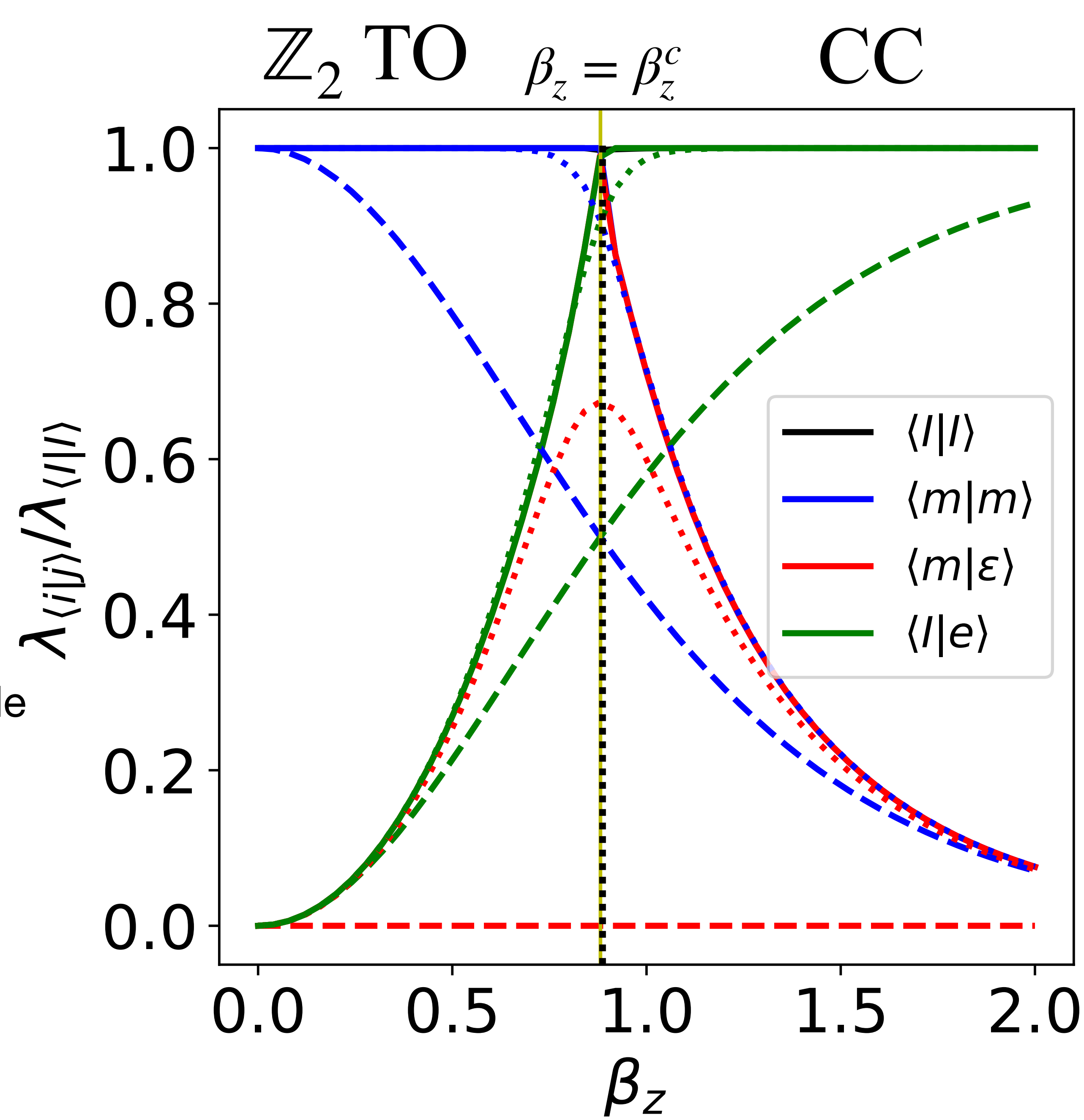
Along $\beta_x = 0$ axis, varying β_z
Only regular and the charge difference TMs are nonzero.

$\beta_z > \beta_z^c$ (CC phase):

- $\lambda_{\langle I|I\rangle} = \lambda_{\langle I|e\rangle} = 1$: $|I\rangle$ and $|e\rangle$ are not orthogonal, indicating the charge anyon **condense** to vacuum.
- $\lambda_{\langle m|\epsilon\rangle} = \lambda_{\langle m|m\rangle} < 1$: $|m\rangle$ and $|\epsilon\rangle$ are not normalizable states, indicating both flux anyon and fermion are **confined**.

	II	IZ	ZI	ZZ
ee	$\langle I\|I\rangle$	$\langle I\|m\rangle$	$\langle m\|I\rangle$	$\langle m\|m\rangle$
eo	$\langle I\|e\rangle$	$\langle I\|\epsilon\rangle$	$\langle m\|e\rangle$	$\langle m\|\epsilon\rangle$
oe	$\langle e\|I\rangle$	$\langle e\|m\rangle$	$\langle \epsilon\|I\rangle$	$\langle \epsilon\|m\rangle$
oo	$\langle e\|e\rangle$	$\langle e\|\epsilon\rangle$	$\langle \epsilon\|e\rangle$	$\langle \epsilon\|\epsilon\rangle$



dashed: L = 1, dotted: L = 16, solid: L = 256

Along $\beta_z = 0$ axis, varying β_x
Only regular and the flux difference TMs are nonzero.

No Other MES can become $|\epsilon\rangle$

	II	IZ	ZI	ZZ
ee	$\langle I\|I\rangle$	$\langle I\|m\rangle$	$\langle m\|I\rangle$	$\langle m\|m\rangle$
eo	$\langle I\|e\rangle$	$\langle I\|\epsilon\rangle$	$\langle m\|e\rangle$	$\langle m\|\epsilon\rangle$
oe	$\langle e\|I\rangle$	$\langle e\|m\rangle$	$\langle \epsilon\|I\rangle$	$\langle \epsilon\|m\rangle$
oo	$\langle e\|e\rangle$	$\langle e\|\epsilon\rangle$	$\langle \epsilon\|e\rangle$	$\langle \epsilon\|\epsilon\rangle$

- CC phase: $|e\rangle$ becomes $|I\rangle$ while $|m\rangle$ and $|\epsilon\rangle$ are confined.
- FC phase: $|m\rangle$ becomes $|I\rangle$ while $|e\rangle$ and $|\epsilon\rangle$ are confined.
- Other possibilities? Can $|e\rangle$ or $|m\rangle$ become $|\epsilon\rangle$? Can $|\epsilon\rangle$ become $|I\rangle$? Can $|e\rangle$ become $|m\rangle$?
- Based on the TM structure, one can show that no other MES can become $|\epsilon\rangle \rightarrow$ fermion cannot condense.

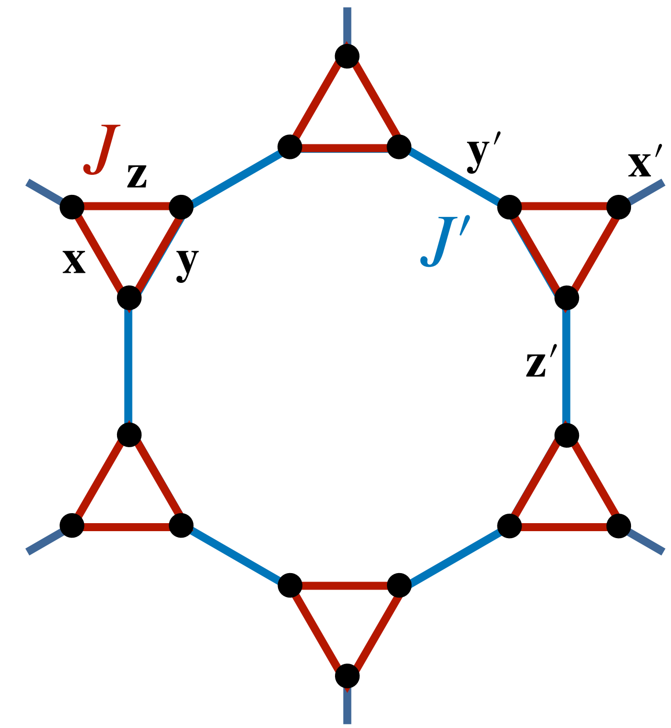
We are left with the only possibility that $|e\rangle$ becomes $|m\rangle$

Kitaev Model on the Star Lattice

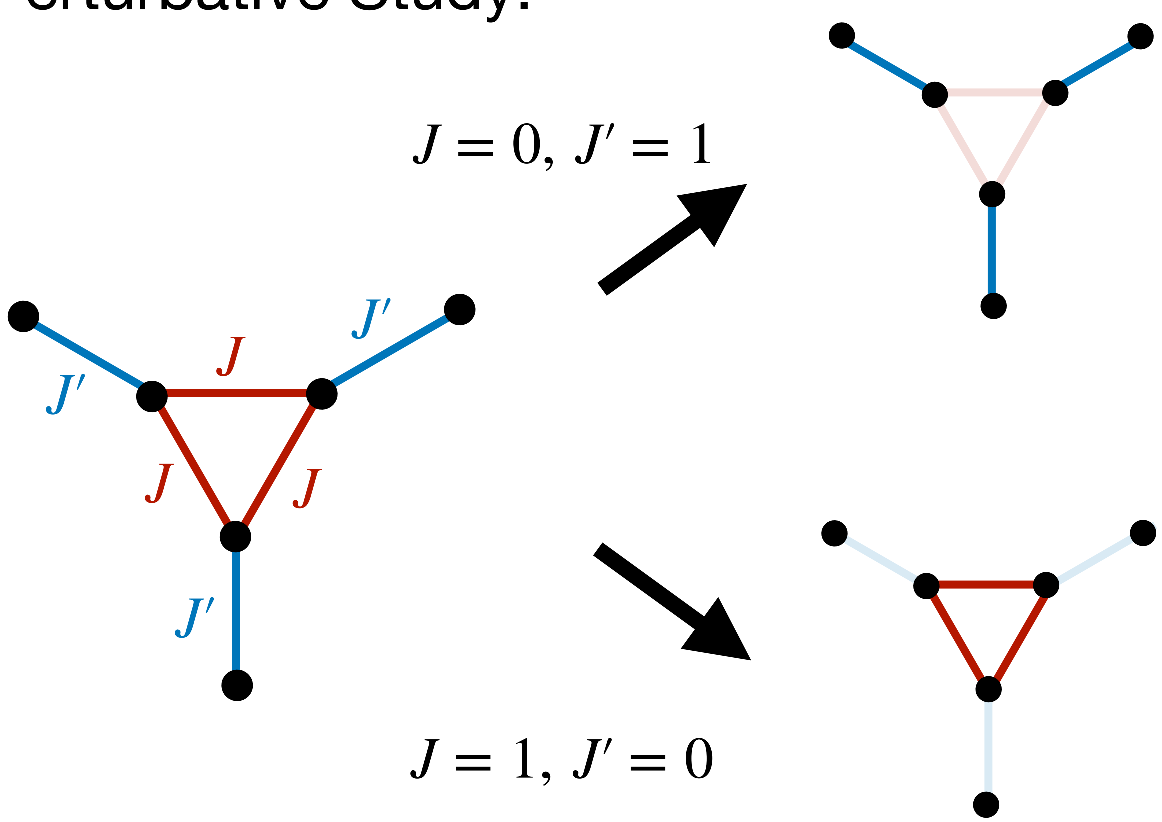
$$H = -J \sum_{\langle i,j \rangle_\gamma} S_i^\gamma S_j^\gamma - J' \sum_{\langle ij \rangle \in \gamma'} S_i^{\gamma'} S_j^{\gamma'}$$

$\gamma = x, y, z$: Intrabond

$\gamma' = x', y', z'$: Interbond



Perturbative Study:



Isolated-dimer limit: Toric code ground state $\rightarrow \mathbb{Z}_2$ topological order: $|I\rangle, |e\rangle, |m\rangle, |\epsilon\rangle$.

Isolated-triangle limit: Kitaev's honeycomb model $\rightarrow$ Ising anyon: $|I\rangle$, $|\sigma\rangle$, $|\epsilon\rangle$.

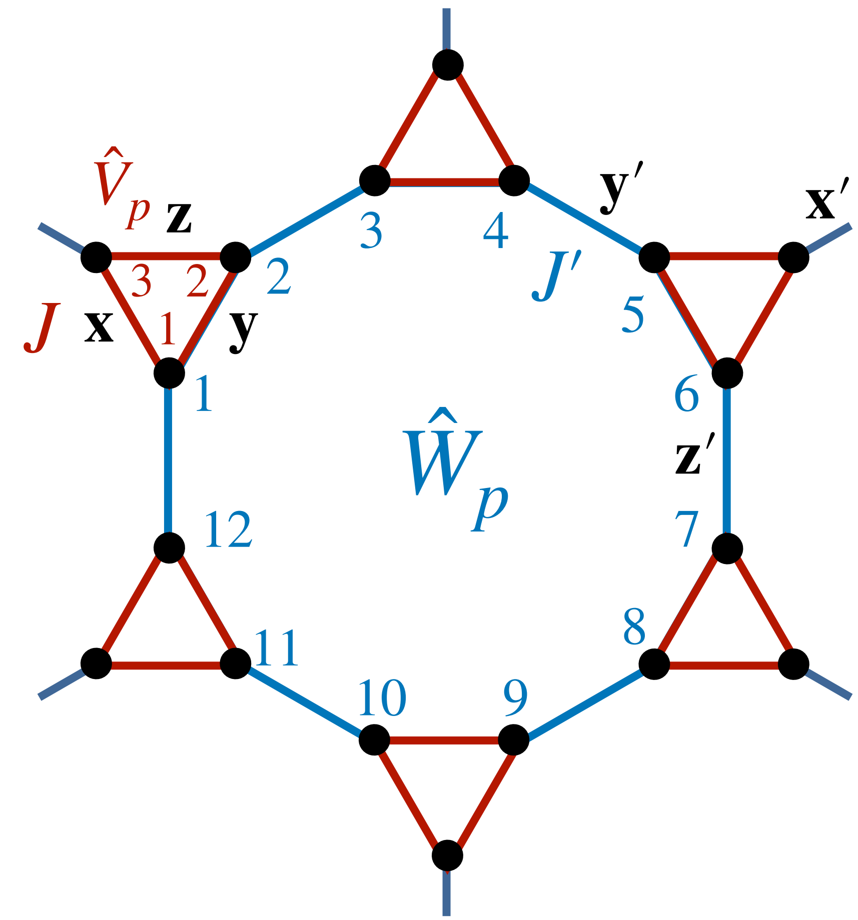
H. Yao and S. A. Kivelson, Phys Rev Lett 99, 247203 (2007).
S. Dusuel, K. P. Schmidt, J. Vidal, and R. L. Zaffino, Phys Rev B 78, 125102 (2008).

Kitaev Model on the Star Lattice

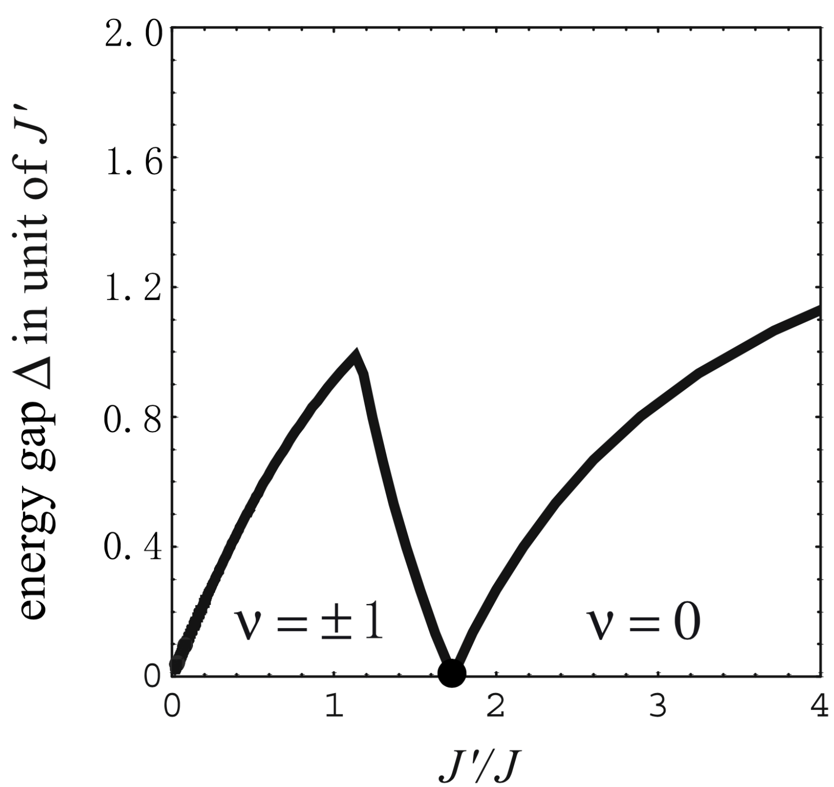
$$H = -J \sum_{\langle i,j \rangle_\gamma} S_i^\gamma S_j^\gamma - J' \sum_{\langle ij \rangle \in \gamma'} S_i^{\gamma'} S_j^{\gamma'}$$

$\gamma = x, y, z$: Intrabond

$\gamma' = x', y', z'$: Interbond



Exact Results



- $J'/J > \sqrt{3}$: $\mathbb{Z}_2$ topological order, 4-flold degeneracy on the torus.
- $J'/J < \sqrt{3}$: Ising Anyon, 3-flold degeneracy on the torus.
- Two types of flux operators: $\hat{W}_p = \hat{S}_1^x \hat{S}_2^z \hat{S}_3^y \ldots \hat{S}_{12}^y$, $\hat{V}_p = \hat{S}_1^z \hat{S}_2^x \hat{S}_3^y$
- Both ground states live in vortex-free sector: $\{W_p = +1, V_p = +1\}$

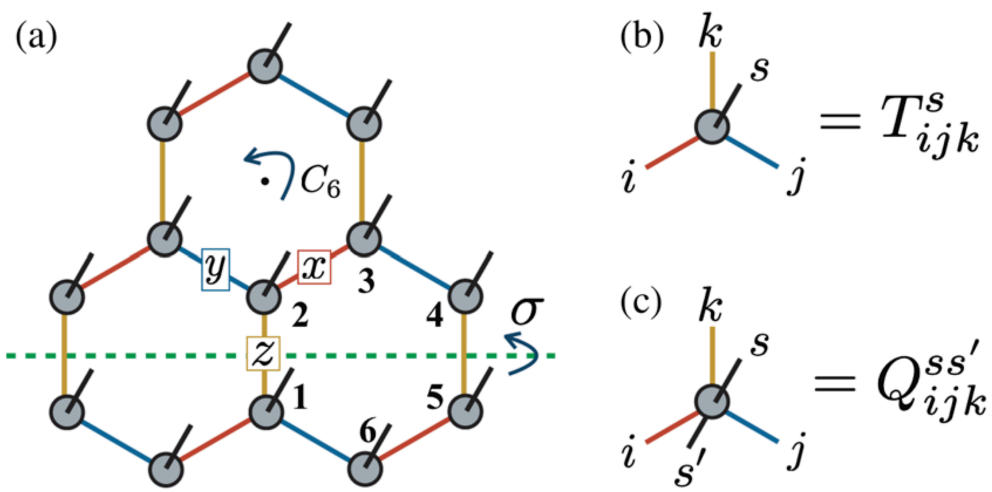
H. Yao and S. A. Kivelson, Phys Rev Lett 99, 247203 (2007).

Loop Gas Projector Gives $\mathbb{Z}_2$-injective Ansatz

Originally defined on the honeycomb lattice

$$\hat{Q}_{LG} = \mathrm{tTr}\prod_{\alpha} Q^{ss'}_{i_\alpha j_\alpha k_\alpha} |s\rangle\langle s'|$$

$$Q_{000} = I, \quad Q_{011} = -iU^x, \quad Q_{101} = -iU^y, \quad Q_{110} = -iU^z$$



Applying $\hat{Q}_{LG}$ on any injective PEPS gives $\mathbb{Z}_2$-injective PEPS

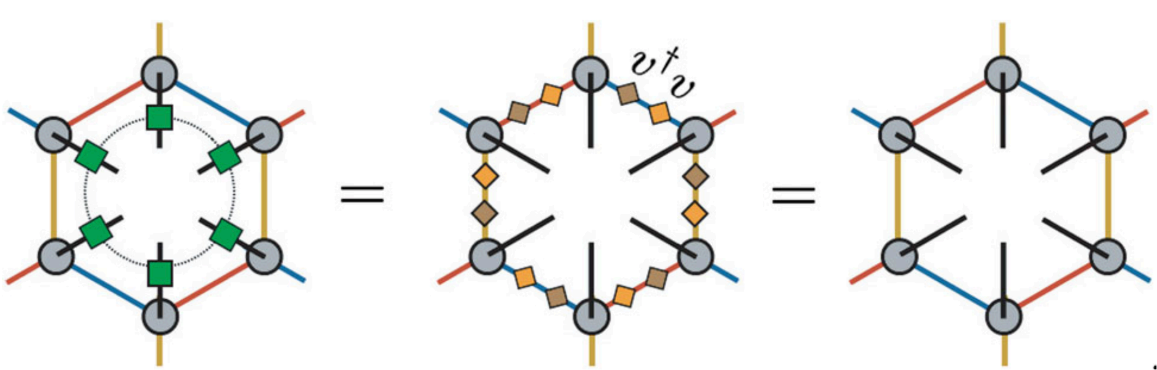
- Loop gas (LG) states (parameter θ):

$$|\psi_{LG}(\theta)\rangle = \hat{Q}_{LG}|\psi(\theta)\rangle, \quad |\psi(\theta)\rangle = \otimes_\alpha |\theta, \gamma_\alpha\rangle$$

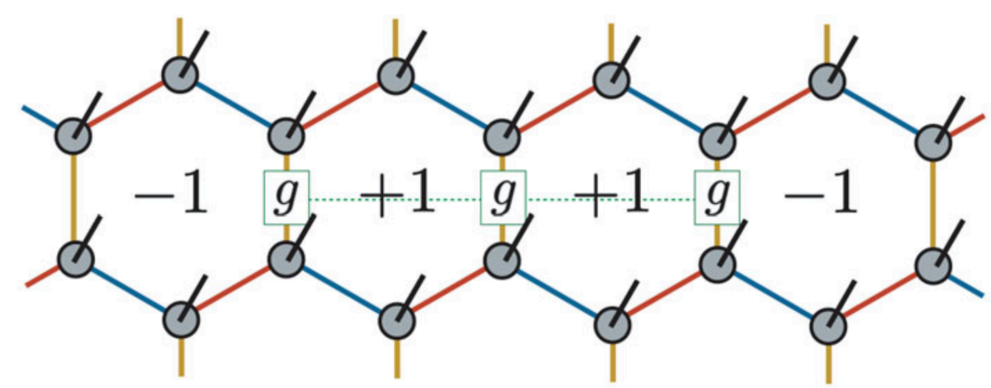
- String gas (SG) states (parameter α, β):

$$|\psi_{SG}(\alpha,\beta)\rangle = \hat{Q}_{LG}\hat{R}_{DG}(\alpha,\beta)|\psi(\theta = \tan^{-1}\sqrt{2})\rangle$$

Any LG Ansatz satisfies vortex-free conditions

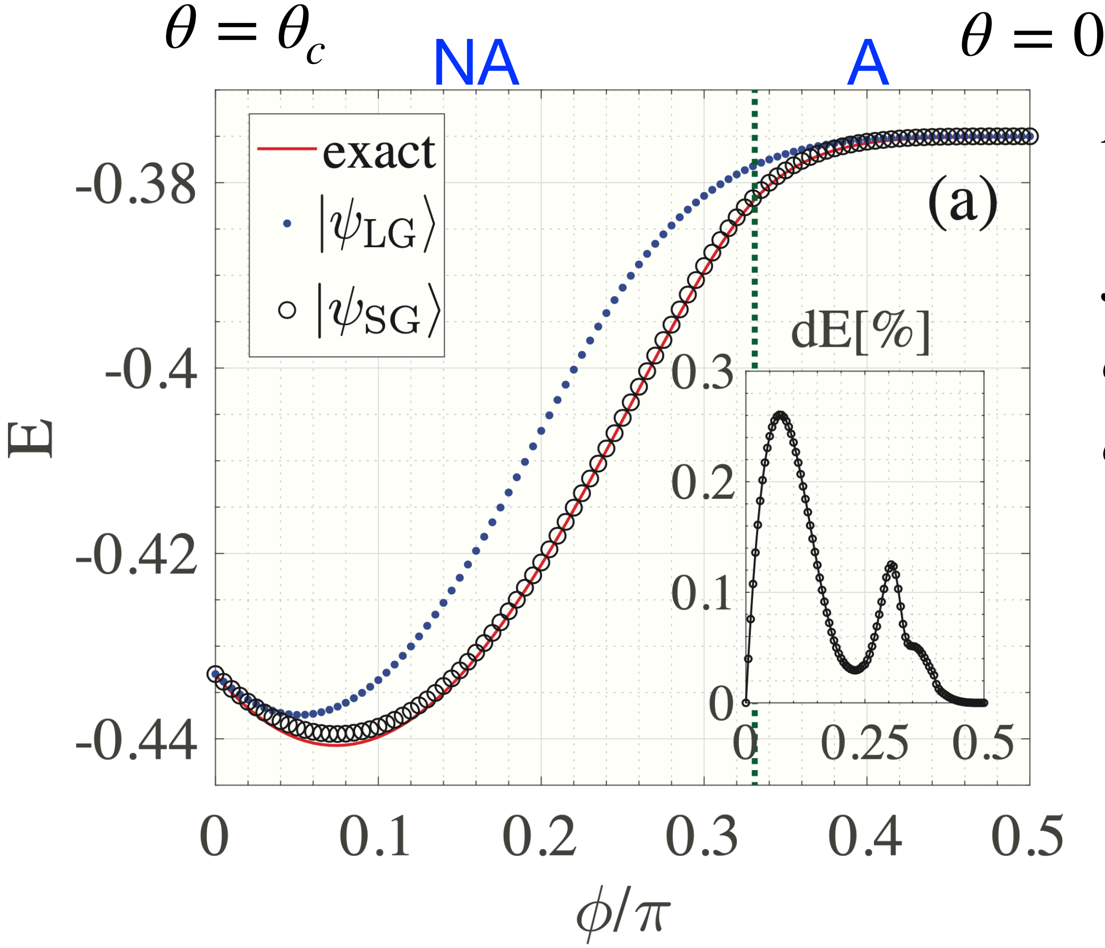


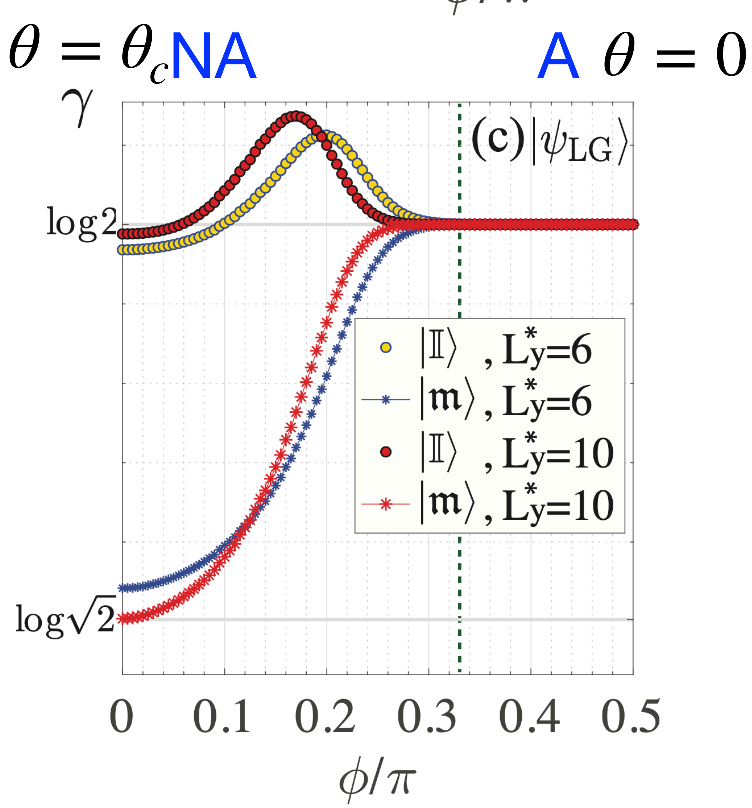
Flux in $\mathbb{Z}_2$-injective PEPS corresponds to physical vortex



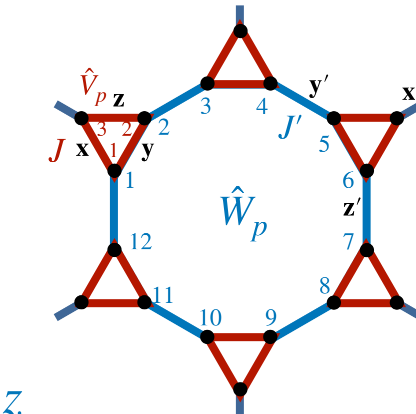
H.-Y. Lee, R. Kaneko, T. Okubo, and N. Kawashima, Phys. Rev. Lett. 123, 087203 (2019).

Loop Gas Projector Gives $\mathbb{Z}_2$-injective Ansatz





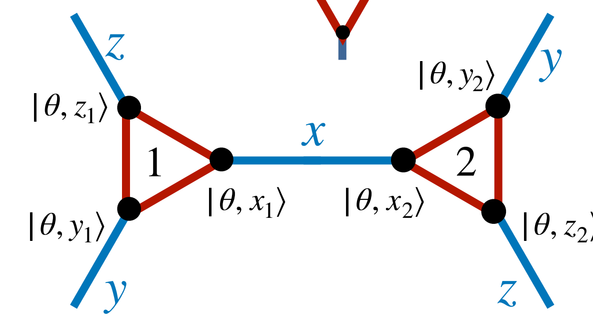
$$H = -J \sum_{\langle i,j\rangle_\gamma} S_i^\gamma S_j^\gamma - J' \sum_{\langle ij\rangle \in \gamma'} S_i^{\gamma'} S_j^{\gamma'}$$



$J = \cos\phi,\ J' = \sin\phi$

$\phi > \tan^{-1}\sqrt{3}$: $\mathbb{Z}_2$ topological order

$\phi < \tan^{-1}\sqrt{3}$: Ising Anyon.



LG states $|\psi_{LG}(\theta)\rangle$:

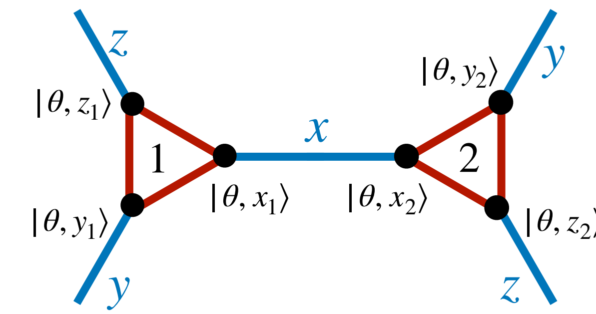
$$|\Psi\rangle = \otimes_\alpha |\psi_\alpha\rangle, |\psi_\alpha(\theta)\rangle = |\theta, x_\alpha\rangle |\theta, y_\alpha\rangle |\theta, z_\alpha\rangle$$

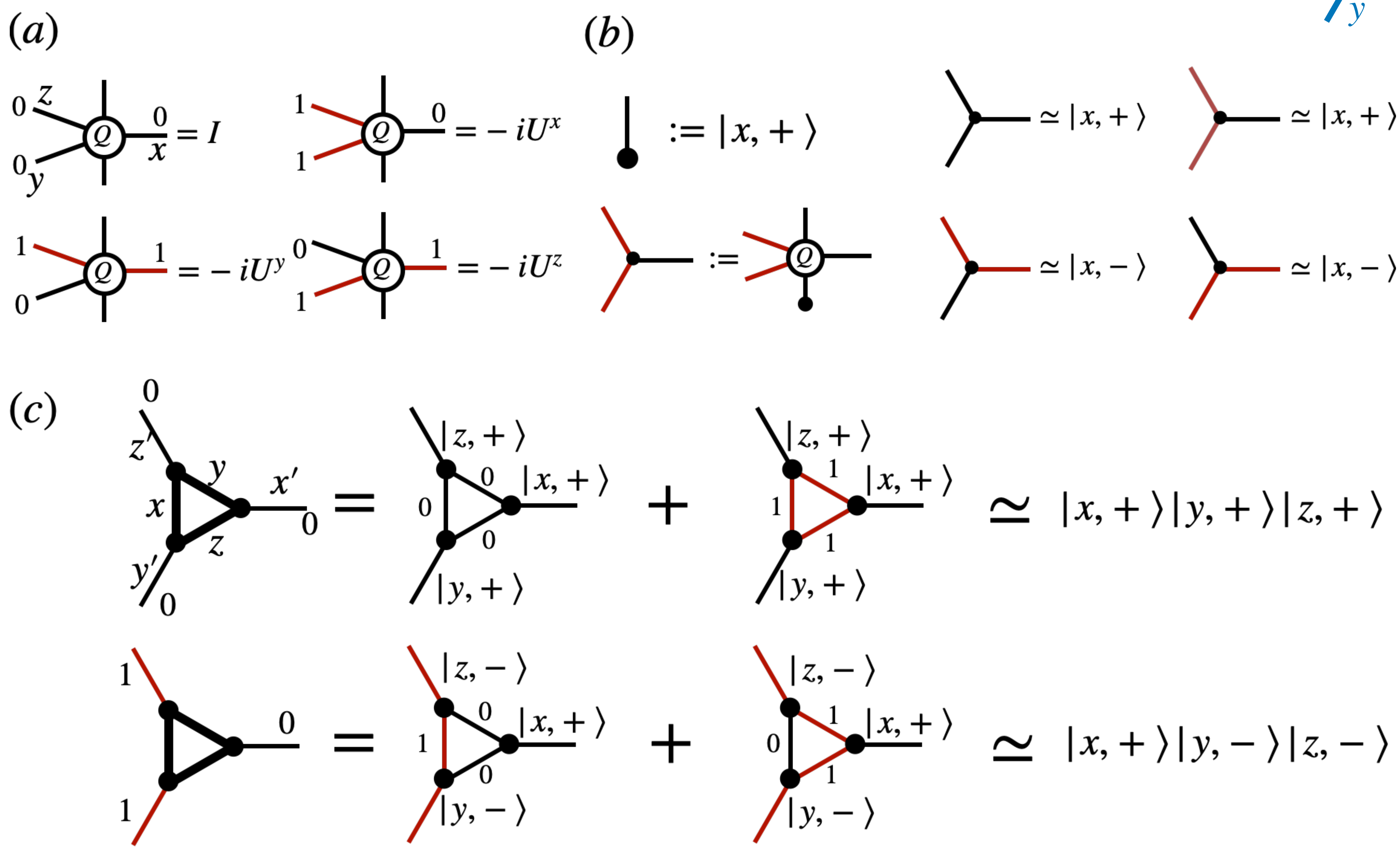
$$\langle \theta, \gamma | \sigma^{\gamma'} | \theta, \gamma\rangle = \delta_{\gamma'\gamma}\cos\theta + (1-\delta_{\gamma'\gamma})\frac{\sin\theta}{\sqrt{2}}$$

- Yield exact energy in both isolated-dimer ($\phi = \pi/2$) and isolated-triangle ($\phi = 0$) limit.
- Large energy deviation for $0 < \phi < \pi/2$. $\rightarrow$ Label them using variational parameter θ.
- In the isolated-dimer limit ($\theta = 0, \phi = \pi/2$), LG and SG states are $\mathbb{Z}_2$-isomteric, i.e., they corresponds to toric code ground states.

H.-Y. Lee, R. Kaneko, T. Okubo, and N. Kawashima, Phys Rev B 101, 035140 (2020).
Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

$\mathbb{Z}_2$-Isometry: $\theta = 0$





$$|\Psi\rangle = \otimes_\alpha |\psi_\alpha\rangle, |\psi_\alpha(\theta)\rangle = |\theta, x_\alpha\rangle |\theta, y_\alpha\rangle |\theta, z_\alpha\rangle$$

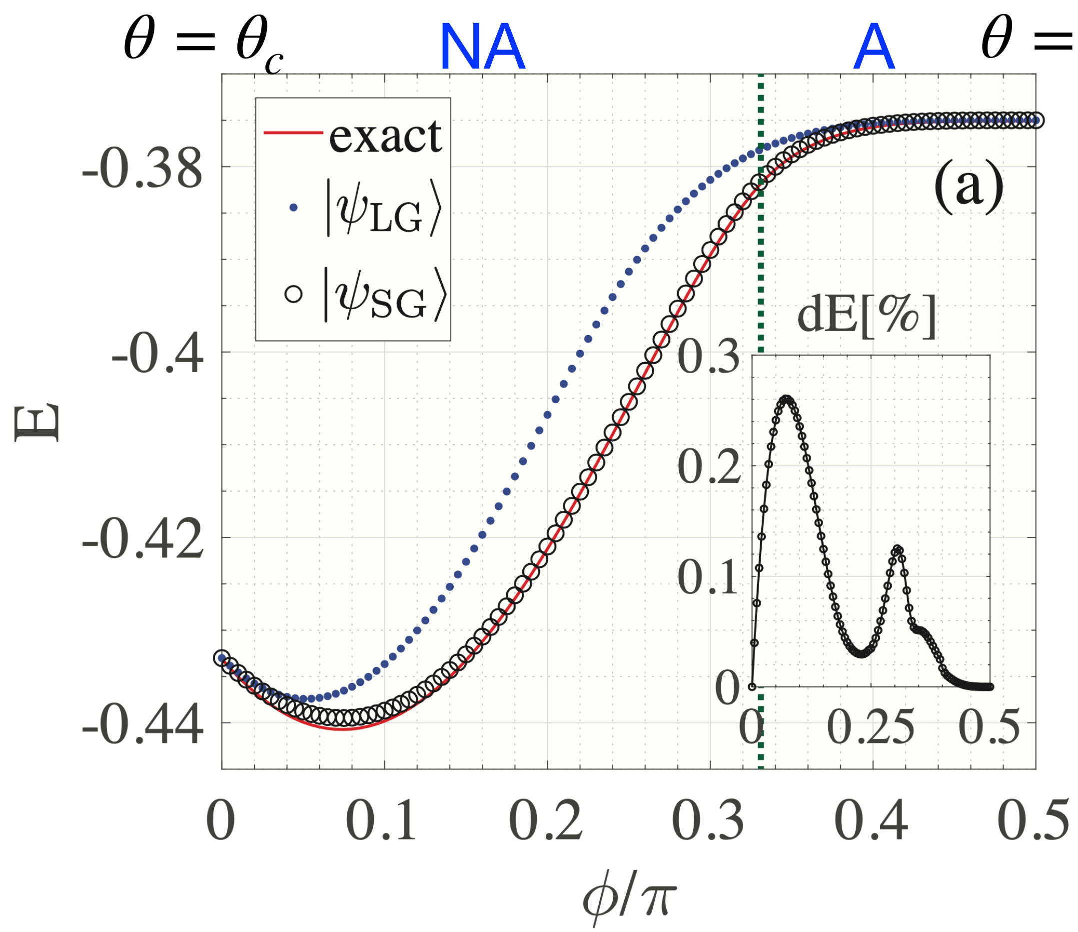
$$\sigma^\gamma |\gamma, \pm\rangle = \pm |\gamma, \pm\rangle$$

$$\langle \theta, \gamma | \sigma^{\gamma'} | \theta, \gamma \rangle = \delta_{\gamma'\gamma} \cos\theta + (1 - \delta_{\gamma'\gamma}) \frac{\sin\theta}{\sqrt{2}}$$

$$|\gamma, +\rangle = |\theta = 0, \gamma\rangle$$

Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

Loop Gas Projector Gives $\mathbb{Z}_2$-injective Ansatz

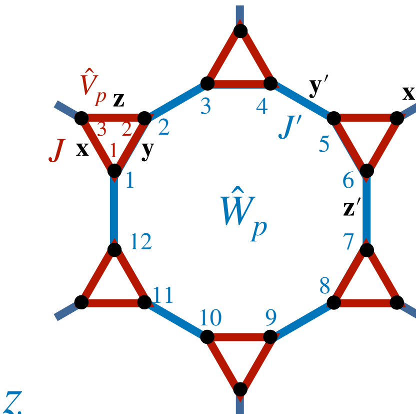


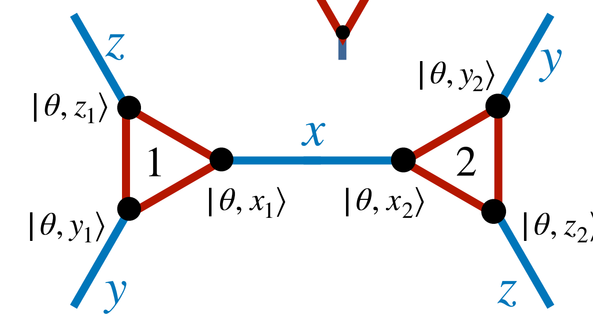
$$H = -J \sum_{\langle i,j \rangle_\gamma} S_i^\gamma S_j^\gamma - J' \sum_{\langle ij \rangle \in \gamma'} S_i^{\gamma'} S_j^{\gamma'}$$

$J = \cos\phi,\ J' = \sin\phi$

$\phi > \tan^{-1}\sqrt{3}$: $\mathbb{Z}_2$ topological order

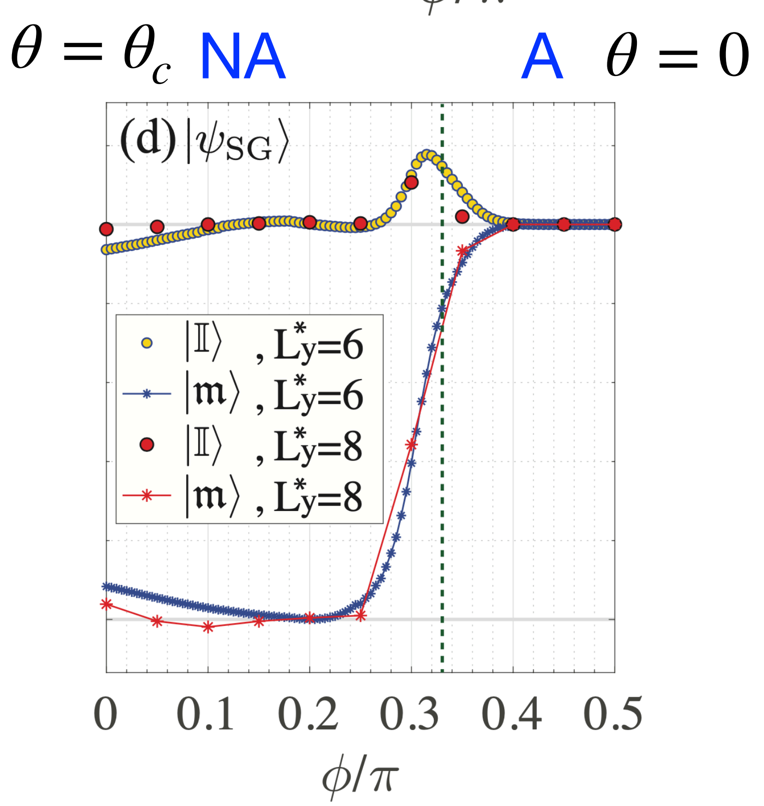
$\phi < \tan^{-1}\sqrt{3}$: Ising Anyon.





SG states $|\psi_{SG}(\alpha, \beta)\rangle$:

- Yield accurate energy for all different ϕ.→ Label them using Hamiltonian's parameter ϕ: $|\psi_{SG}\rangle = |\psi_{SG}(\theta)\rangle$
- Topological entanglement entropy shows that flux anyon ($\gamma = \ln(2)$) transmutes into σ anyon ($\gamma = \ln(\sqrt{2})$).
- Ground state subspace
  - Abelian: 4-fold degeneracy
  - Non-Abelian: 3-fold degeneracy (incompatible with $\mathbb{Z}_2$ -injective classification !)



H.-Y. Lee, R. Kaneko, T. Okubo, and N. Kawashima, Phys Rev B 101, 035140 (2020).
Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

Results: LG States

Only regular and fermion difference TMs are nonzero.

$\theta \neq \theta_c$:

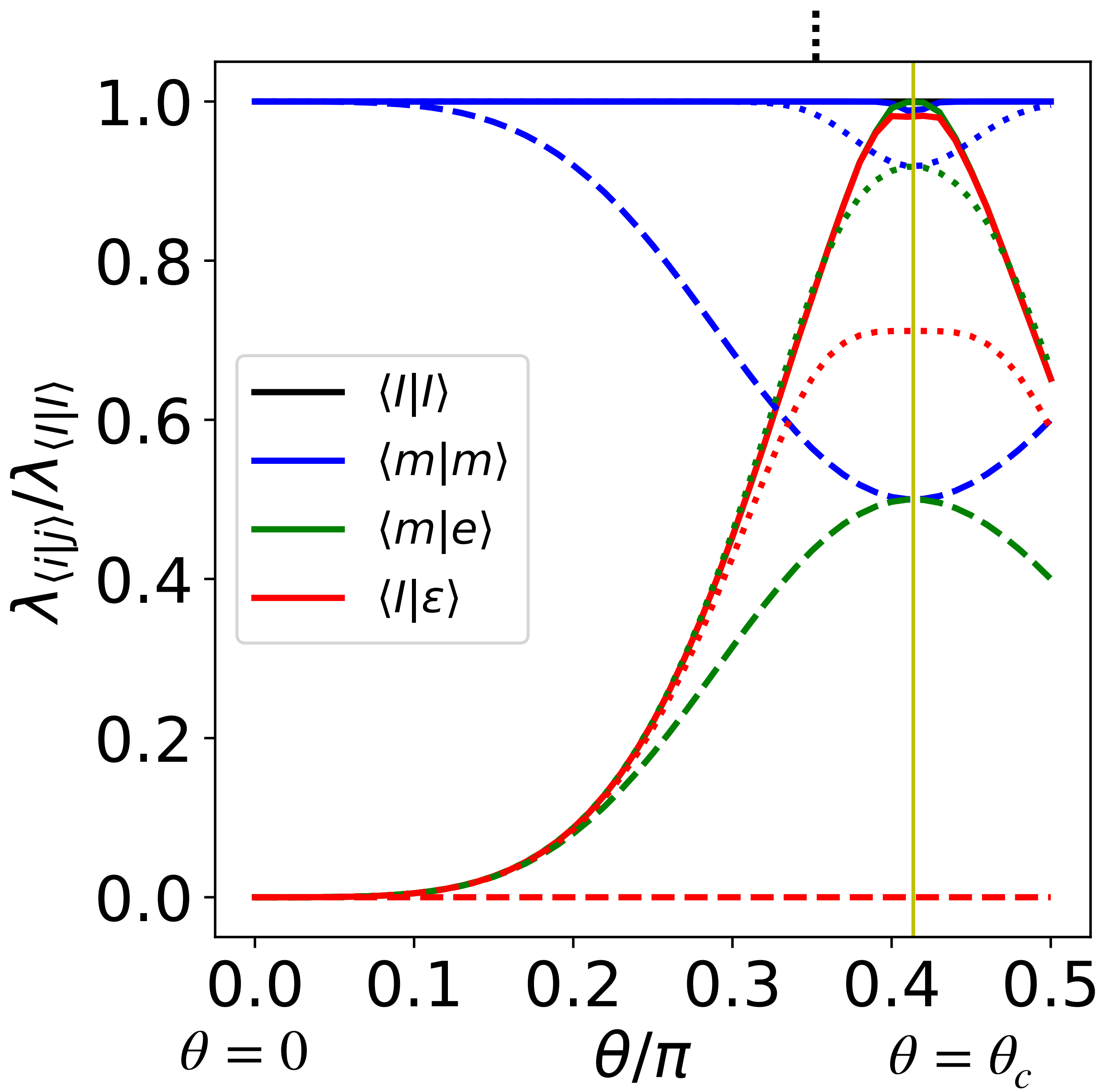
- $\lambda_{\langle m|e\rangle} = \lambda_{\langle I|\epsilon\rangle} < 1, \;\; \lambda_{\langle I|I\rangle} = \lambda_{\langle m|m\rangle} = 1$:
  Four MES are orthogonal and normalizable, indicating $\mathbb{Z}_2$ TO phase.

$\theta = \theta_c$:

- $\lambda_{\langle m|e\rangle} = \lambda_{\langle m|m\rangle} > \lambda_{\langle I|\epsilon\rangle}$ for all L.
- As L increases, $\lambda_{\langle m|e\rangle}, \lambda_{\langle m|m\rangle}, \lambda_{\langle I|\epsilon\rangle} \to 1 \Rightarrow$ Power-law decay correlation.

- One-parameter family of LG states $|\psi_{LG}(\theta)\rangle$ are all in $\mathbb{Z}_2$ TO phase except for $\theta = \theta_c$.

	II	IZ	ZI	ZZ
ee	$\langle I\|I\rangle$	$\langle I\|m\rangle$	$\langle m\|I\rangle$	$\langle m\|m\rangle$
eo	$\langle I\|e\rangle$	$\langle I\|\epsilon\rangle$	$\langle m\|e\rangle$	$\langle m\|\epsilon\rangle$
oe	$\langle e\|I\rangle$	$\langle e\|m\rangle$	$\langle \epsilon\|I\rangle$	$\langle \epsilon\|m\rangle$
oo	$\langle e\|e\rangle$	$\langle e\|\epsilon\rangle$	$\langle \epsilon\|e\rangle$	$\langle \epsilon\|\epsilon\rangle$



$\theta = 0$ $\theta = \theta_c$

Isolated-dimer isolated-triangle

dashed: L = 1, dotted: L = 16, solid: L = 256

Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

Results: SG States

$\phi > 0.24\pi$:

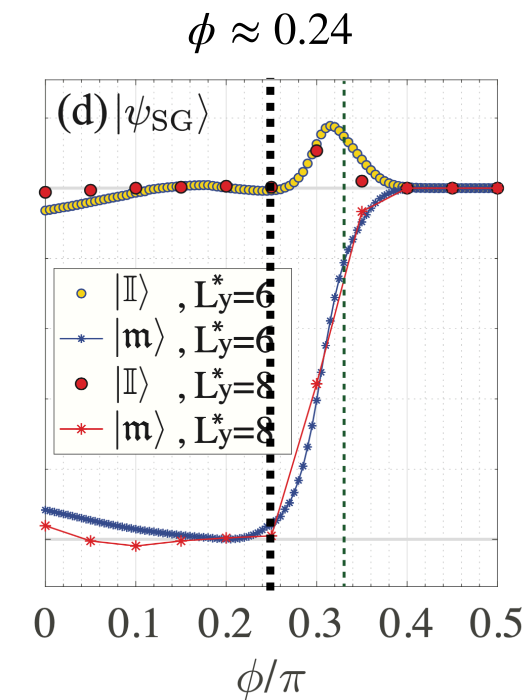
- $\lambda_{\langle m|e\rangle} = \lambda_{\langle I|\epsilon\rangle} < 1, \;\; \lambda_{\langle I|I\rangle} = \lambda_{\langle m|m\rangle} = 1 \rightarrow$ $\mathbb{Z}_2$ TO phase (Similar to LG states for $\theta \neq \theta_c$).

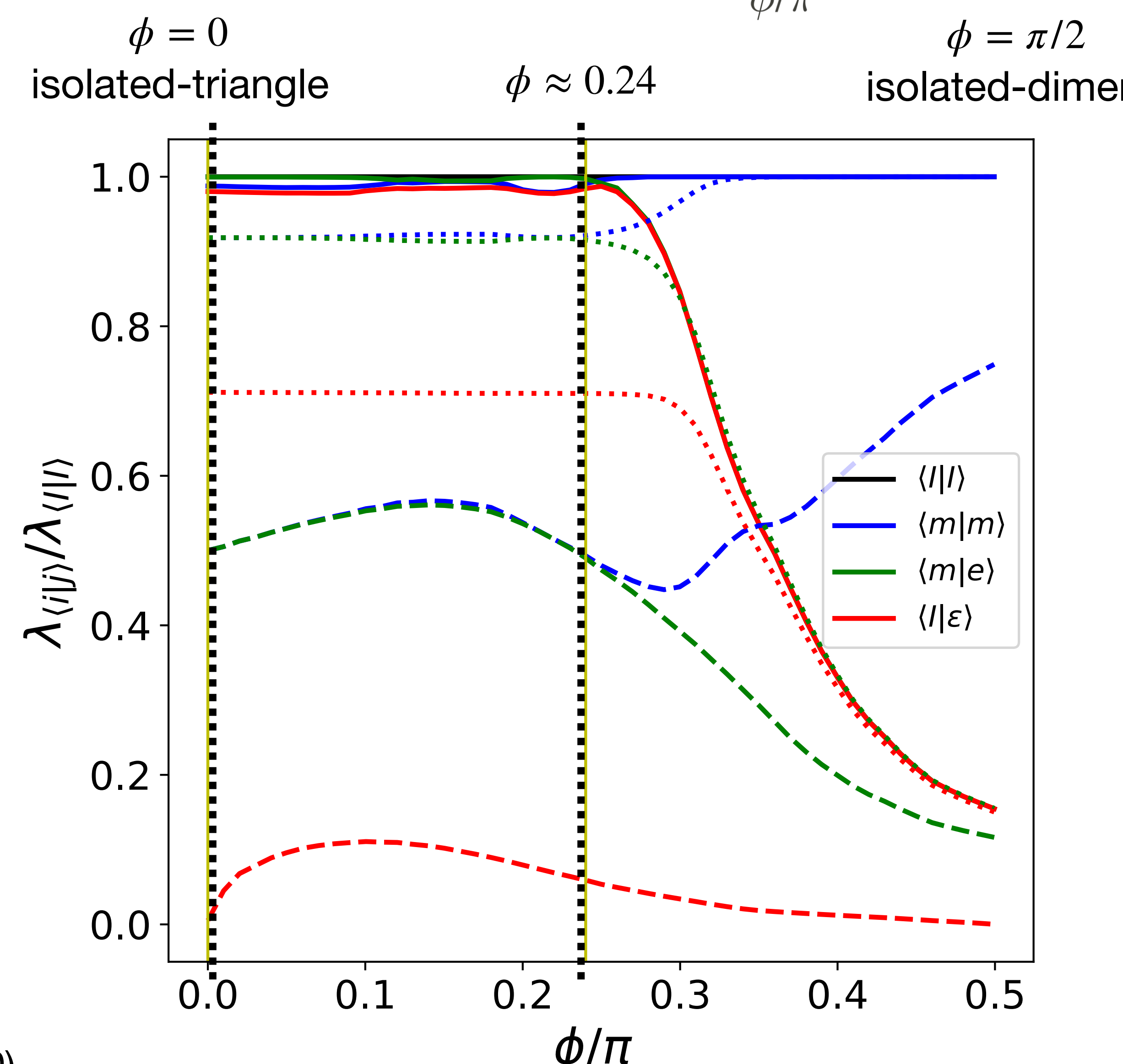
$\phi < 0.24\pi$:

- $\lambda_{\langle m|e\rangle} = \lambda_{\langle m|m\rangle} > \lambda_{\langle I|\epsilon\rangle}$ for all L.
- As L increases, $\lambda_{\langle m|e\rangle}, \lambda_{\langle m|m\rangle}, \lambda_{\langle I|\epsilon\rangle} \rightarrow 1 \Rightarrow$ Power law decaying correlation (Similar to LG states for $\theta \neq \theta_c$).
- Since no other MES can become $|\epsilon\rangle$, $|I\rangle$ and $|\epsilon\rangle$ should be regarded as different states.

Note that the crossing point coincides the point that the entanglement entropy of $|m\rangle$ change from $\log 2$ to $\log \sqrt{2}$.

Can we identify $|e\rangle$ and $|m\rangle$ as the same states?



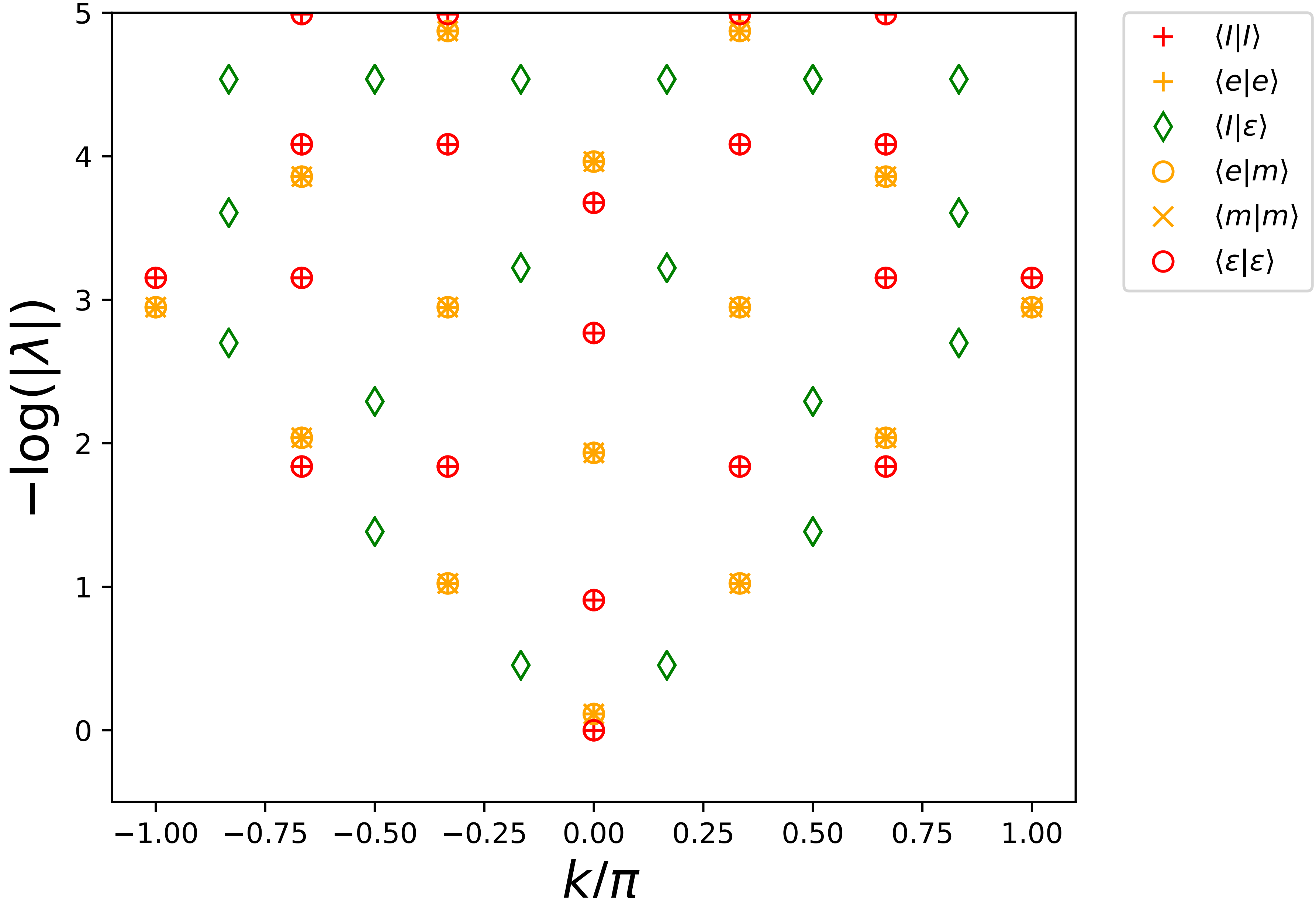


dashed: L = 1, dotted: L = 16, solid: L = 256

H.-Y. Lee, et al., Phys Rev B 101, 035140 (2020).
Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

Results: TM Spectrum

TM spectrum of LG at $\theta = \theta_c$ with $L = 6$ shows that $\langle e \,|\, e \rangle$, $\langle e \,|\, m \rangle$ and $\langle m \,|\, m \rangle$ have exactly the same spectrum: $|\, e \rangle$ and $|\, m \rangle$ are the same states $\Rightarrow$
The ground state is 3-fold degenerate in non-Abelian phase.



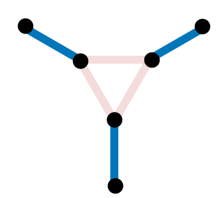
Y.-H. Chen, C.-Y. Huang, YJK arXiv: 2102.10980

Physical Interpretation

Kitaev's honeycomb model: $H = -J_x \sum_{\langle i,j \rangle_x} S_i^x S_j^x - J_y \sum_{\langle i,j \rangle_y} S_i^y S_j^y - J_z \sum_{\langle i,j \rangle_z} S_i^z S_j^z$

$|J_i| > |J_j| + |J_k|$

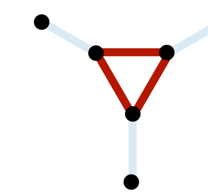
$\mathbb{Z}_2$TO: I, e, m, ϵ



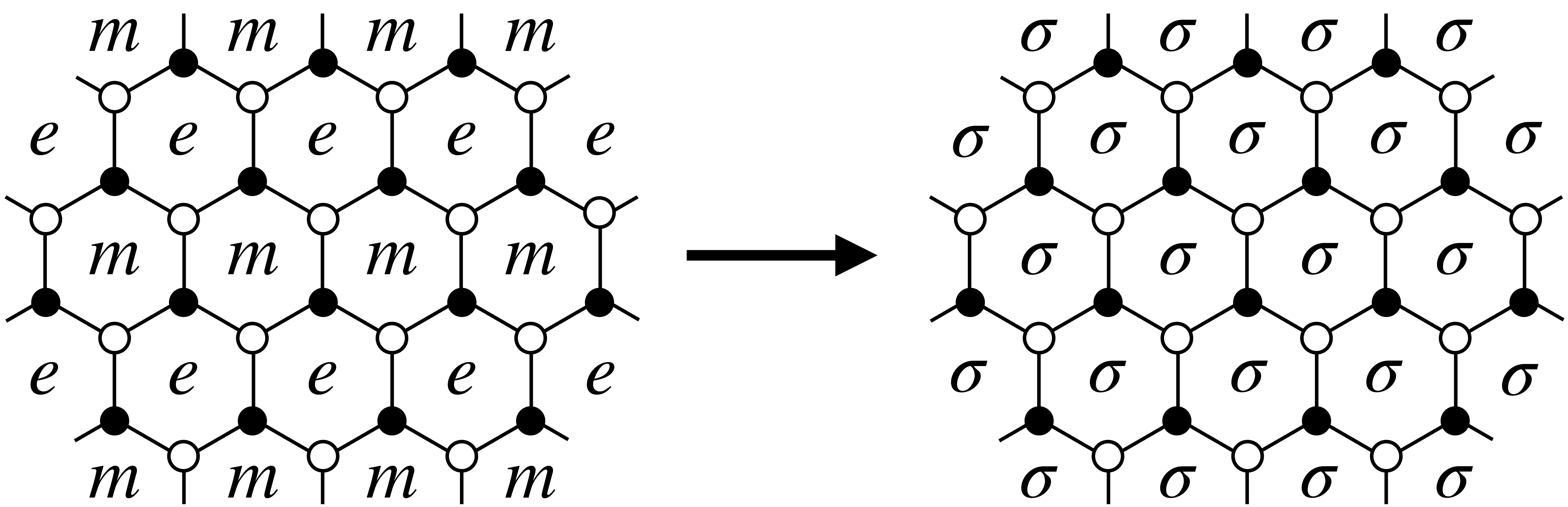
Isolated-dimer

$|J_i| \leq |J_j| + |J_k|$

Ising anyon: I, σ, ϵ



Isolated-triangle



A. Kitaev, Annals of Physics **321**, 2–111 (2006)

Conclusion

- Within the framework of $\mathbb{Z}_2$-injective PEPS, we can understand the transition from Abelian to non-Abelian topological order phase as charge and flux anyons transmuting into σ anyon.

- Since no other MES can become $|\epsilon\rangle$, we conclude that $\mathbb{Z}_2$-injective PEPS can only describe three kinds of anyonic transitions: $|e\rangle = |I\rangle$, $|m\rangle = |I\rangle$, and $|e\rangle = |m\rangle$.

- Our study shows that $\mathbb{Z}_2$-injective PEPS can unify the transition from $\mathbb{Z}_2$-TO phase to both trivial phases (product states) and nontrivial phases (non-Abelian phase).

- Generalization to other symmetry groups? MPO-injective PEPS?