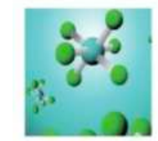
Entanglement in Strongly Correlated Systems
2021, Feb 15 -- Feb 26

Organizers:
F. Mila (Ecole Polytechnique Federale de Lausanne)
R. Orús (DIPC)
D. Poilblanc (CNRS / U. Toulouse)
N. Schuch (Max-Planck-Institute of Quantum Optics / MCQST)



QUANTUM MAGNETS ON SMALL-WORLD NETWORKS

arXiv:2102.04919

Nicolas Laflorencie & **Maxime Dupont**

Laboratoire de Physique Théorique CNRS Toulouse

Lawrence Berkeley National Laboratory Berkeley, USA

What is a Small World ?

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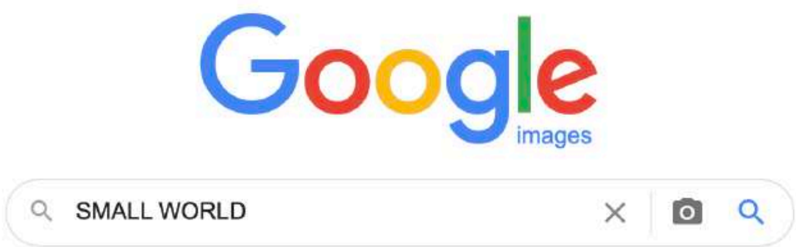

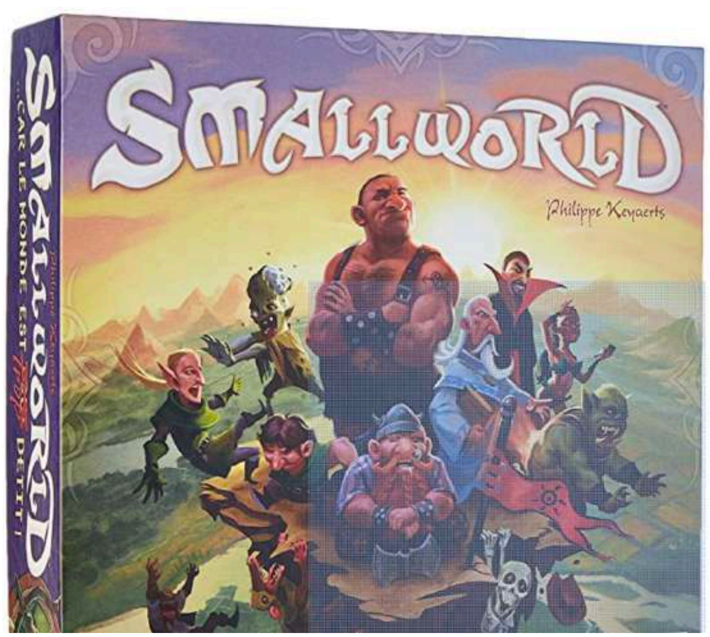

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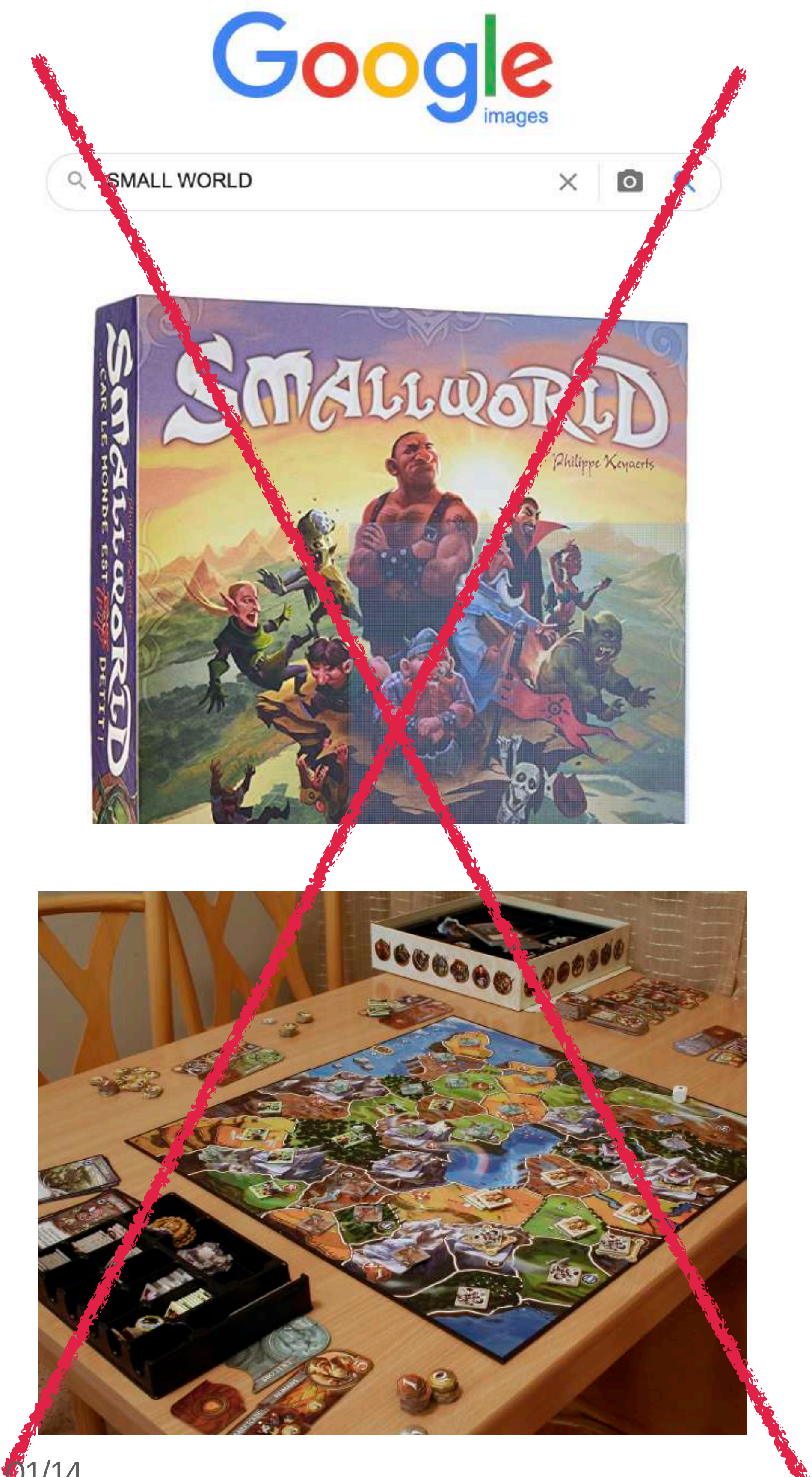
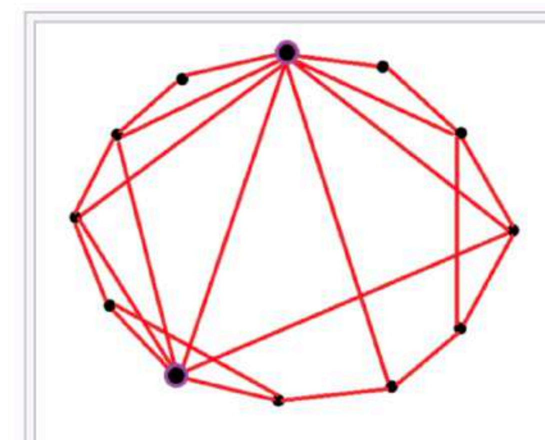


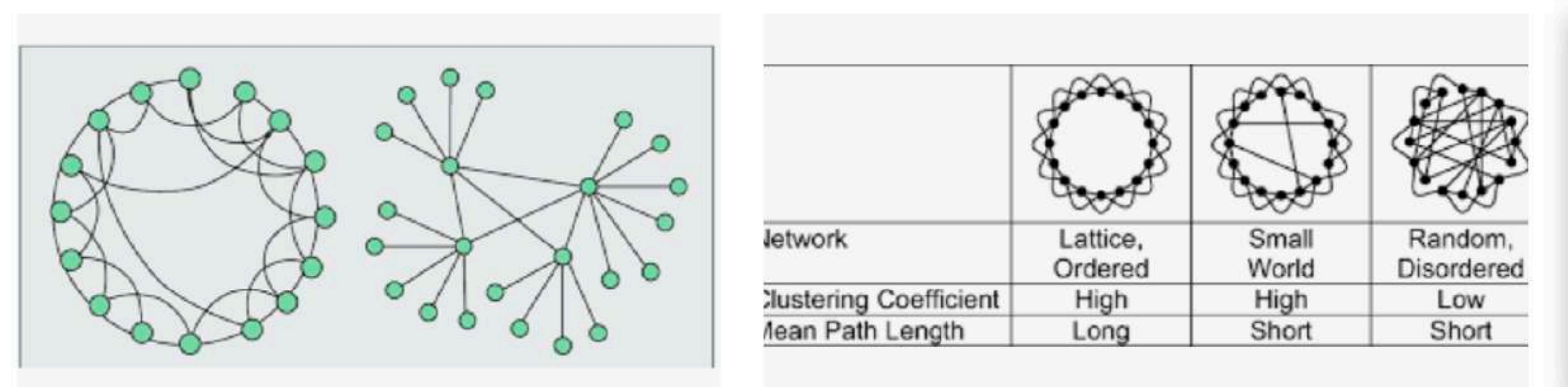
Small-world network

From Wikipedia, the free encyclopedia

A **small-world network** is a type of mathematical graph in which most nodes are not neighbors of one another, but the neighbors of any given node are likely to be neighbors of each other and most nodes can be reached from every other node by a small number of hops or steps. Specifically, a small-world network is defined to be a network where the typical distance L between two randomly chosen nodes (the number of steps required) grows proportionally to the logarithm of the number of nodes N in the network, that is:[1]

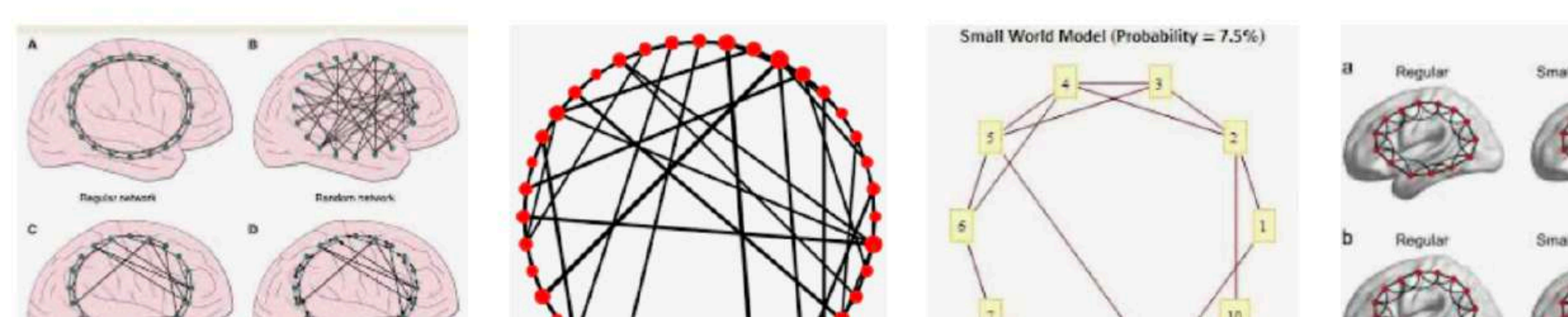
$$L \propto \log N$$





An example of a) a small-world network generated using ...
researchgate.net

Is the Urban World Small? The Evidence for Small World Struct...
link.springer.com



The Small-World Effect

Random graph of N nodes, fluctuating connectivity

$$\text{Diameter } \overline{\ell} \sim \ln N \quad \Rightarrow d_{\text{eff}} = \infty$$

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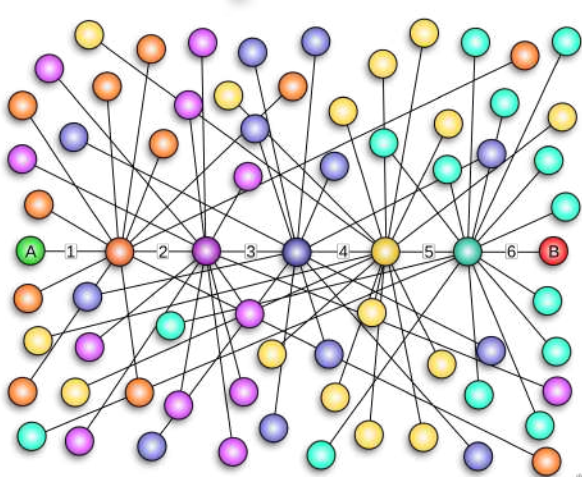
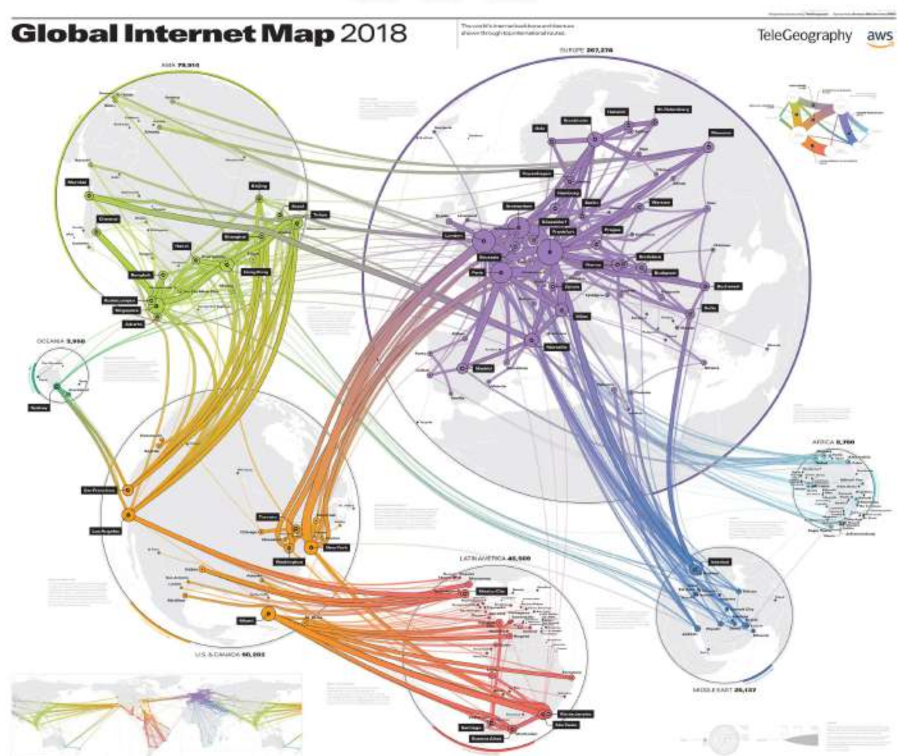
Social network analysis

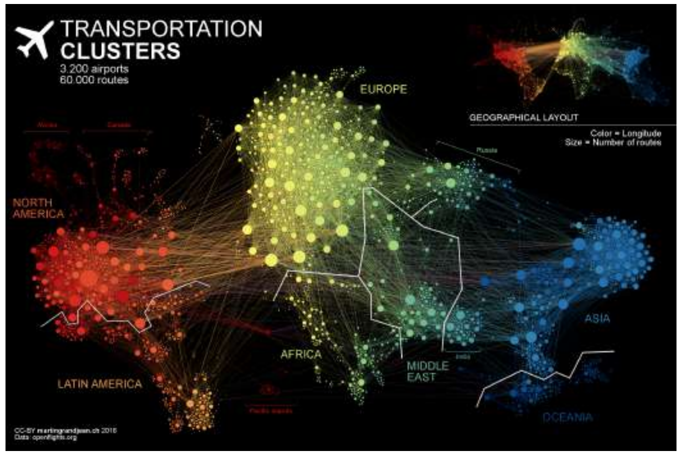
The "six degrees of separation"



www



airlines connections



epidemic spreading

Small-World models

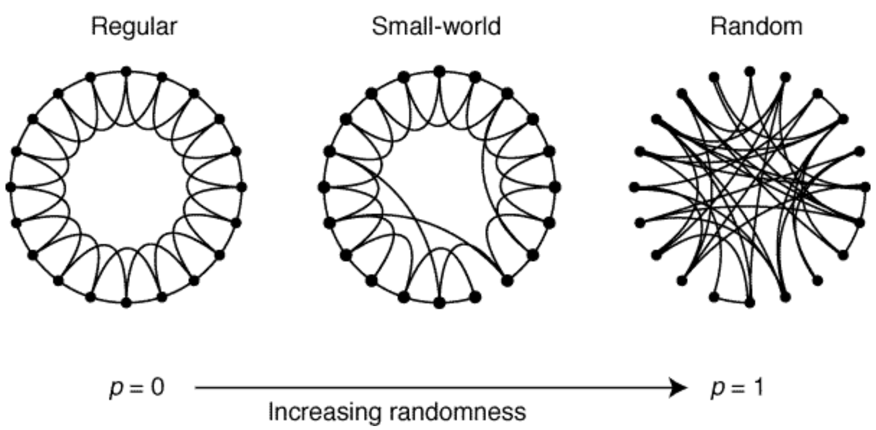
The Watts-Strogatz model

NATURE | VOL 393 | 4 JUNE 1998

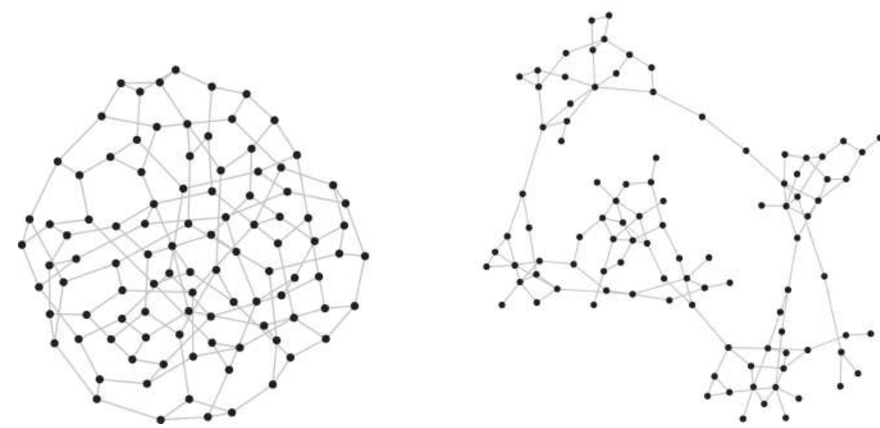
Collective dynamics of 'small-world' networks

Duncan J. Watts* & Steven H. Strogatz

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, New York 14853, USA



Random regular graphs (Erdős–Rényi)

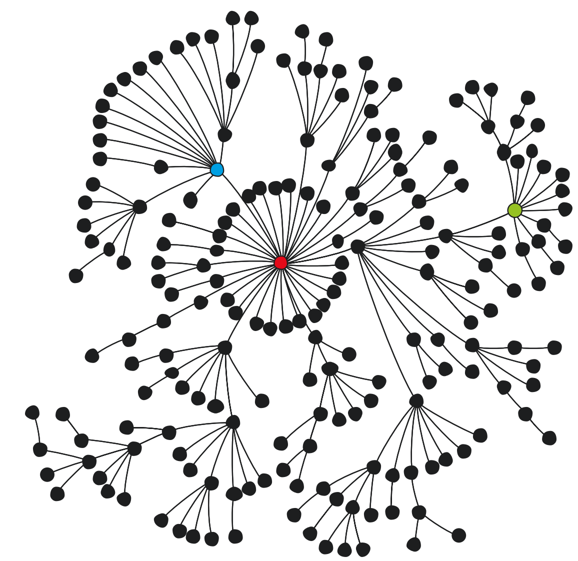


Scale-free networks

SCIENCE VOL 286 15 OCTOBER 1999

Emergence of Scaling in Random Networks

Albert-László Barabási* and Réka Albert



power-law distributed connectivity

$$P(z) \sim z^{-\alpha}$$

$$\bar{l} \propto \ln \ln N$$

<u>ultra-small worlds</u>

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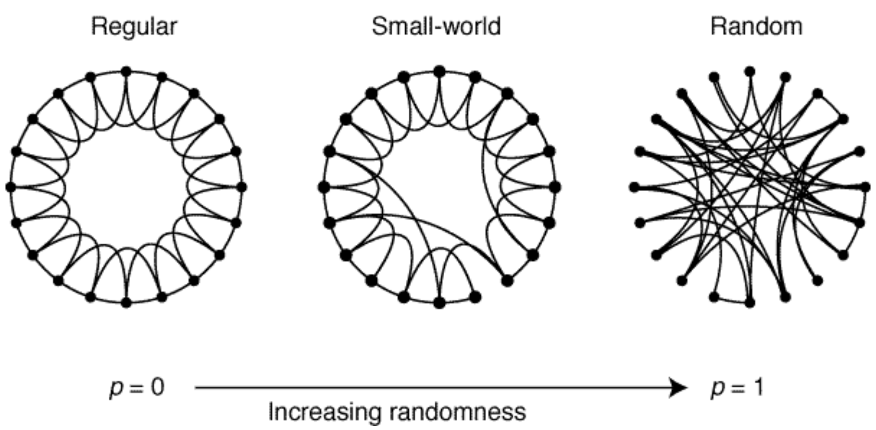

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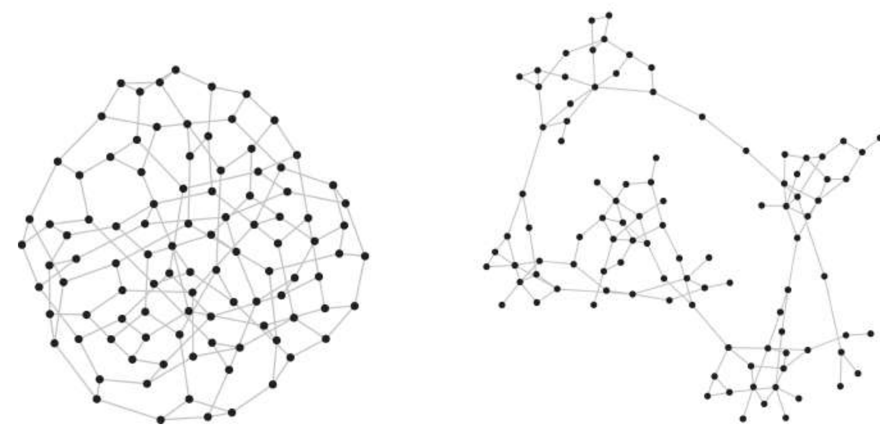

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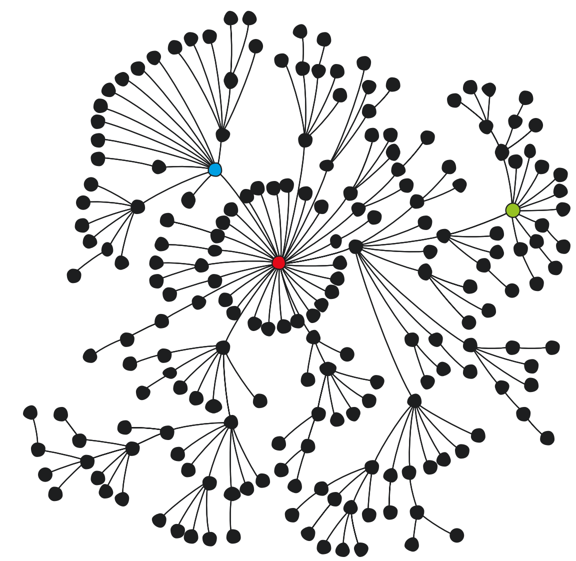

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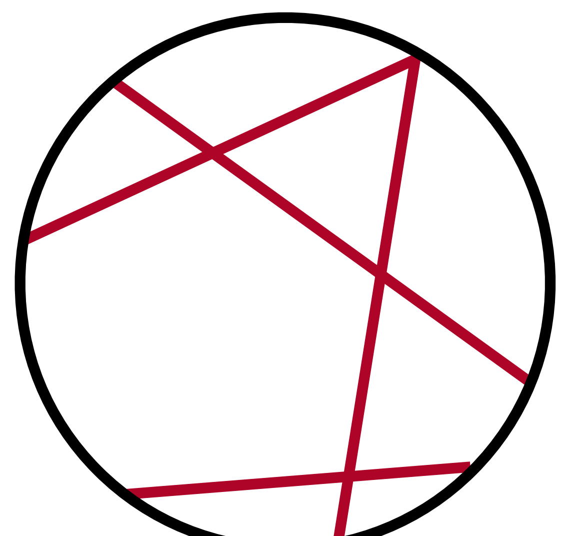
The undiluted Watts-Strogatz model

- R. Monasson EPJ B (1999)
- M. E. J. Newman & D. J. Watts, Phys. Lett. A (1999)



(branching probability p)

further motivations

further motivations

Relevant for Anderson localization in random graphs and MBL physics

- De Luca, Altshuler, Kravtsov, Scardicchio PRL 2014
- Tikhonov, Mirlin PRB 2016
- Garcia-Mata *et al*. PRL 2017
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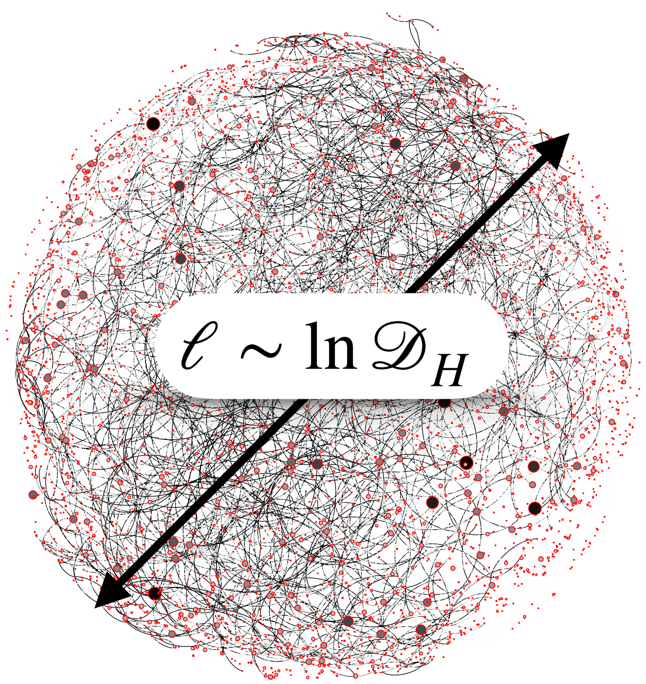
- Tikhonov, Mirlin PRB 2019
- Mace, Alet, NL, PRL 2019

The many-body Hilbert space looks like a random graph



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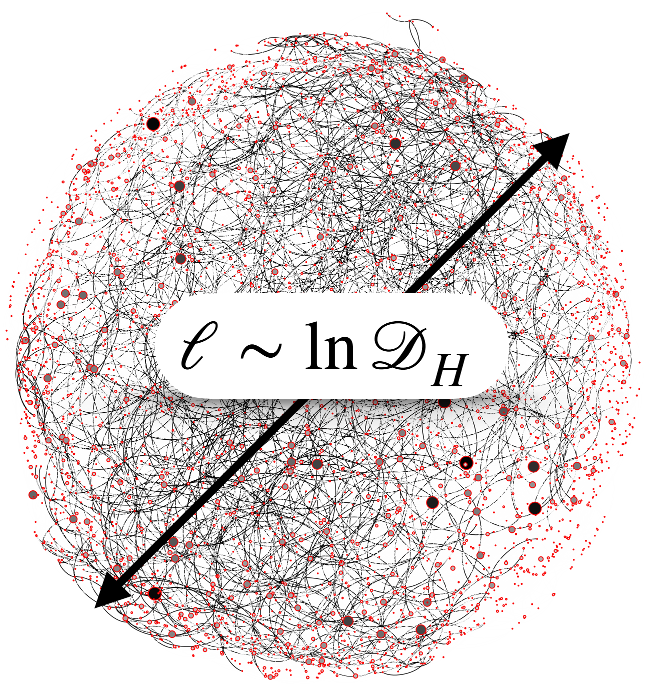


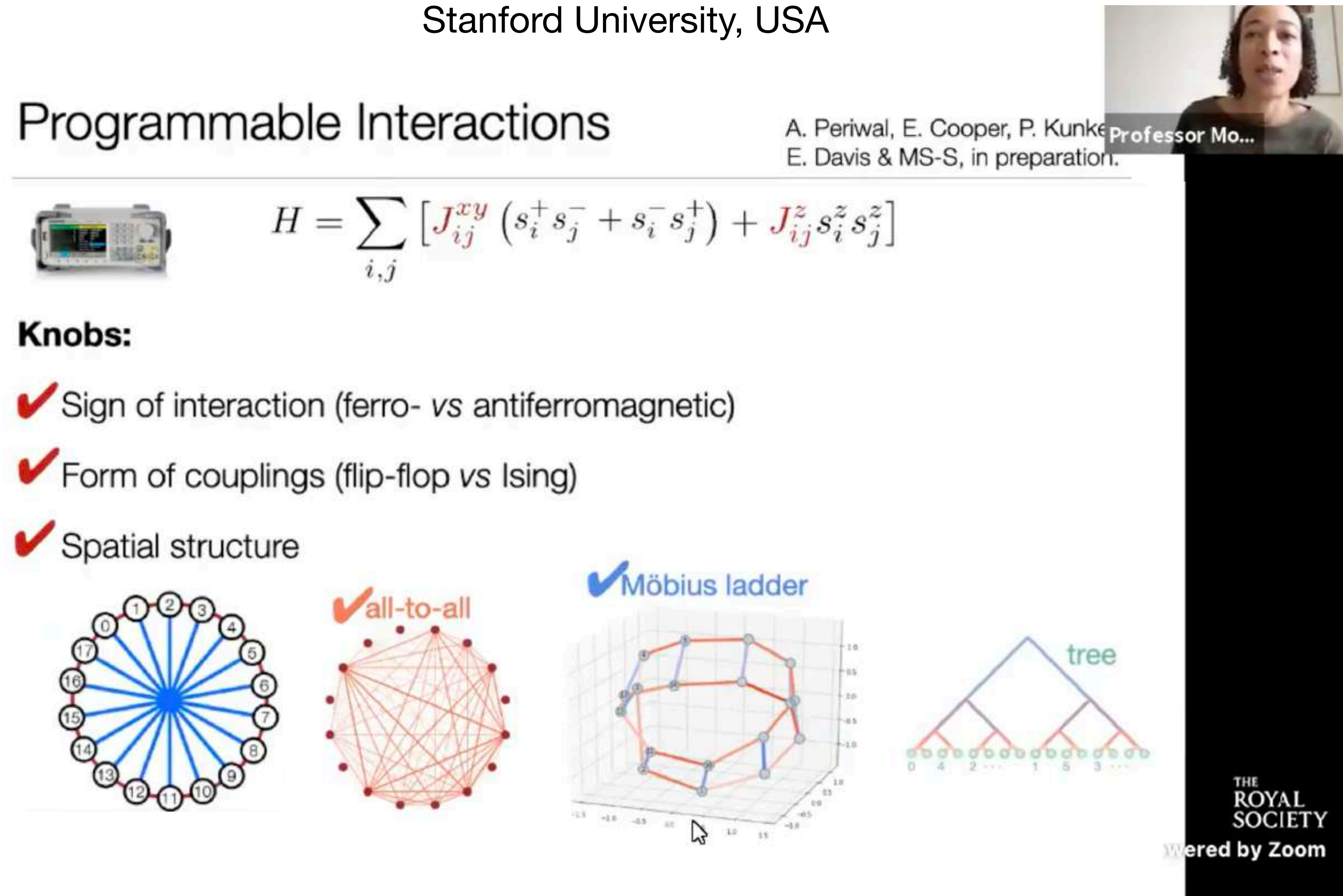
Cold atom setup coupled to an optical cavity

- G. Bentsen, I.-D. Potirniche, V. B. Bulchandani, T. Scaffidi,X. Cao, X.-L. Qi, M. Schleier-Smith, and E. Altman, PRX (2019)
- G. Bentsen, T. Hashizume, A. S. Buyskikh, E. J. Davis, A. J. Daley, S. S. Gubser, and M. Schleier-Smith, PRL (2019)
- E. J. Davis, A. Periwal, E. S. Cooper, G. Bentsen, S. J. Evered, K. Van Kirk, and M. H. Schleier-Smith, PRL (2020)

Optically Programmable Interactions for Quantum Simulation

- Scientific meeting: New perspectives on quantum many-body chaos (8-11 Feb 2021)

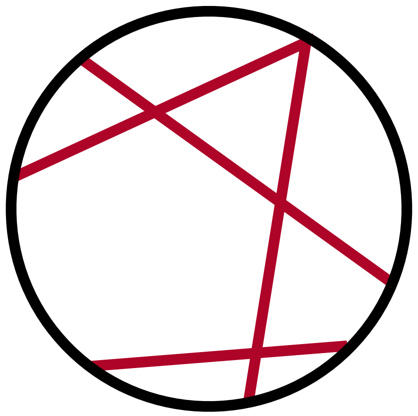
Monika Schleier-Smith
Stanford University, USA



Classical magnetism on SW networks

FM Ising model

$$\mathscr{H} = \underbrace{-J\sum_i S_i^z S_{i+1}^z}_{\text{1D}} \underbrace{-J\sum_{i,j}\epsilon_{ij} S_i^z S_j^z}_{\text{long-range}}$$



$$\epsilon_{ij} = \begin{cases} 1 & (p) \\ 0 & (1-p) \end{cases}$$

- A.Barrat & M. Weigt EPJB (2000)
- M. Gitterman J. Phys. A (2000)

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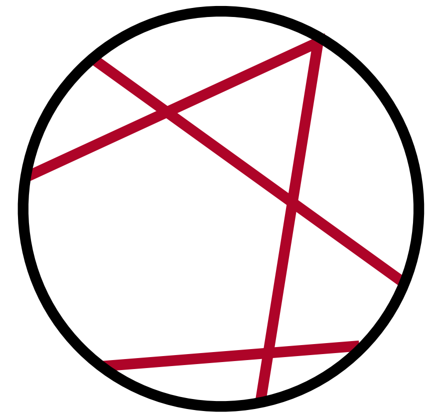

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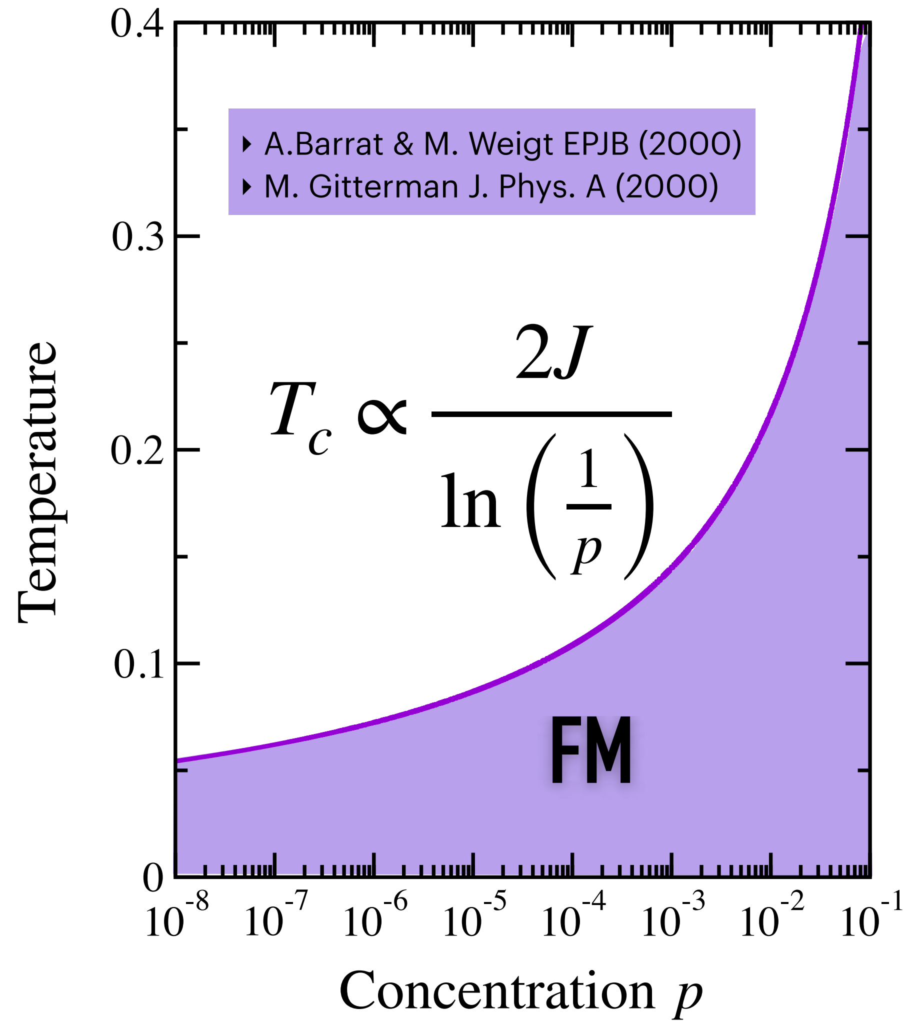
Finite temperature transition



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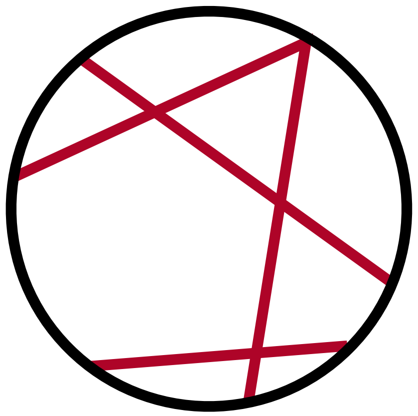

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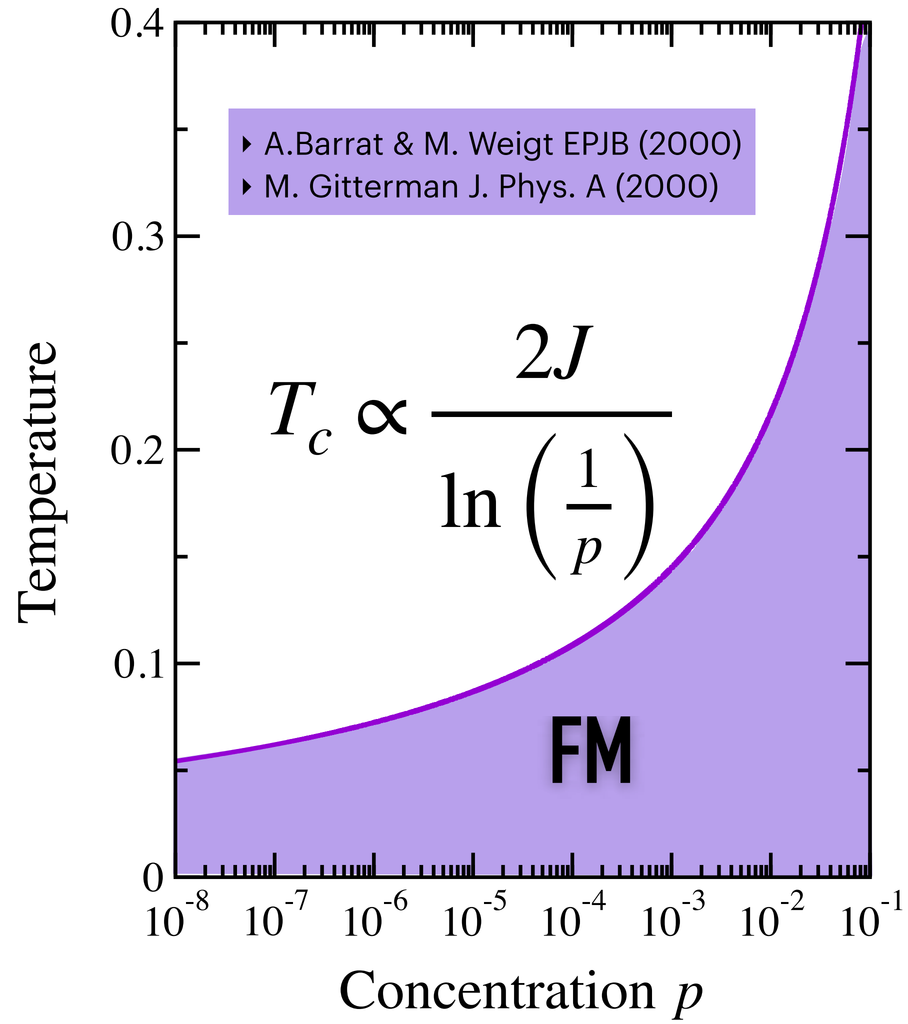


Simple mean-field (MF) argument

$\xi(T) \sim \exp\left(\frac{2J}{T}\right)$ = 1D correlation length

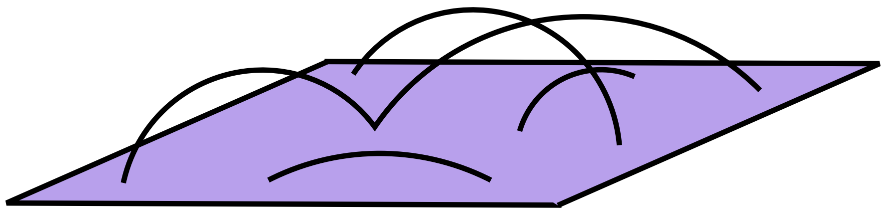
$\zeta_p \sim 1/p$ = mean distance between shortcuts

we expect a transition when

$$\xi(T) \approx \zeta_p \;\Rightarrow\; T_c \propto \frac{2J}{\ln\left(\frac{1}{p}\right)}$$

$T_c \to 0$ very slowly, only when $p \to 0$

Beyond 1D

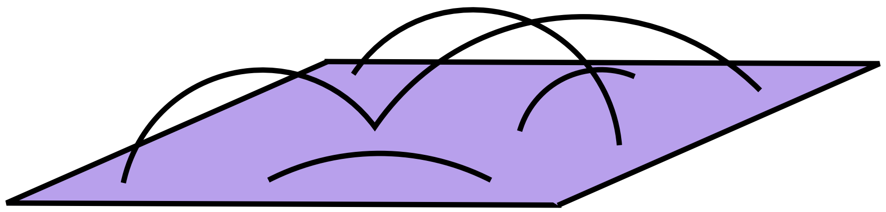


$\xi(T) \sim \left| T - T_c(0) \right|^{-\nu}$ = bare correlation length

$\zeta_p \sim p^{-1/d}$ = mean distance between shortcuts

$$\xi(T) \approx \zeta_p \quad \Rightarrow \quad T_c(p) - T_c(0) \propto J p^{\frac{1}{\nu d}}$$

Beyond 1D



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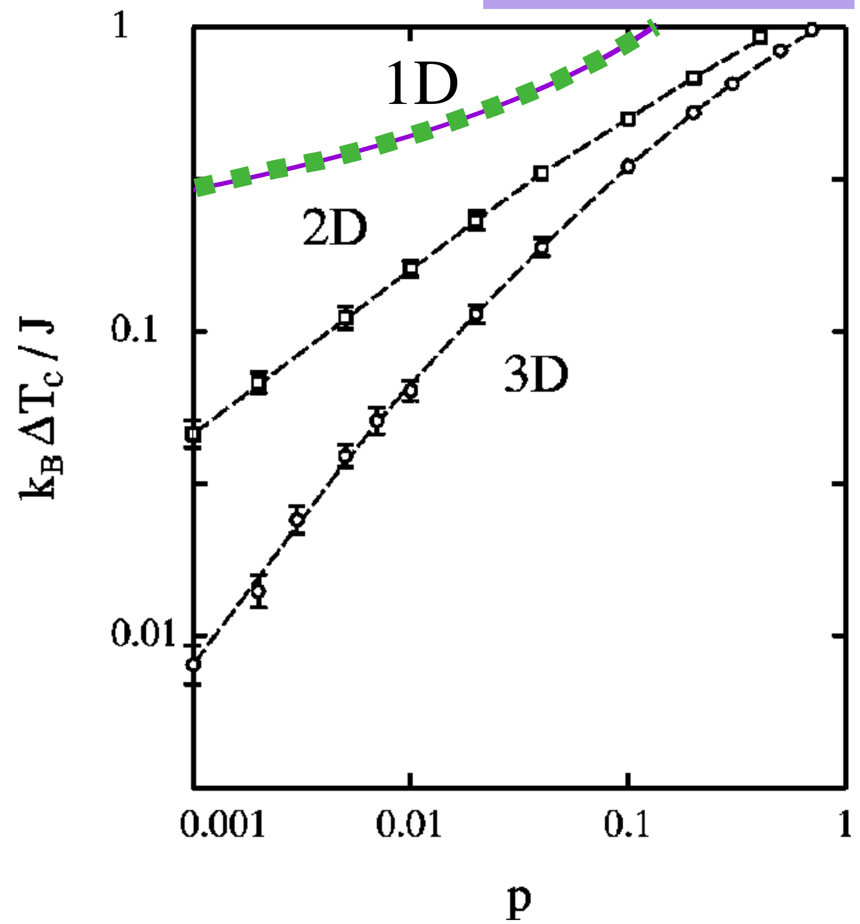
Classical Monte Carlo Ising model

▸ Herrero PRB (2002)



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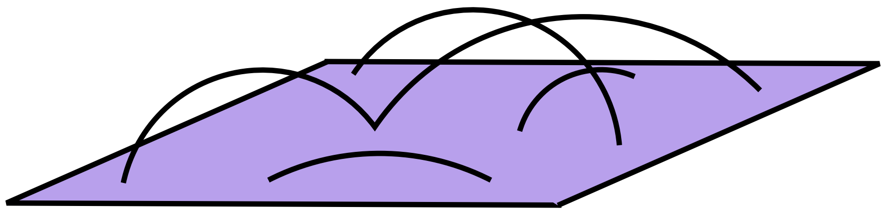

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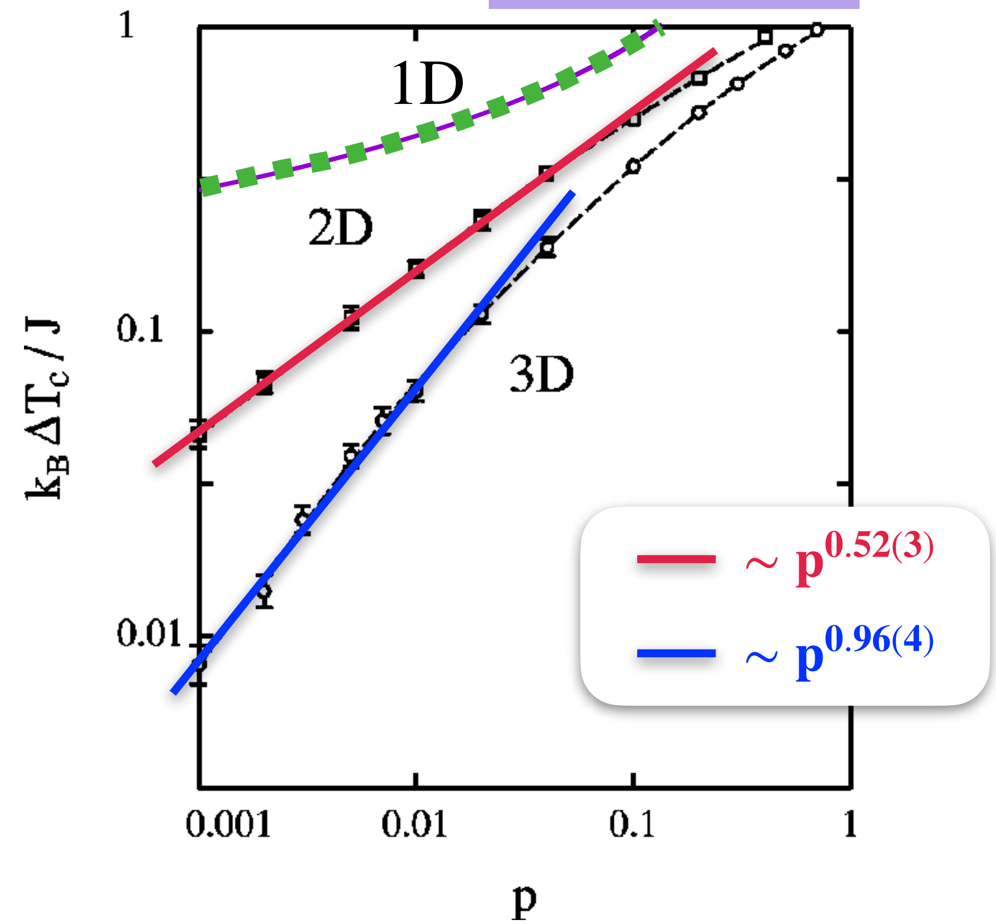

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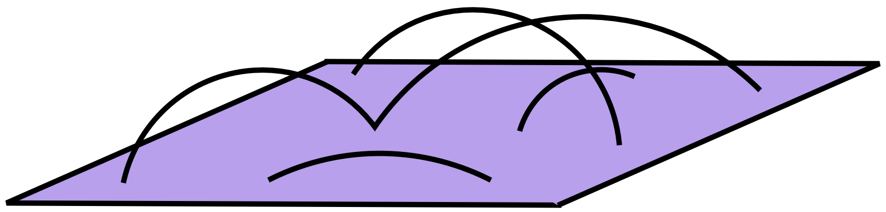
$d=1$	$\exp\left(\frac{2J}{T}\right)$	0	✓
$d=2$	$\left(\frac{1}{T-T_c}\right)^{\nu=1}$	0.5	✓
$d=3$	$\left(\frac{1}{T-T_c}\right)^{\nu\approx 0.63}$	0.53	✗

Beyond 1D



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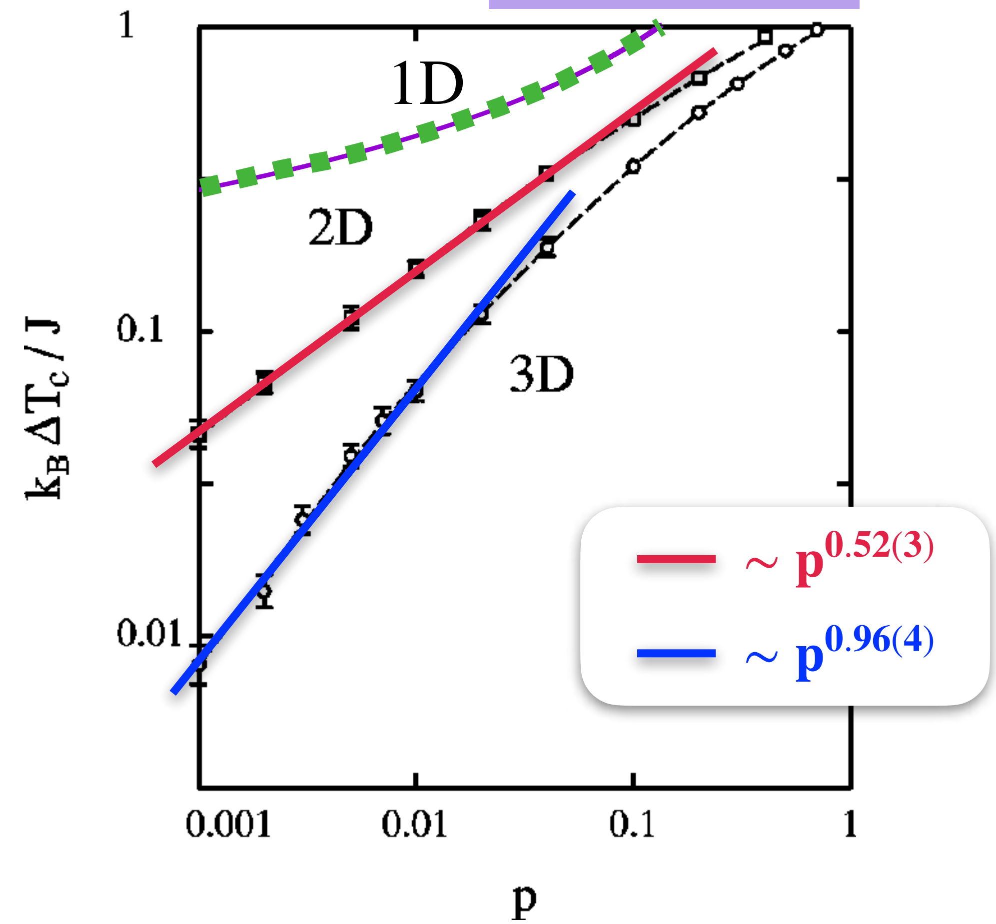
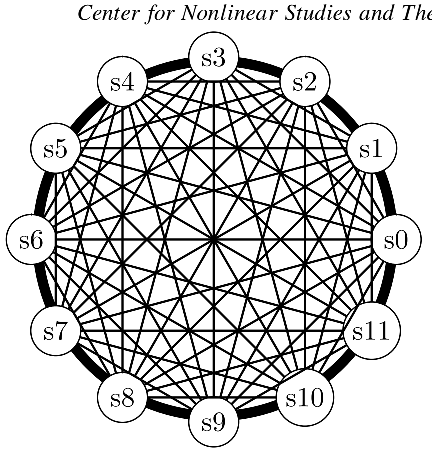


VOLUME 91, NUMBER 9 PHYSICAL REVIEW LETTERS week ending 29 AUGUST 2003

Mean-Field and Anomalous Behavior on a Small-World Network

M. B. Hastings*
Center for Nonlinear Studies and Theoretical Division, Los Alamos National Laboratory, Los Alamos, New Mexico 87545, USA
(Received 28 April 2003; published 27 August 2003)



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$$\Rightarrow \quad T_c(p) - T_c(0) \propto Jp^{\frac{1}{\gamma}}$$ (γ = susceptibility exponent)

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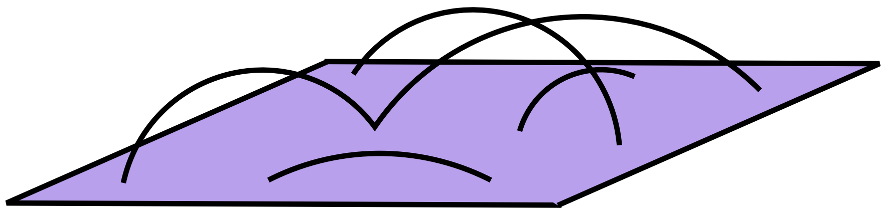

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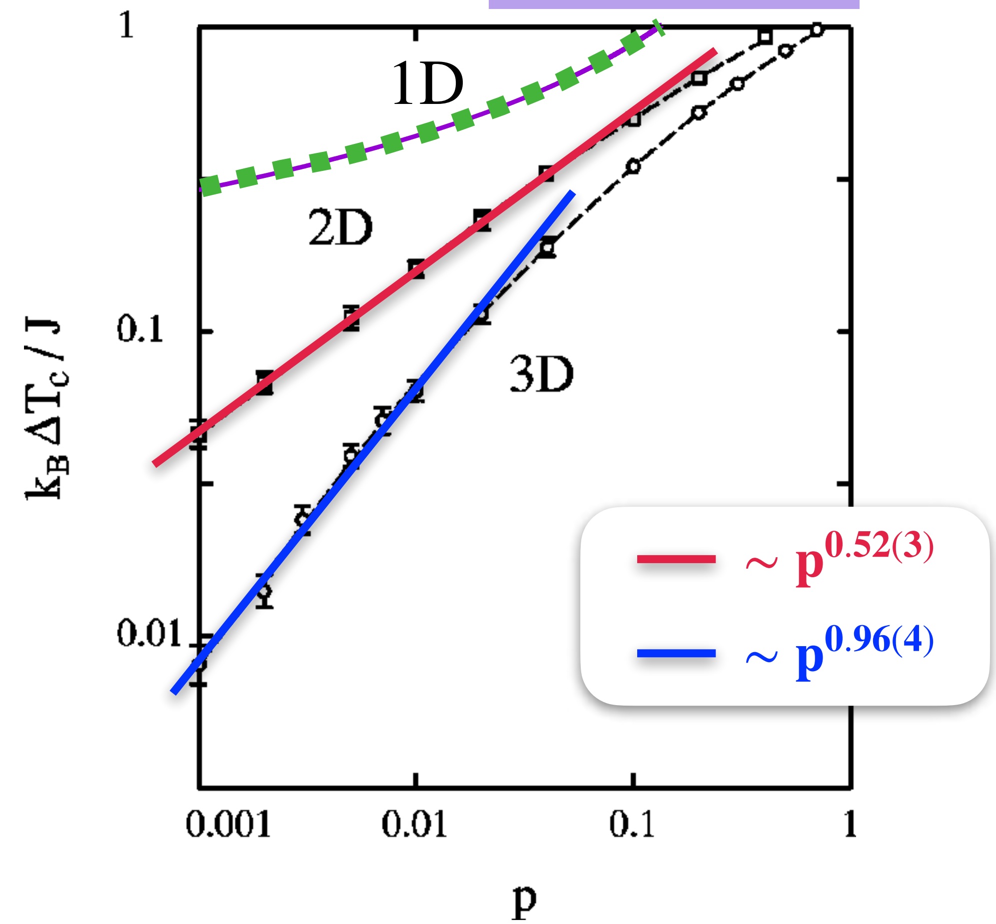

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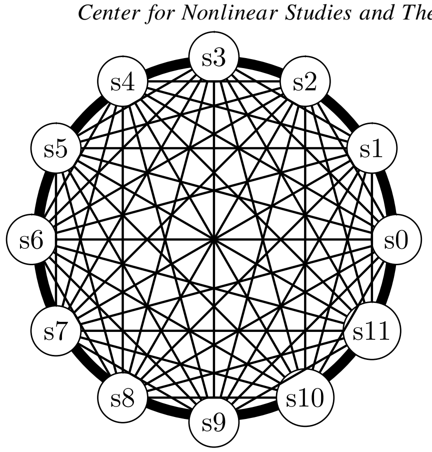
$d=2$	$\left(\frac{1}{T-T_c}\right)^{\nu=1}$	0.5	✓	$\left(\frac{1}{T-T_c}\right)^{\gamma=7/4}$	0.57	✓
$d=3$	$\left(\frac{1}{T-T_c}\right)^{\nu\approx 0.63}$	0.53	✗	$\left(\frac{1}{T-T_c}\right)^{\gamma\approx 1.24}$	0.81	✓



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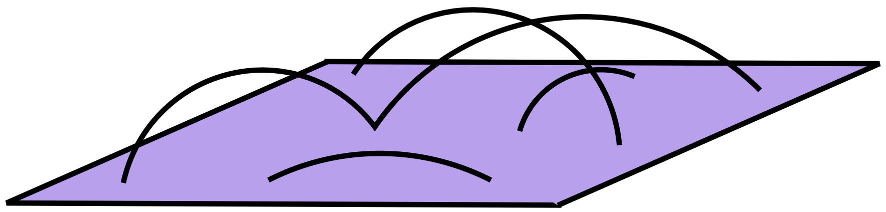

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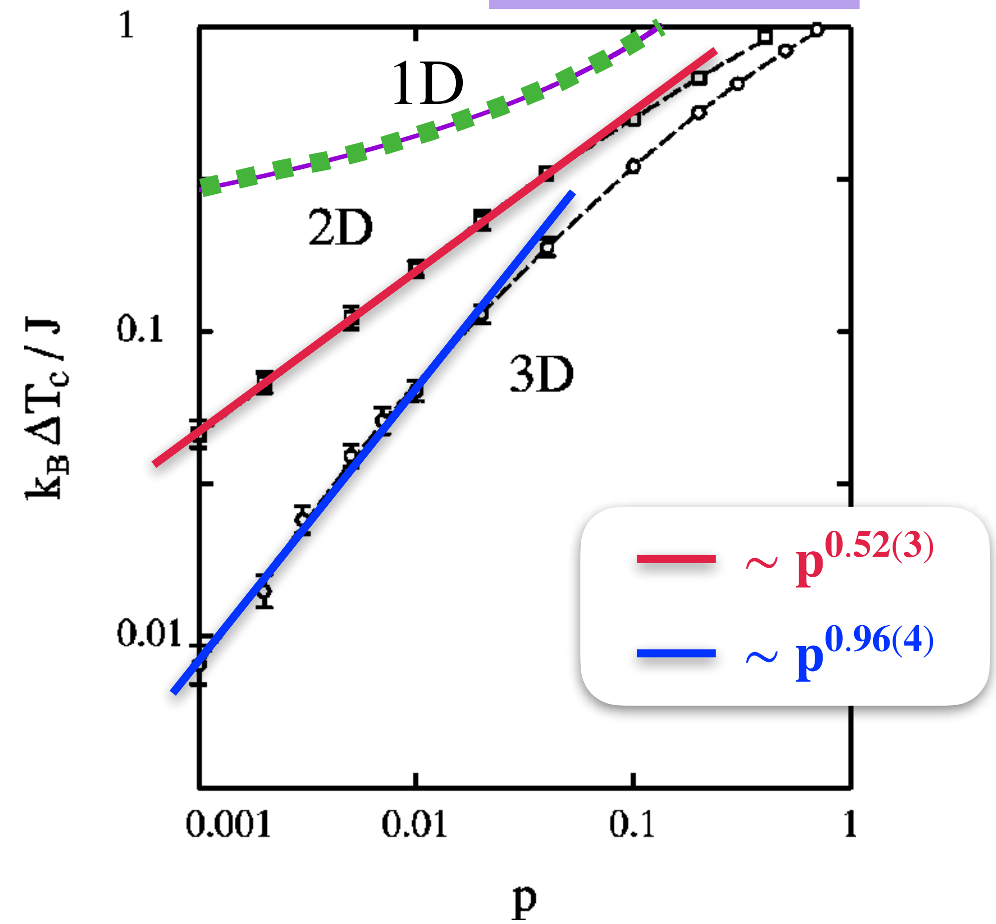

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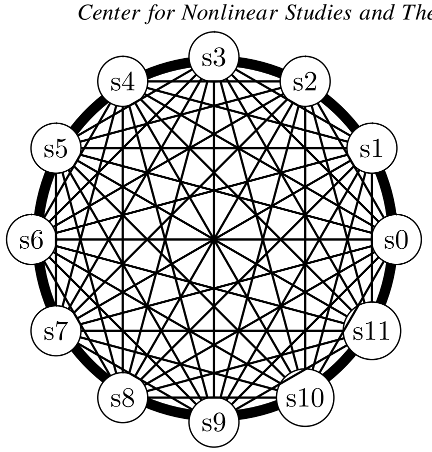

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$d=1$	$\exp\left(\frac{2J}{T}\right)$	0	✓	$\frac{1}{T}\exp\left(\frac{2J}{T}\right)$	0	✓
$d=2$	$\left(\frac{1}{T-T_c}\right)^{\nu=1}$	0.5	✓	$\left(\frac{1}{T-T_c}\right)^{\gamma=7/4}$	0.57	✓
$d=3$	$\left(\frac{1}{T-T_c}\right)^{\nu\approx 0.63}$	0.53	✗	$\left(\frac{1}{T-T_c}\right)^{\gamma\approx 1.24}$	0.81	✓

We want to compare $\dfrac{1}{\nu d}$ with $\dfrac{1}{\gamma = \nu(2-\eta)}$

$$\frac{1}{\nu d} < \frac{1}{\gamma} \quad \Leftrightarrow \quad \eta + d - 2 > 0$$

Universality class	η
ISING $d=3$	0.036
XY $d=3$	0.038
HEISENBERG $d=3$	0.035

Is $\eta + d - 2 > 0$ always true ?

Critical correlations at Tc:

$$\langle S^z(0) S^z(r) \rangle_{T_c} \sim \frac{1}{r^{\eta+d-2}} > 0 \quad \checkmark$$

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Critical correlations at Tc:

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Quantum criticality (T=0):

$$\langle S^\alpha(0) S^\alpha(r) \rangle_{\mathrm{GS}} \sim \frac{1}{r^{\eta+d+z-2}}$$

$$\eta + d + z - 2 > 0$$

$$\not\Downarrow$$

$$\eta + d - 2 > 0$$

Is $\eta + d - 2 > 0$ always true ?

Critical correlations at Tc:

$$\langle S^z(0) S^z(r) \rangle_{T_c} \sim \frac{1}{r^{\eta+d-2}} > 0 \checkmark$$

Quantum criticality (T=0):

$$\langle S^\alpha(0) S^\alpha(r) \rangle_{\mathrm{GS}} \sim \frac{1}{r^{\eta+d+z-2}}$$

$$\eta + d + z - 2 > 0$$

$$\cancel{\Downarrow}$$

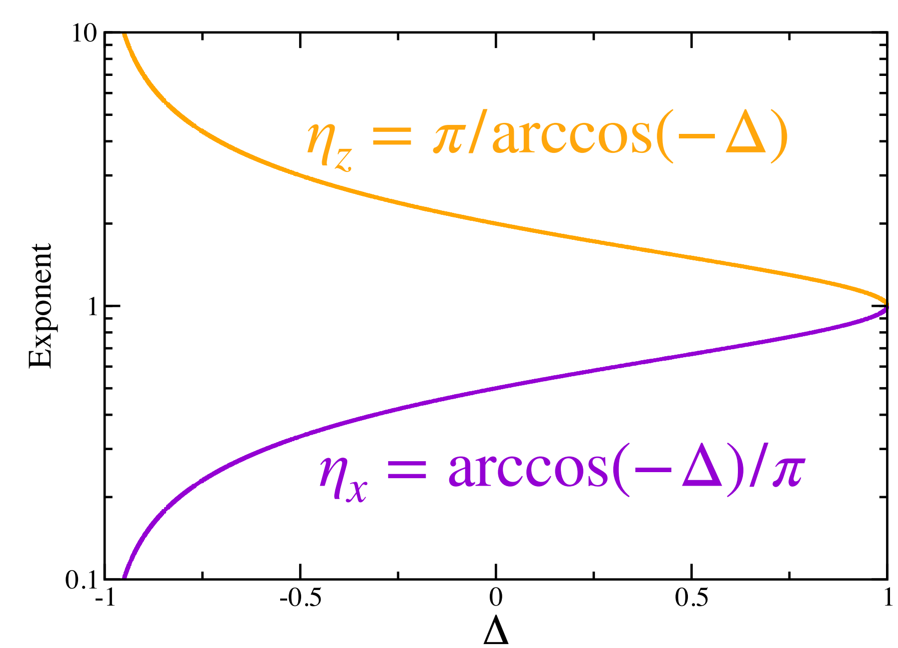
$$\eta + d - 2 > 0$$

Tomonaga – Luttinger liquids ! $\quad (d = z = 1)$

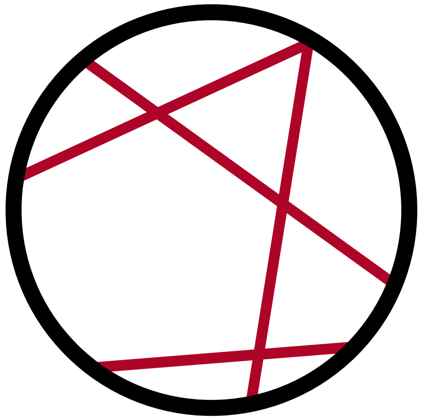
S=1/2 XXZ chains

$$\mathcal{H}_{XXZ} = J \sum_i \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

$$\langle S^\alpha(0) S^\alpha(r) \rangle_{\mathrm{GS}} \sim \frac{1}{r^{\eta_\alpha}} \qquad \eta_x \leq 1 \, !$$



1D small-world with quantum spin-1/2



$$\mathscr{H}_{\mathrm{SW}} = \mathscr{H}_{\mathrm{1D}}^{XXZ} + \mathscr{H}_{\mathrm{LR}}$$

$$\mathscr{H}_{\mathrm{1D}}^{XXZ} = J\sum_{i=1}^{N} h_{i,i+1}^{\Delta}, \quad \text{with } h_{i,i+1}^{\Delta} = S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z$$

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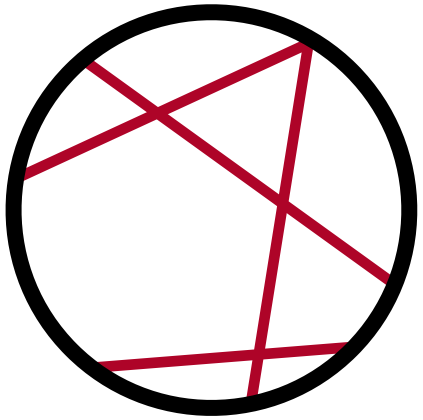

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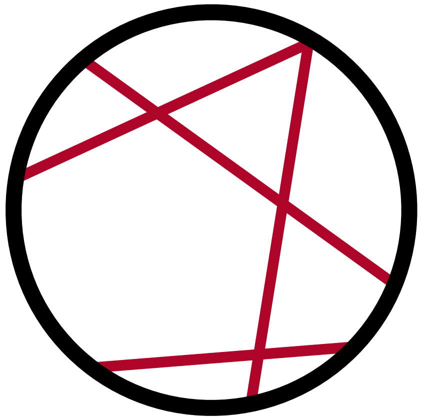

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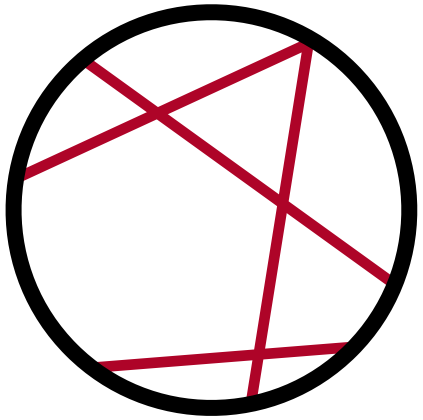

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both models are unfrustrated $\Rightarrow$ quantum Monte Carlo (QMC) simulations are possible

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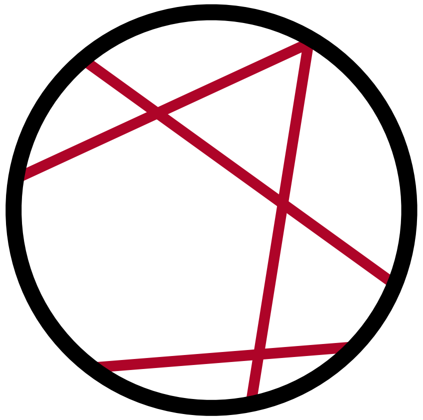

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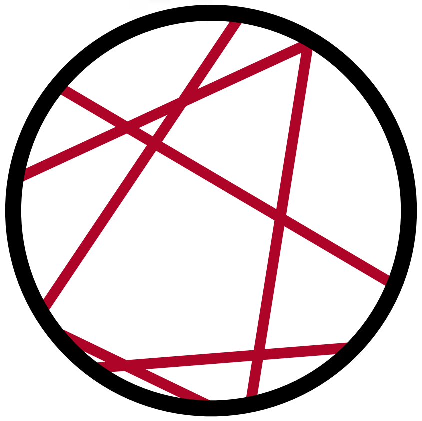
Undiluted SW
(branching probability p)



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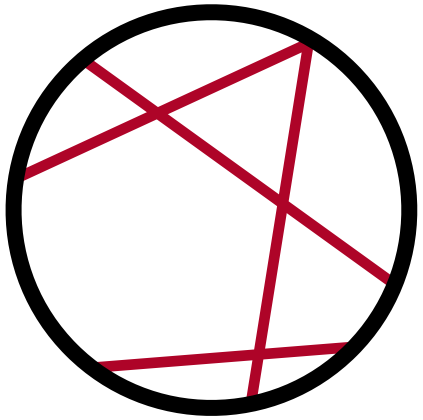

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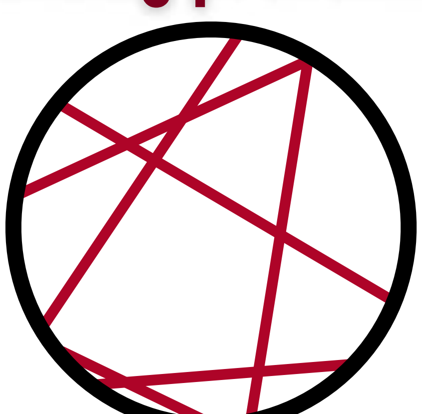
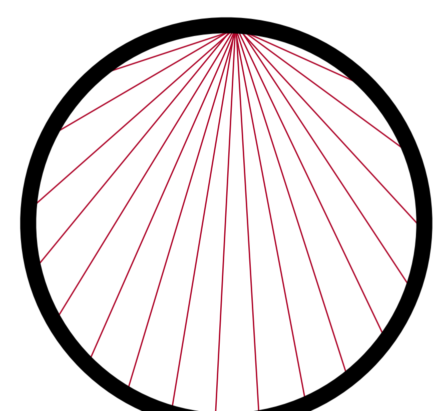


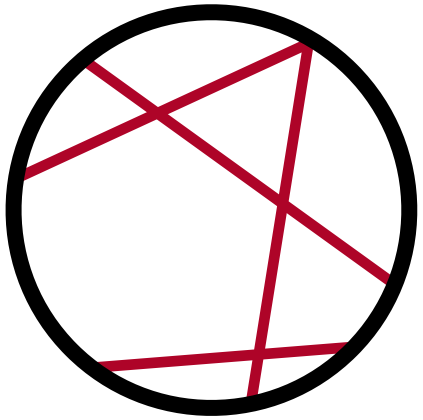
Hastings model

▸ Hastings, PRL 2003



$$J^{\rm LR} \propto J\frac{p}{N}$$

1D small-world with quantum spin-1/2



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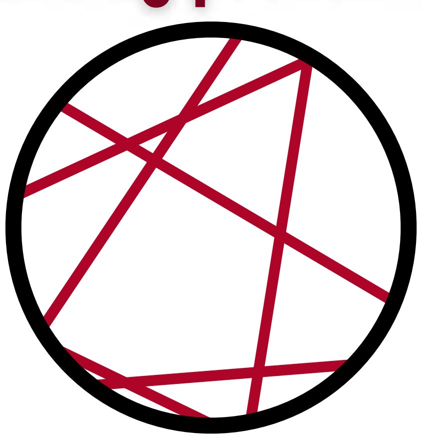

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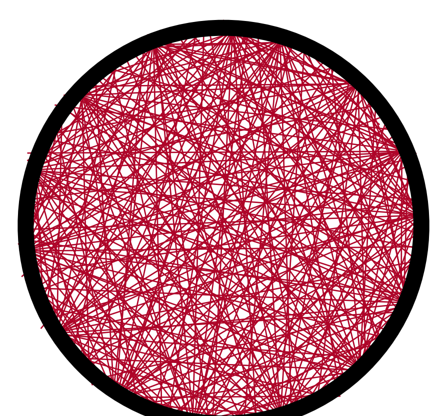
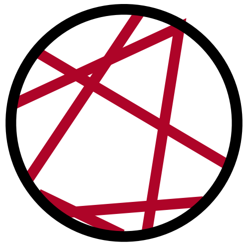
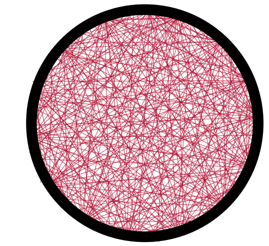


$$J^{\mathrm{LR}} \propto J\,\frac{p}{N}$$



Mean-Field Theories



1d Small-World $p \ll 1$

1d Hastings Small-World

Mean-Field Theories

1d Small-World $p \ll 1$

We repeat the simple MF argument

$\xi(T) \sim \dfrac{uJ}{T}$ Luttinger liquid correlation length

$\zeta_p \sim 1/p$ = mean distance between shortcuts

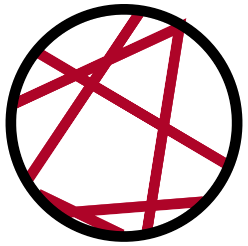
we expect a transition when

$$\xi(T) \approx \zeta_p \Rightarrow T_c^{\mathrm{MF}} \propto uJp$$

agree with

$$T_c(p) - T_c(0) \propto Jp^{\frac{1}{\nu d}} \quad \text{with } \nu = d = 1$$

1d Hastings Small-World



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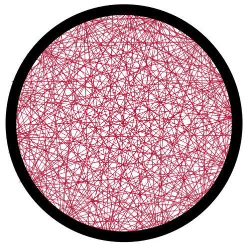

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Random Phase approximation

- D. J. Scalapino, Y. Imry, and P. Pincus, PRB (1975).
- H. J. Schulz, PRL (1996).

$$\mathcal{H}_{\mathrm{eff}}^{\mathrm{XXX}} = J\sum_i \boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + \sum_{i,j} J_{ij}^{\mathrm{LR}} \langle S_j^z \rangle S_i^z$$

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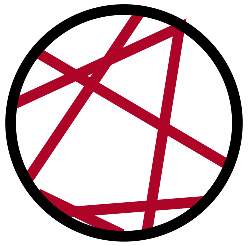
treat the LR part in MF



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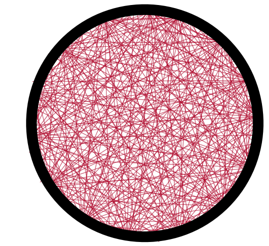

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treat the LR part in MF

order parameter

$$\langle m \rangle = \chi^{1\mathrm{D}} \left(h_{\mathrm{ext}}^{sb} + \overline{J^{\mathrm{LR}}} \langle m \rangle \right) = \frac{\chi^{1\mathrm{D}}}{1 - \overline{J^{\mathrm{LR}}}\chi^{1\mathrm{D}}} h_{\mathrm{ext}}^{sb} = \chi^{\mathrm{RPA}} h_{\mathrm{ext}}^{sb}$$

ordering transition when

$$\chi^{\mathrm{RPA}}(T_c^{\mathrm{RPA}}) \to \infty \qquad T_c^{\mathrm{RPA}} = \left[\chi^{1\mathrm{D}}\right]^{-1} \left(\frac{1}{\overline{J^{\mathrm{LR}}}} \right)$$

Mean-Field Theories

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Divergence of the 1D susceptibility $$J\chi^{1\mathrm{D}}(T) \propto (T/J)^{-\gamma}$$

$$T_{\mathrm{c}}^{\mathrm{RPA}} \propto J \left(\frac{\overline{J^{\mathrm{LR}}}}{J} \right)^{\frac{1}{\gamma}} \propto Jp^{\frac{1}{\gamma}}$$

Hastings result recovered

Mean-Field Theories

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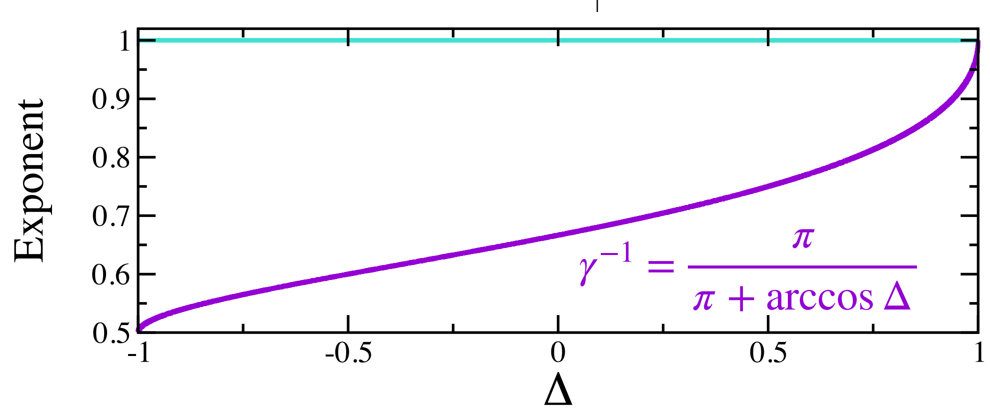
Divergence of the 1D susceptibility $\quad J\chi^{1\mathrm{D}}(T) \propto (T/J)^{-\gamma}$



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Hastings result recovered

1D quantum Hastings: test for the RPA

S=1/2

XXZ chain

$$\mathscr{H}_{XXZ} = J \sum_{i=1}^{N} \left(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z \right)$$

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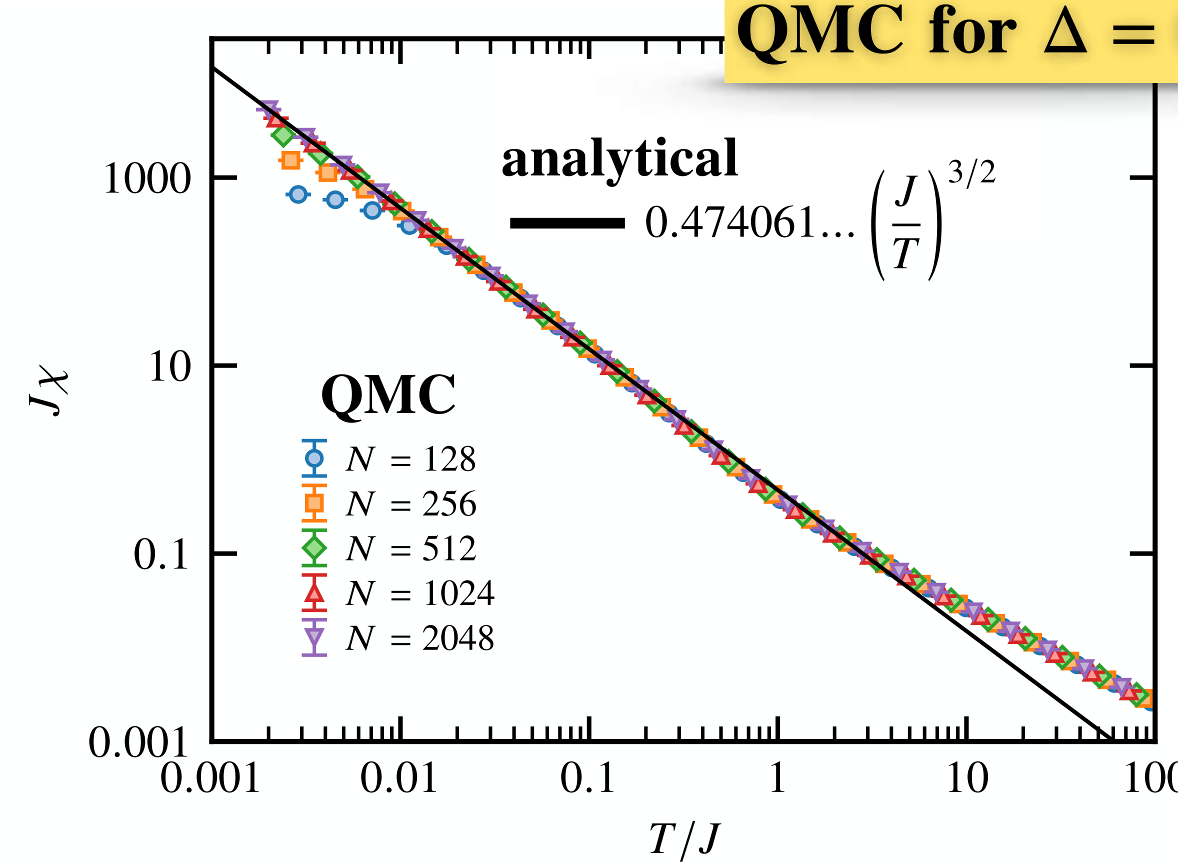
Luttinger liquid theory
Transverse 1D susceptibility

$$J\chi_\perp^{1\mathrm{D}}(T) = f_\Delta \left(\frac{J}{T}\right)^{\frac{1+\frac{\arccos\Delta}{\pi}}{\gamma}}$$

Exact (parameter-free) expression for $|\Delta| < 1$

QMC for $\Delta = 0$



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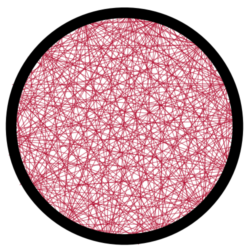
Hastings model



$$J^{\mathrm{LR}} = J\,\frac{p}{N}$$

Random Phase approximation

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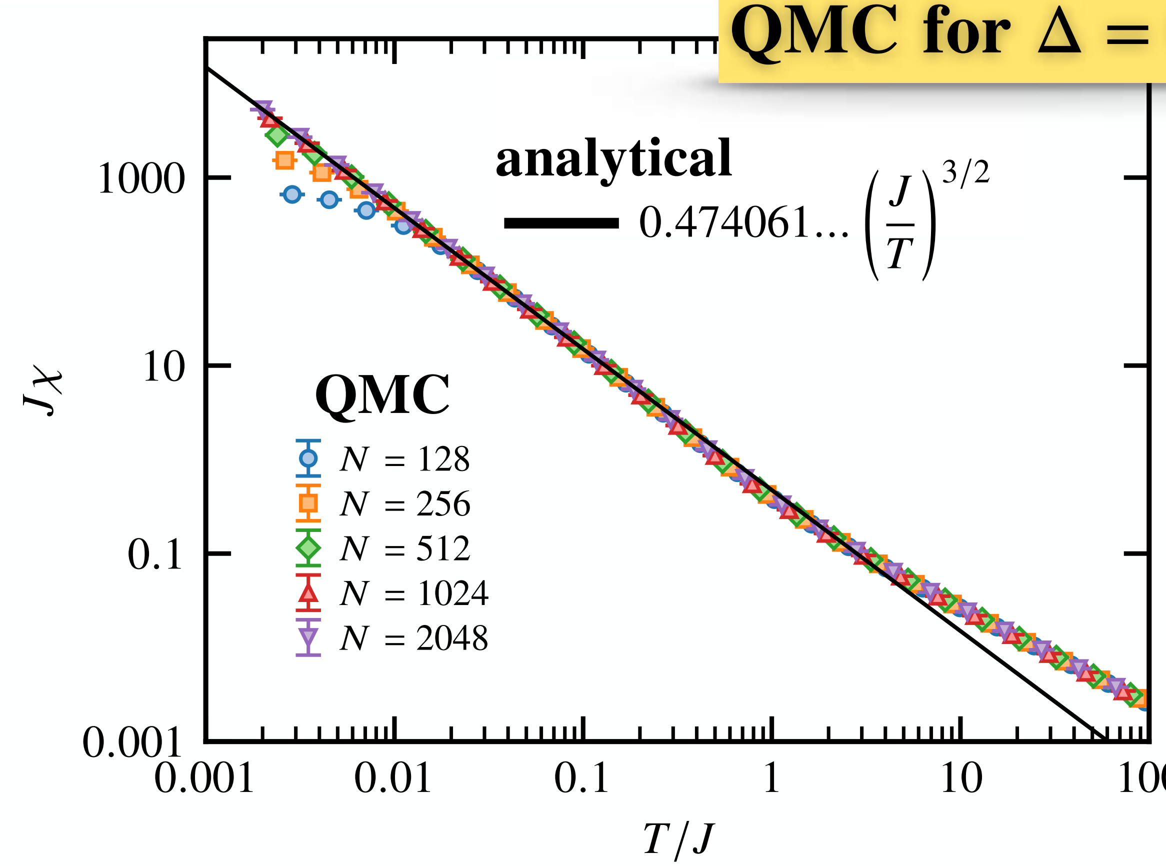

1D quantum Hastings: test for the RPA

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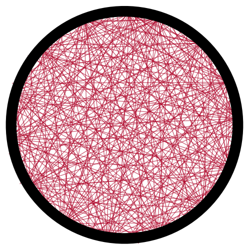

$J^{\mathrm{LR}} = J\,\frac{p}{N}$

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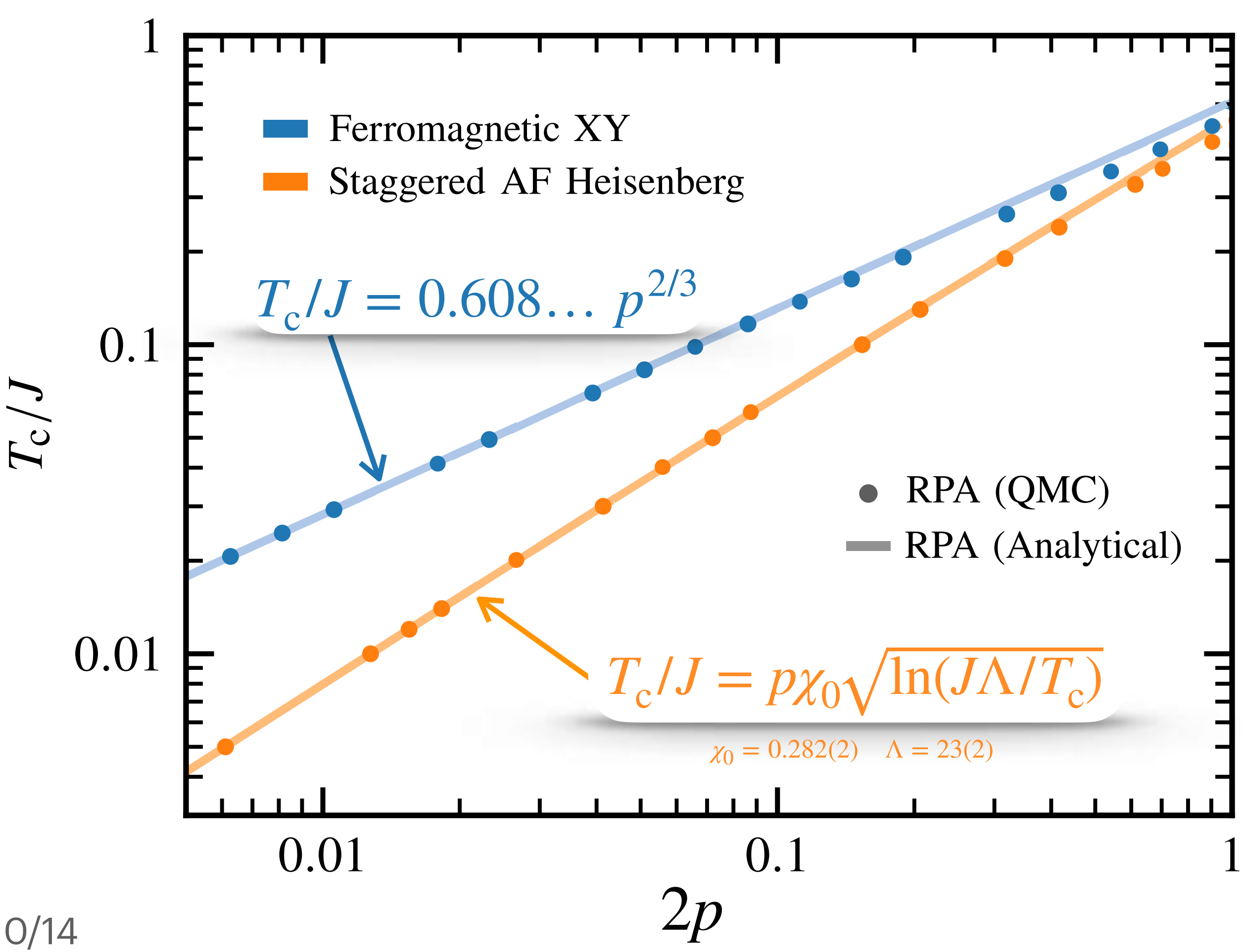

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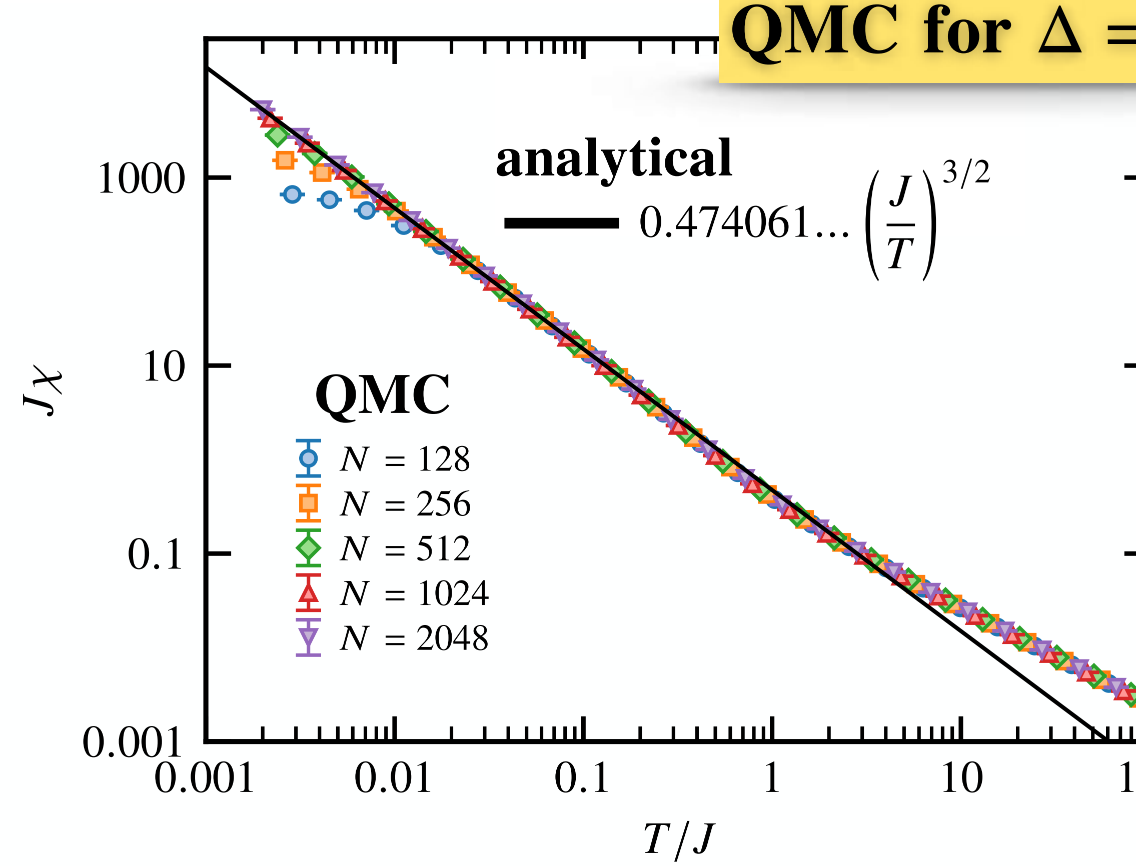
Exact (parameter-free) expression for $|\Delta| < 1$

QMC for $\Delta = 0$

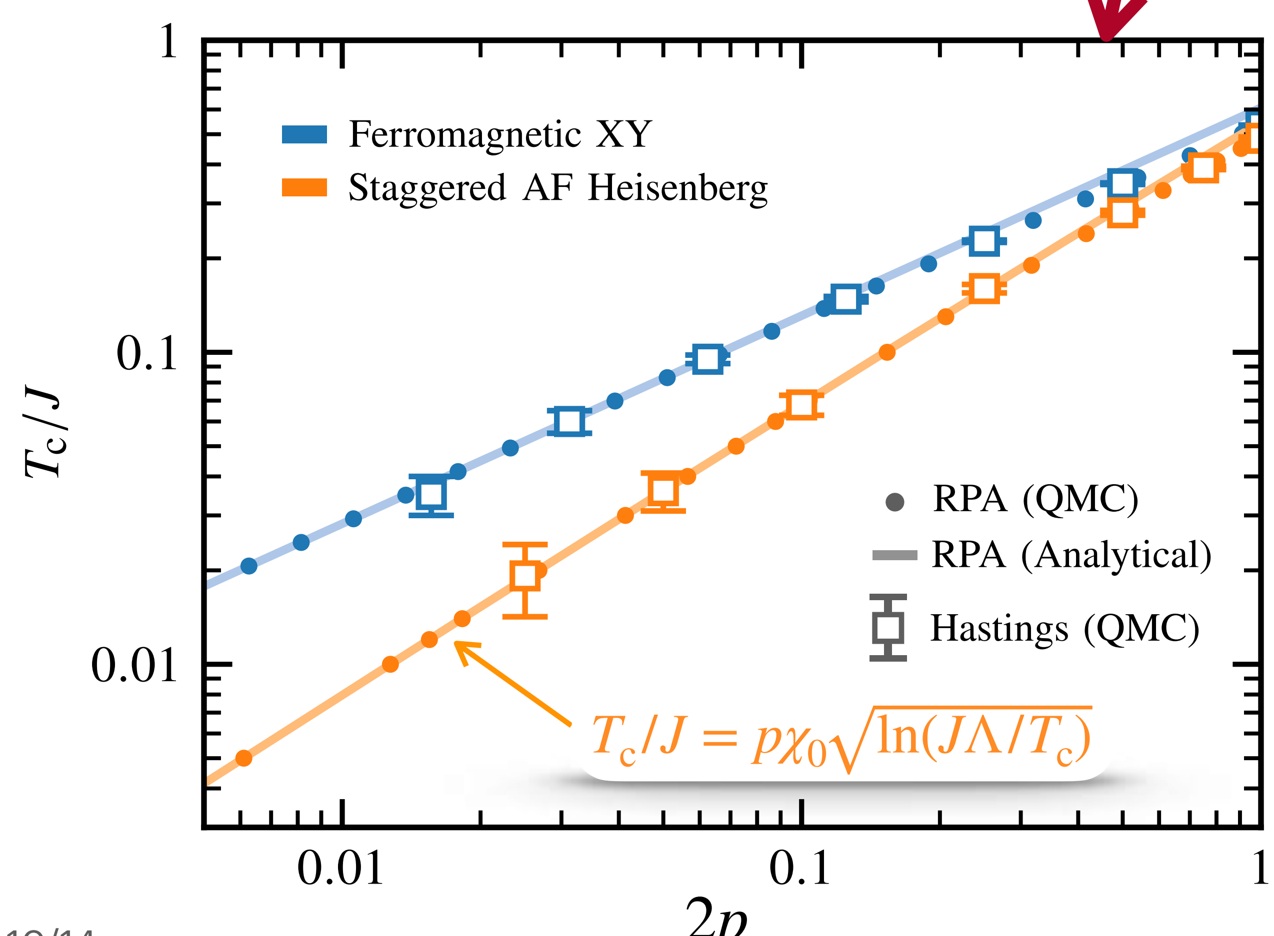


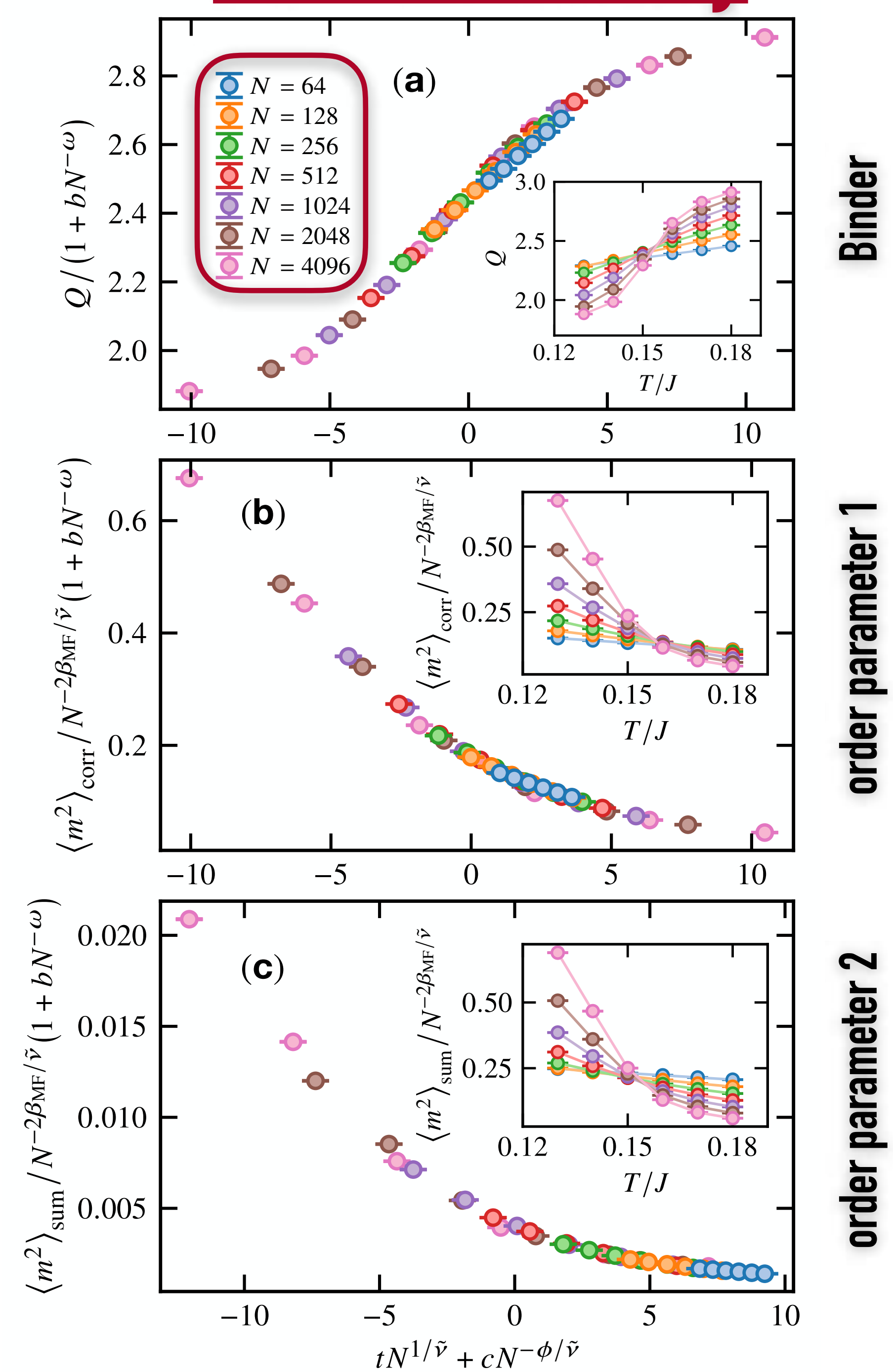
1D quantum Hastings: QMC results

scaling analysis including irrelevant corrections

MF universality

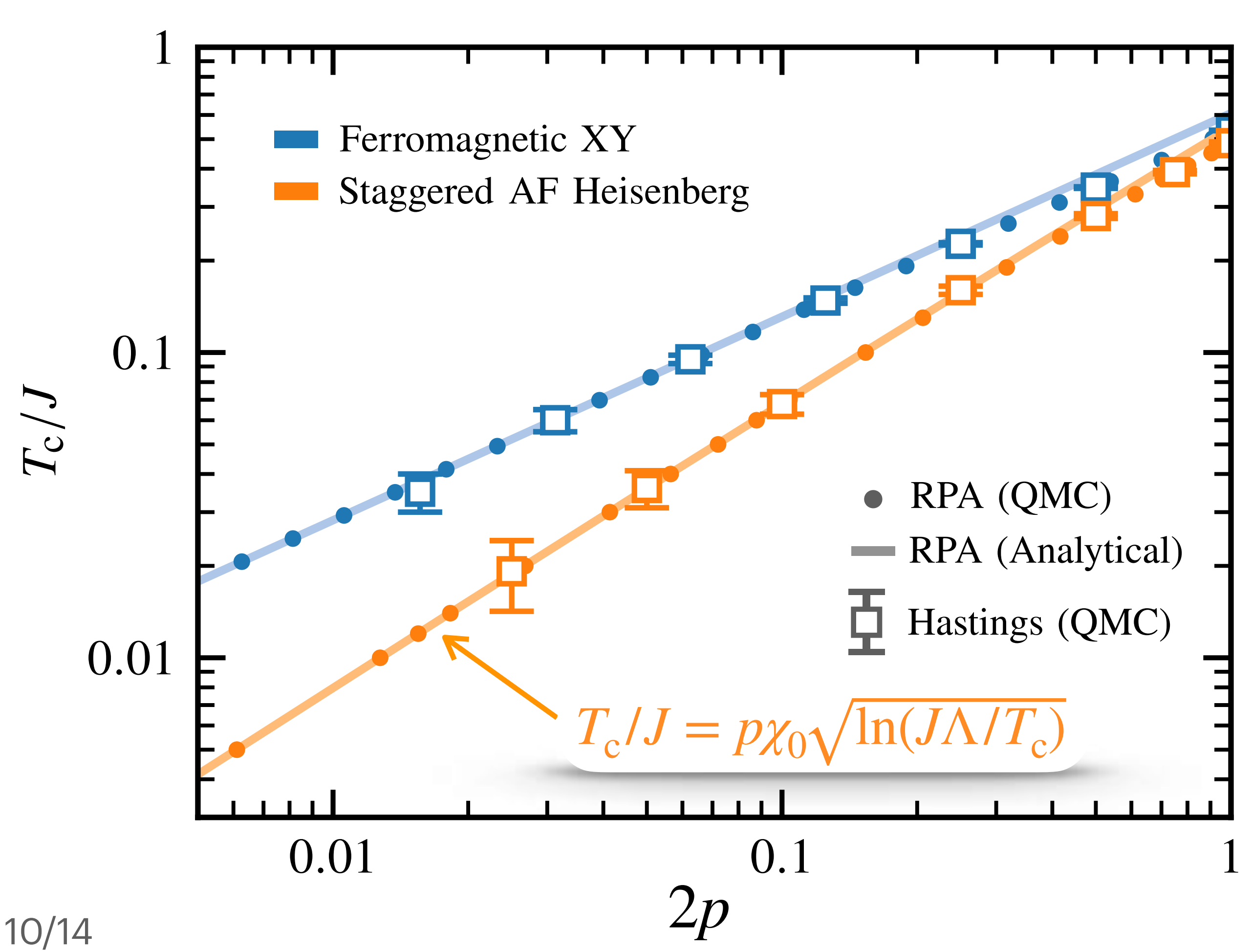
T_c vs. p





1D quantum Hastings: QMC results

- T_c of the Hastings model perfectly agree with RPA estimates
- Universality class is MF
- 1D subtleties (such as log corrections at the SU(2) point) are perfectly well captured by a $d = \infty$ model



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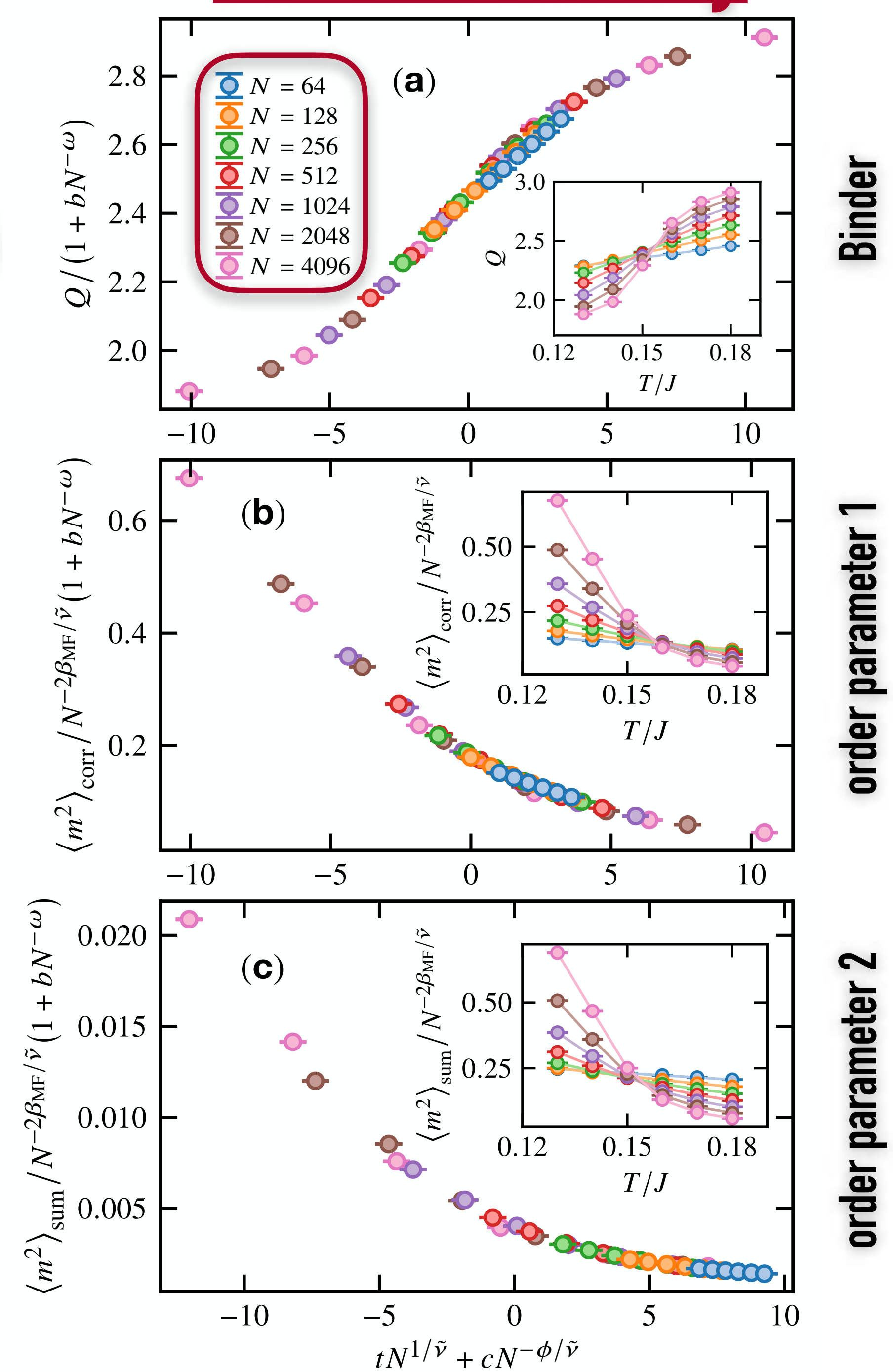
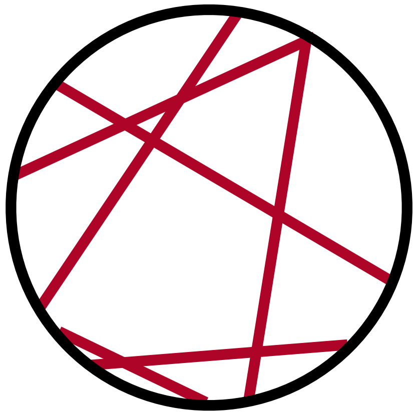


Random branching (i) — Heisenberg AF



$$\mathscr{H} = J\sum_{i=1}^{N} \overrightarrow{S}_i \cdot \overrightarrow{S}_{i+1} - J\sum_{i,j}(-1)^{|i-j|}\epsilon_{ij}\overrightarrow{S}_i \cdot \overrightarrow{S}_j$$

$$\epsilon_{ij} = \begin{cases} 1 & (p) \\ 0 & (1-p) \end{cases}$$

disorder – average ($\mathrm{N_{samples}} \propto 10^{2-3}$)

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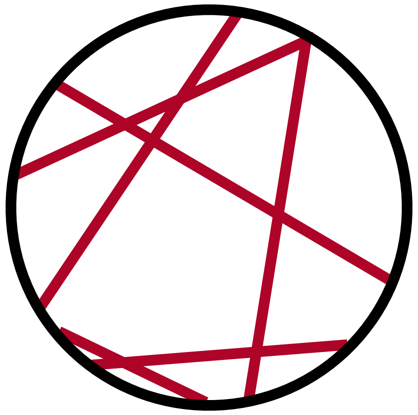

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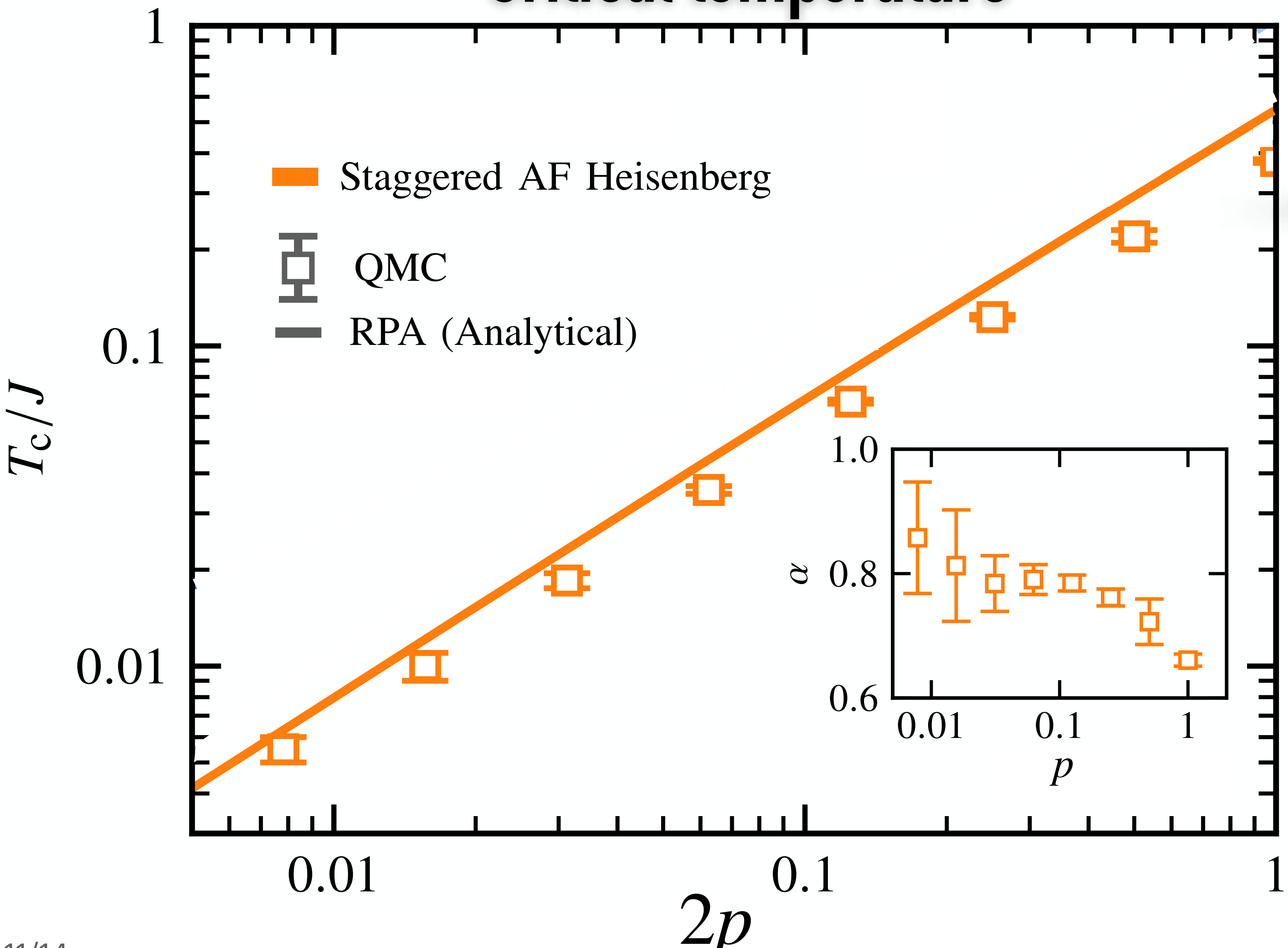
Critical temperature



RPA estimate is correct, but with a renormalized coupling $\overline{J^{\mathrm{LR}}} \to \alpha\overline{J^{\mathrm{LR}}}$

$$T_c^{\mathrm{QMC}} = \left[\chi^{1\mathrm{D}}\right]^{-1}\left(\frac{1}{\alpha\overline{J^{\mathrm{LR}}}}\right)$$

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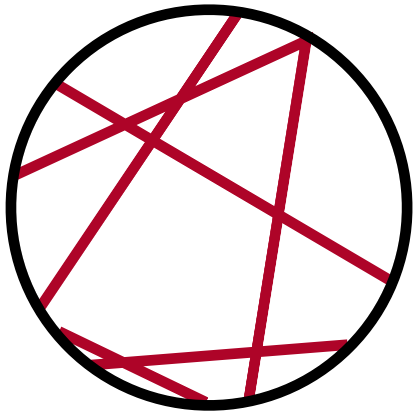

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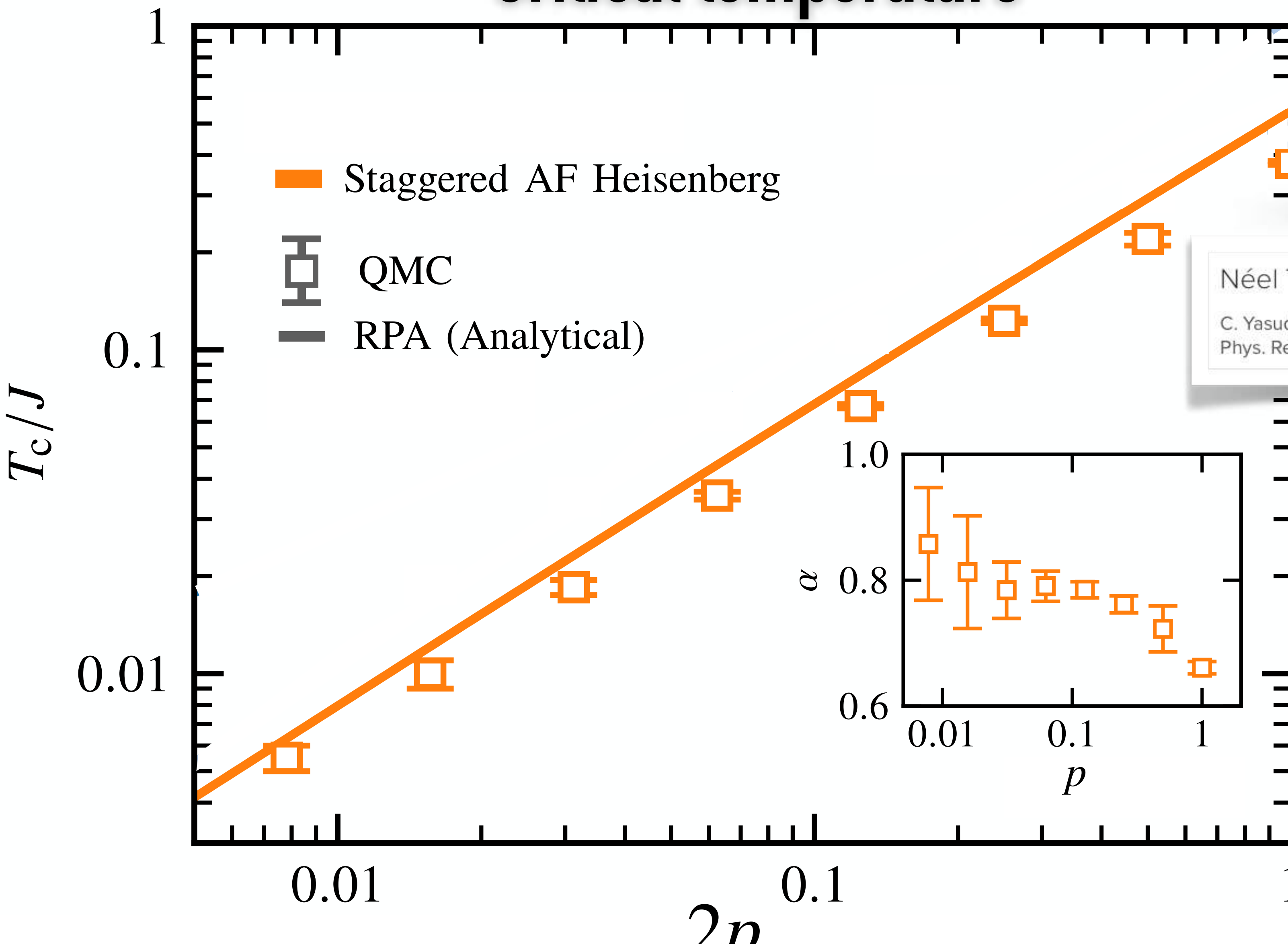
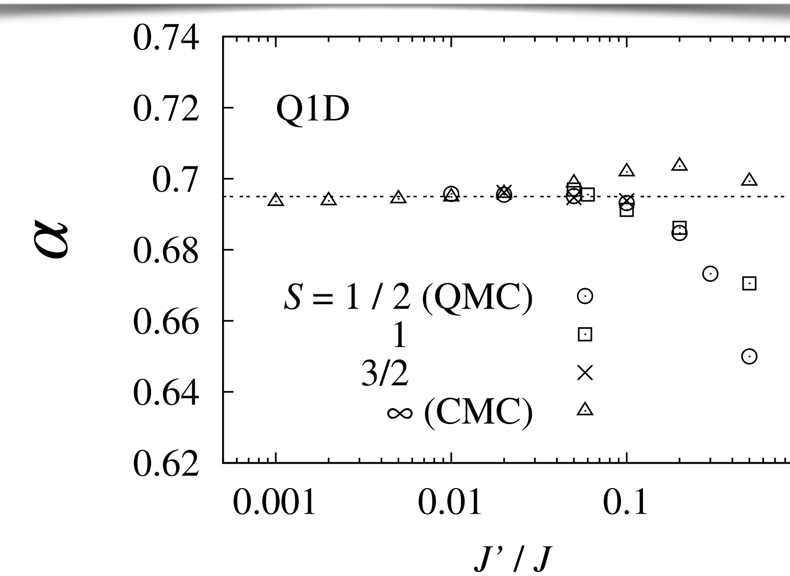


Néel Temperature of Quasi-Low-Dimensional Heisenberg Antiferromagnets

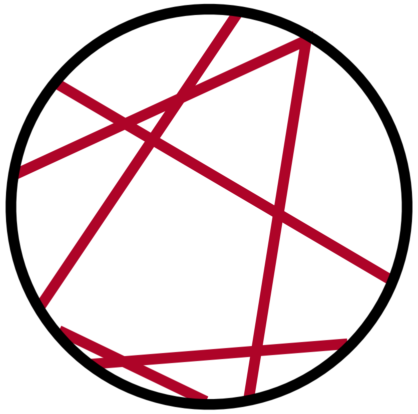
C. Yasuda, S. Todo, K. Hukushima, F. Alet, M. Keller, M. Troyer, and H. Takayama
Phys. Rev. Lett. **94**, 217201 – Published 2 June 2005



see also

- V. Y. Irkhin and A. A. Katanin, PRB (2000)
- A. Praz, C. Mudry, and M. B. Hastings, PRB (2006)
- B. Thielemann, et al, PRB (2009)
- R. Blinder et al. PRB (2017)

Random branching (ii) — XY FM



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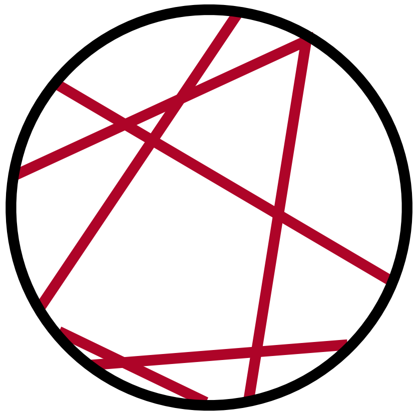

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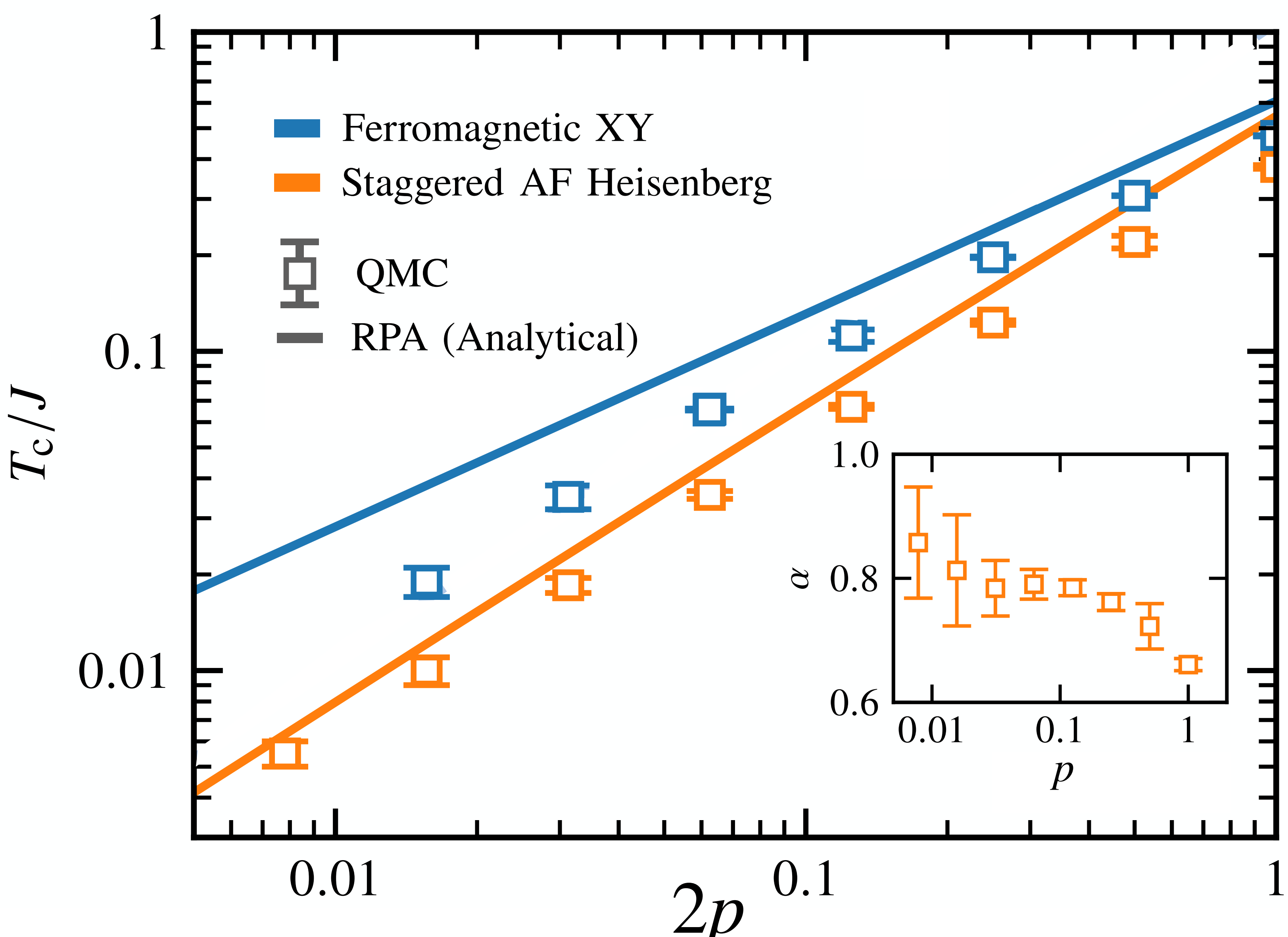

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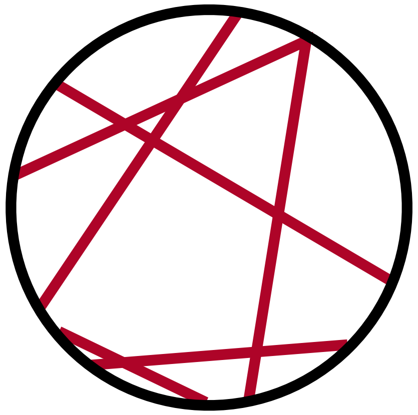

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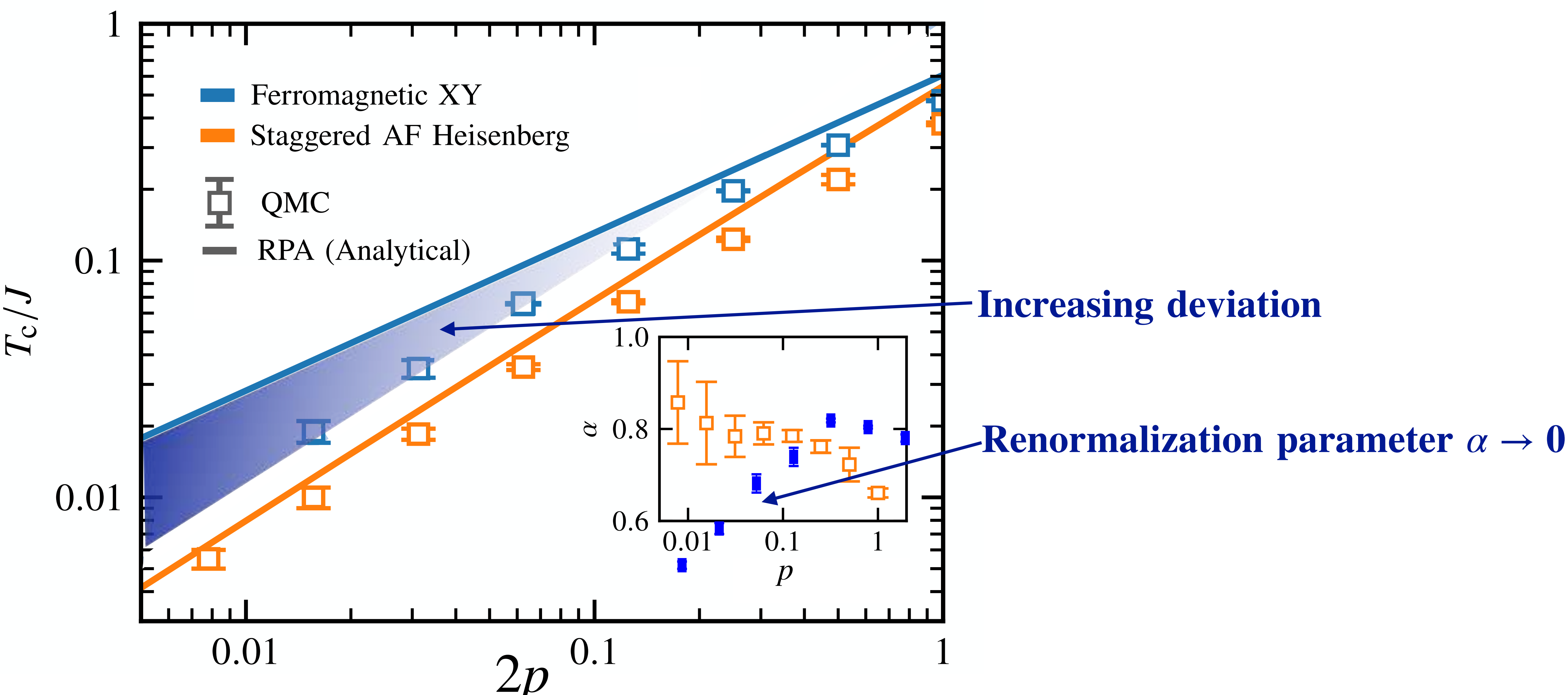

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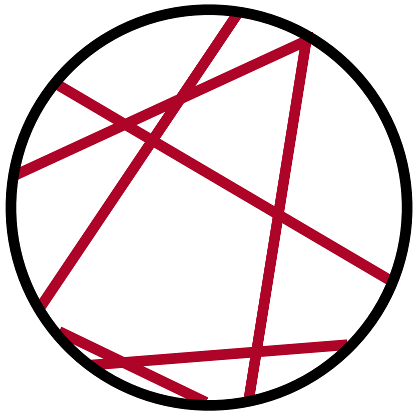

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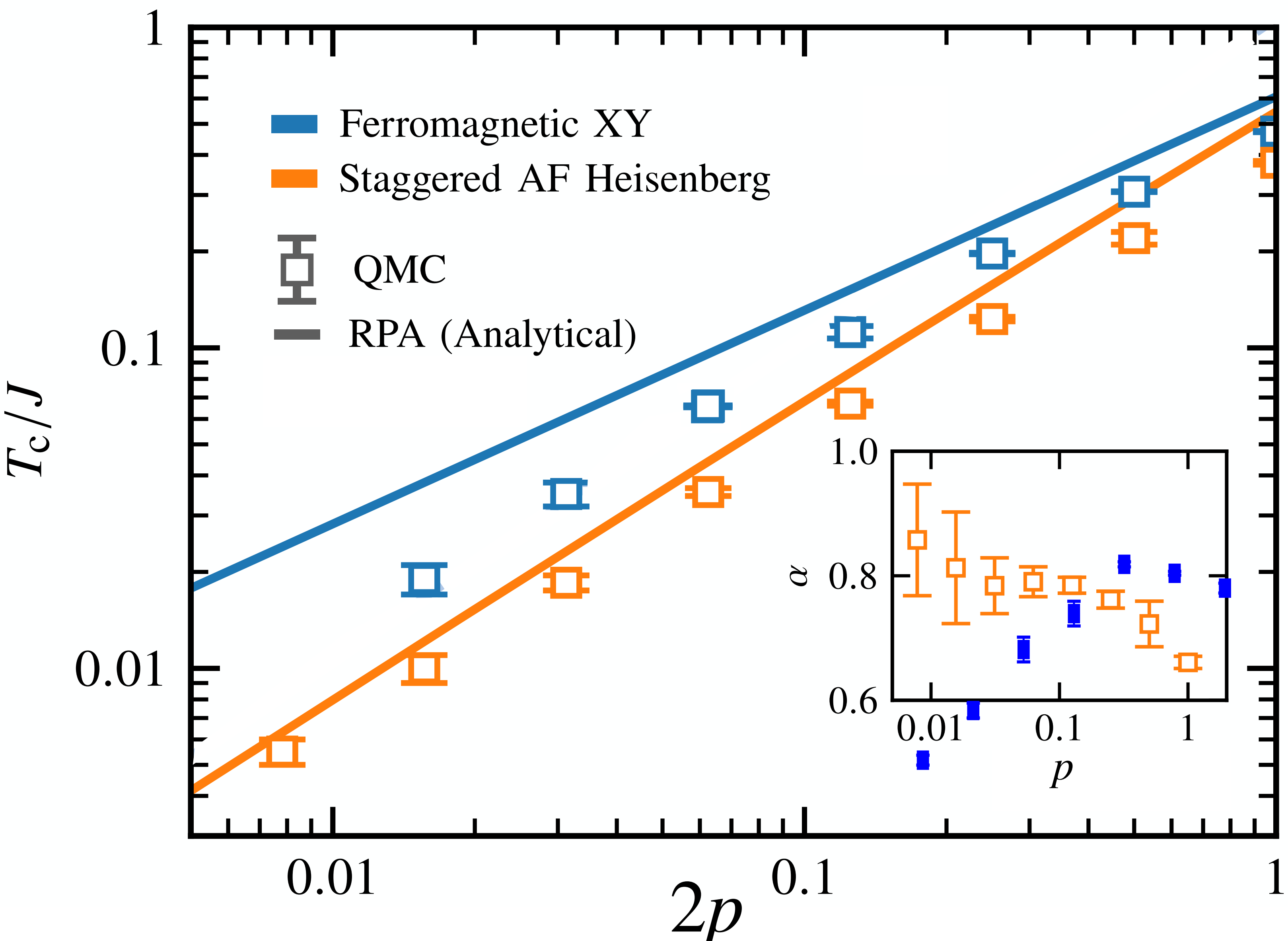

$T_c^{\mathrm{RPA}}/J = 0.608\ldots\ p^{2/3}$

But, at small p

$\xi^{1\mathrm{D}}_{T=T_c^{\mathrm{RPA}}} \sim p^{-2/3} < \zeta_p \sim 1/p$

(average distance between shortcuts)

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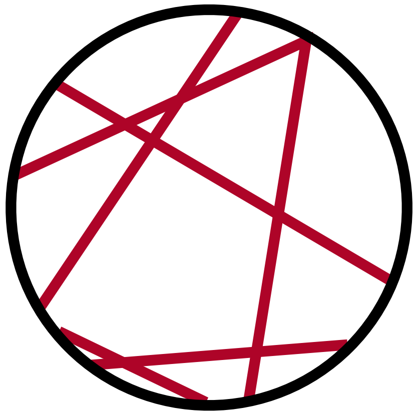

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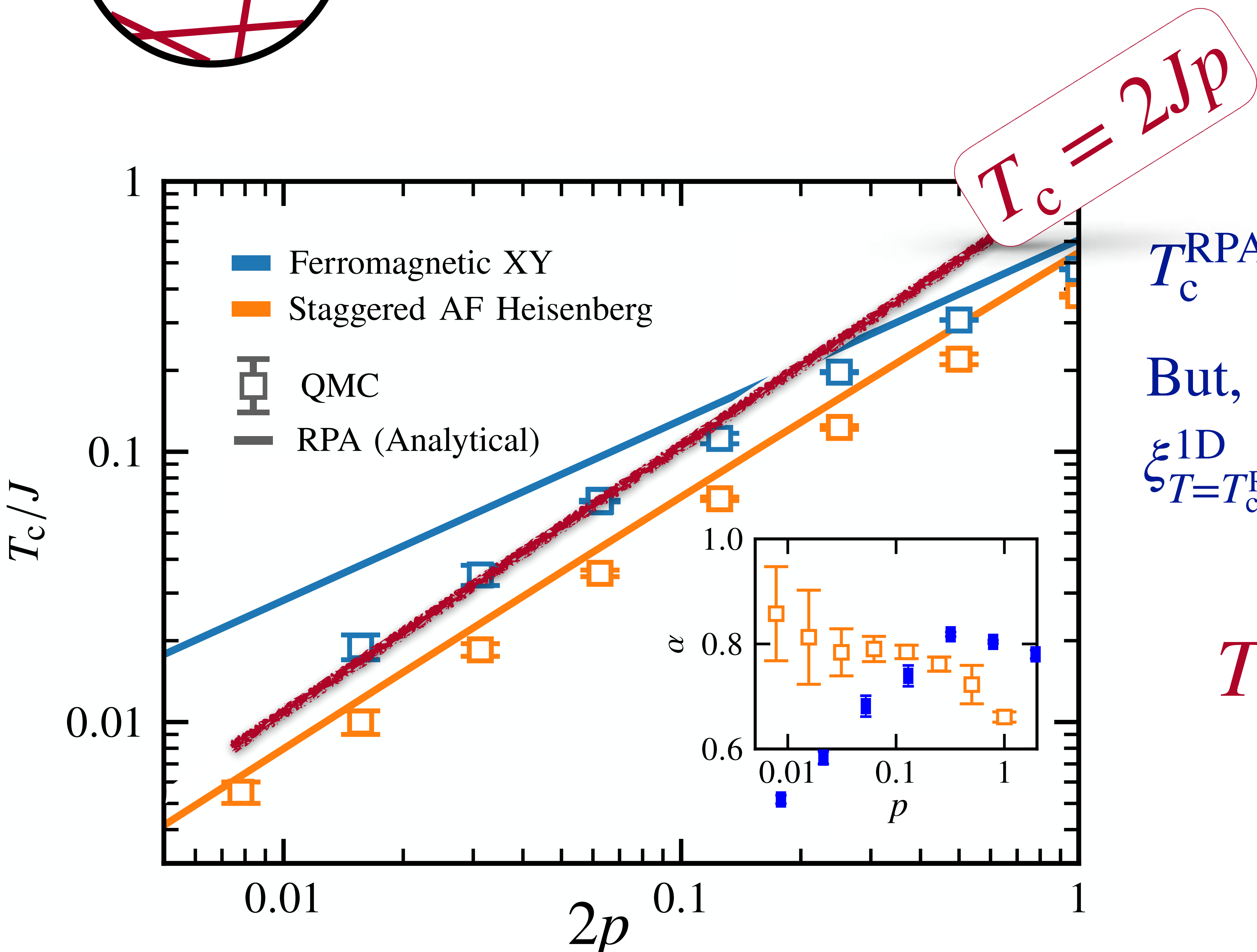

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$T_c \to T_c^{MF} \approx 2Jp$

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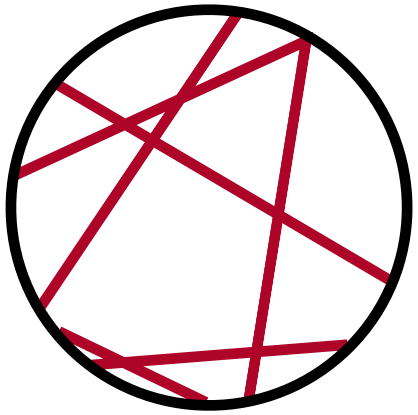

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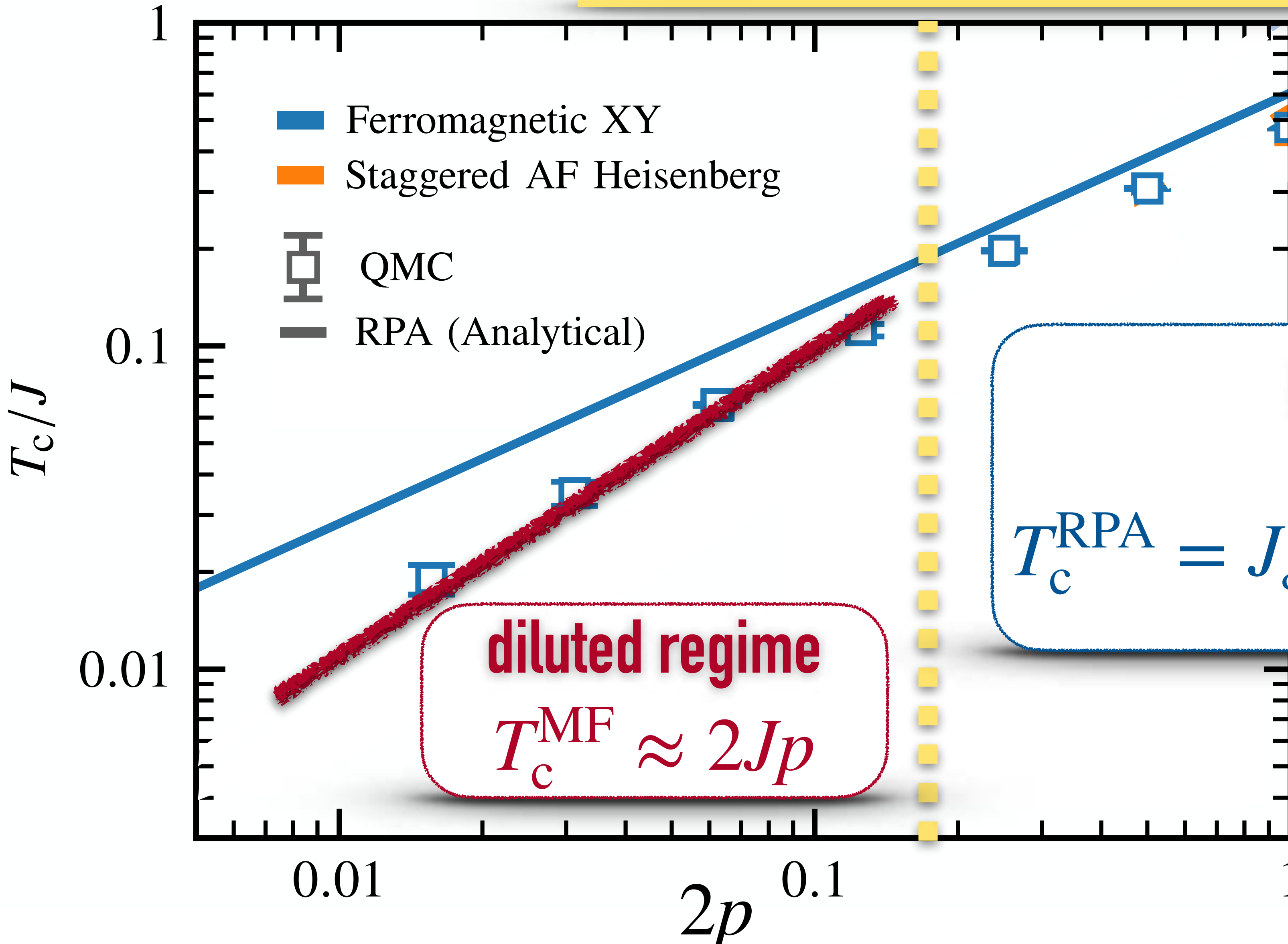
Crossover between 2 distincts MF regimes!



diluted regime

$$T_c^{\mathrm{MF}} \approx 2Jp$$

RPA regime

$$T_c^{\mathrm{RPA}} = Jg_\Delta \times p^{\frac{2K}{4K-1} = \frac{\pi}{\pi + \arccos\Delta}}$$

SUMMARY

‣ Two mean-field approaches for the spin ordering transition on small-world networks

(i) Competition between random branching typical length $\zeta_p \sim p^{-1/d}$ and thermal correlation length $\xi(T)$

$$T_c^{\rm MF}(p) - T_c(0) \propto Jp^{\frac{1}{\nu d}}$$

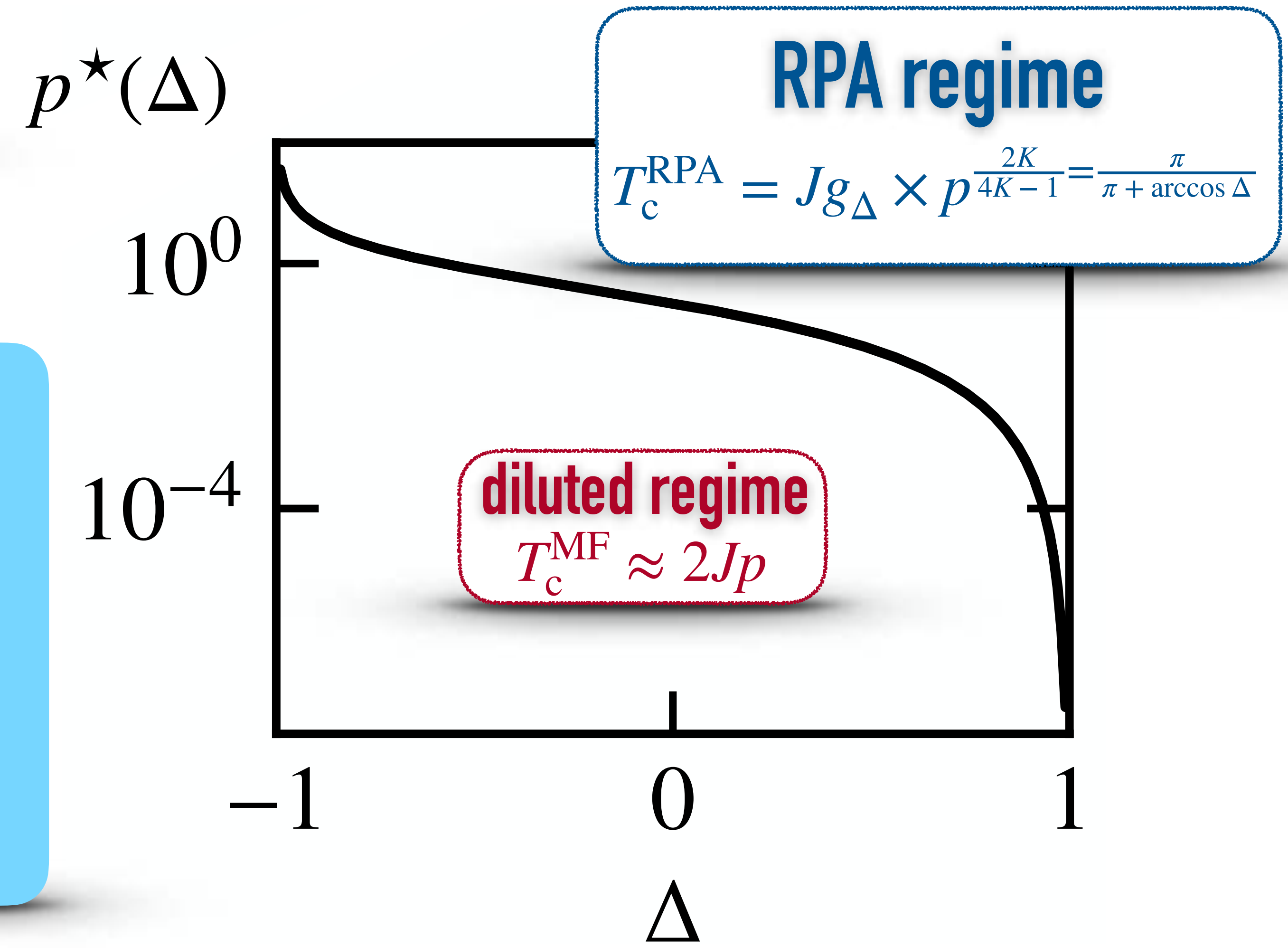
(ii) Chain-mean-field theory RPA analysis

$$\boxed{T_c^{\rm RPA}(p) - T_c(0) \propto Jp^{\frac{1}{\gamma}}}$$

‣ Classical phase transitions $\gamma < \nu d$ RPA is generically expected

Quantum case

- **Luttinger liquids: crossover between two mean-field regimes**
- **Exact analytical expression for the "critical concentration" $p^\star(\Delta)$**



Possible new directions

Influence of a spin gap

Frustration effect

Disorder effect

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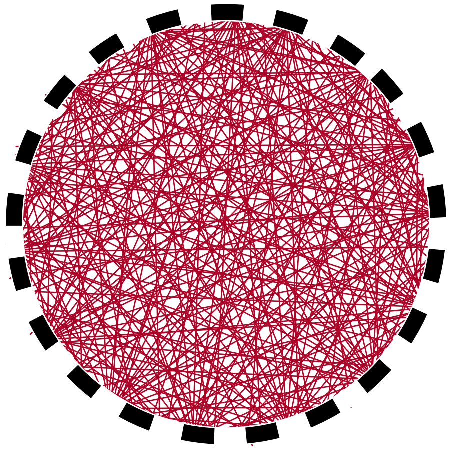

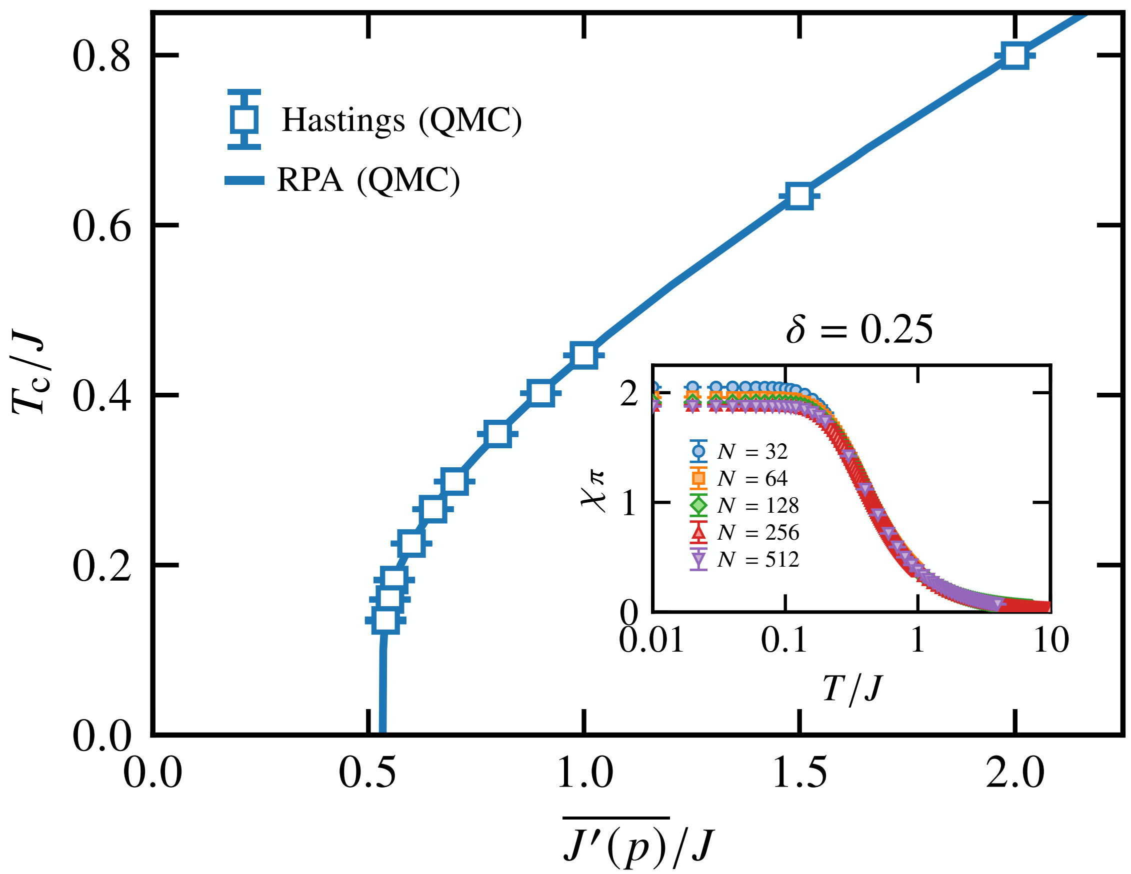

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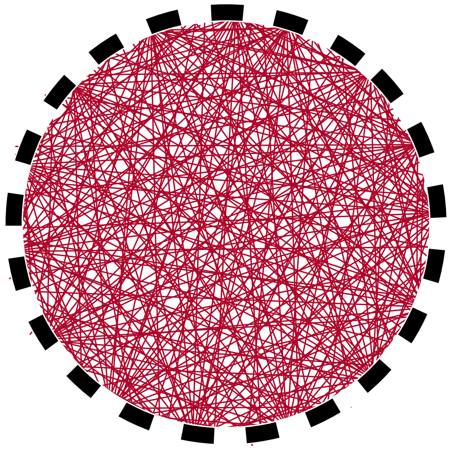
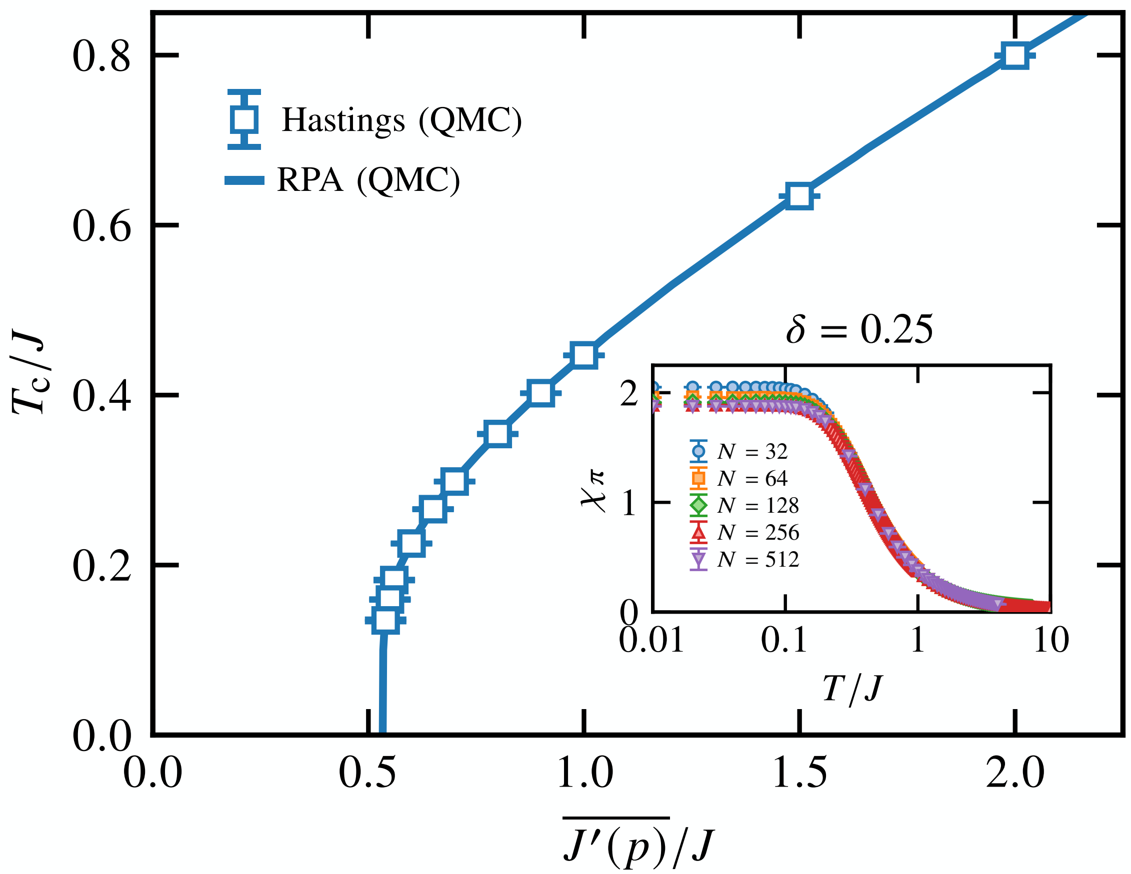
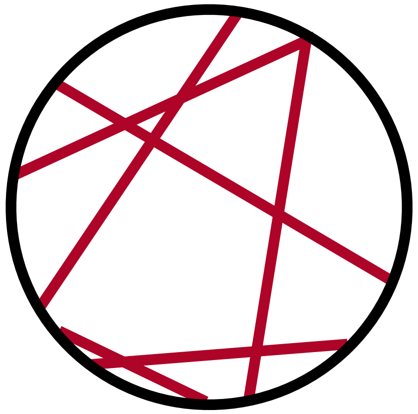




Frustration effect

spin-glass physics?



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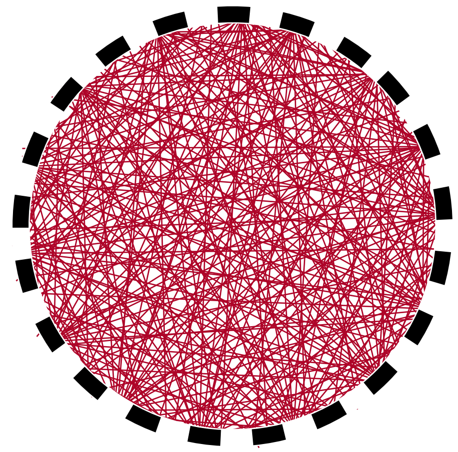

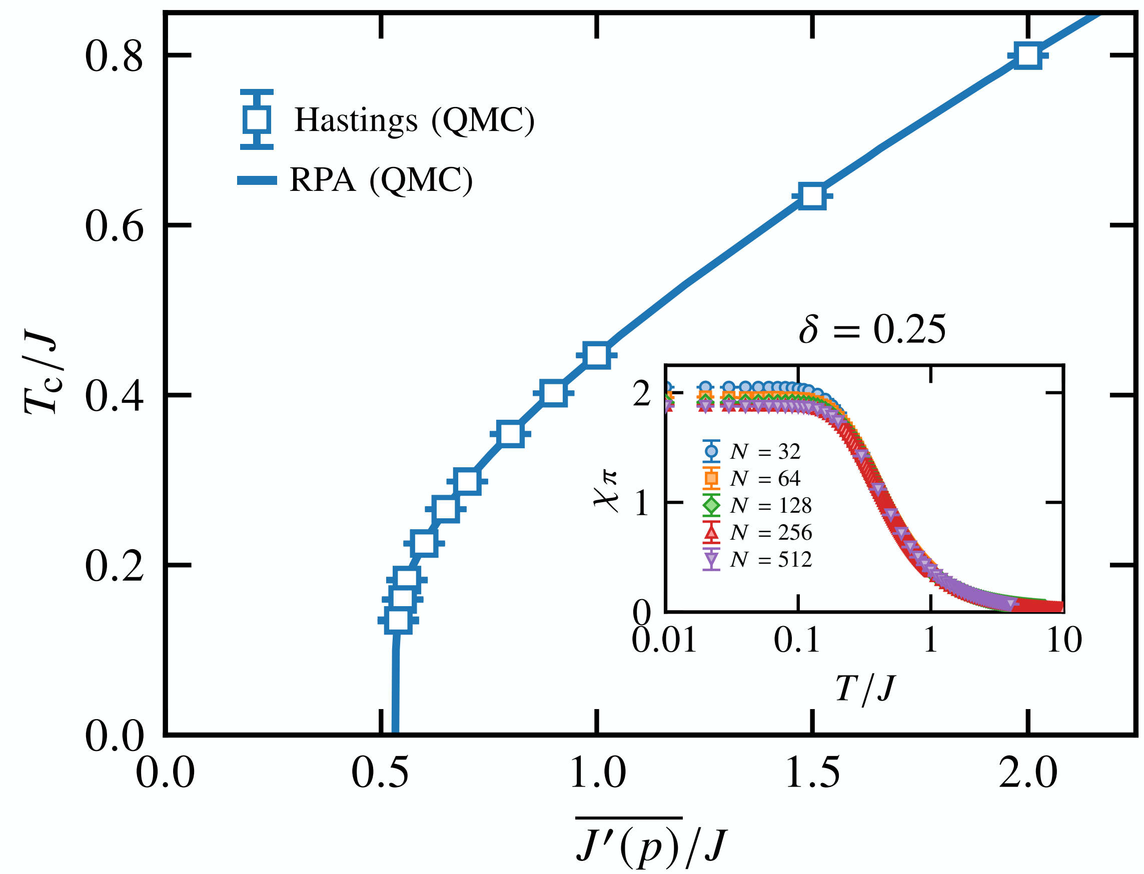

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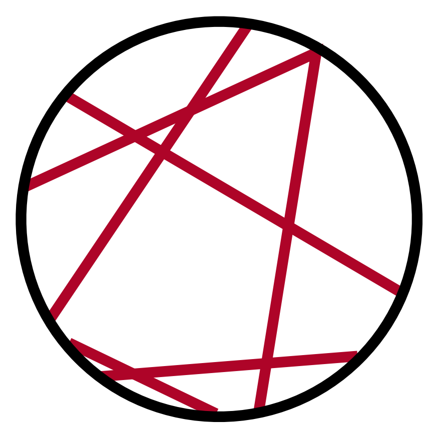

Disorder effect

Random bonds $\Rightarrow d = \infty$ random-singlet physics

Random fields $\Rightarrow d = \infty$ Bose-glass ($T = 0$) / MBL ($T = \infty$)


thank you for
your attention!