Supersymmetric Lattice Models

Kareljan Schoutens
University of Amsterdam

Benasque – 17-19 February 2020

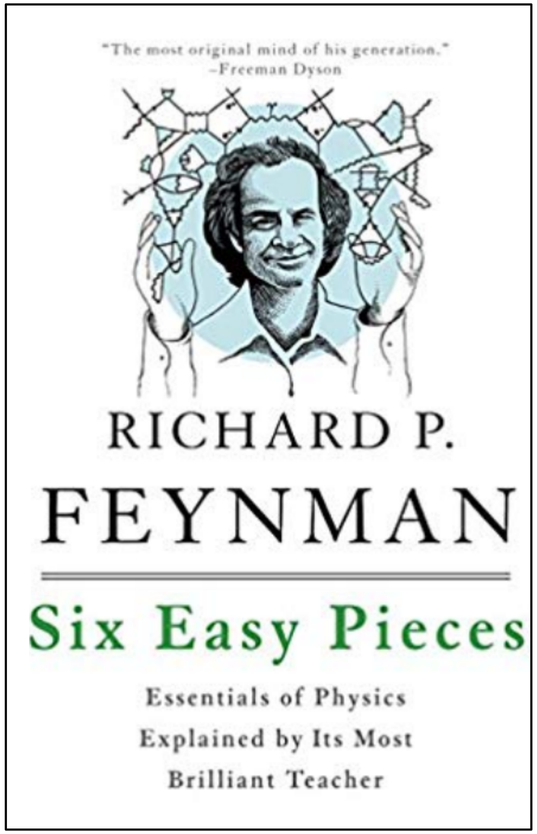
QuSoft
Research Center for Quantum Software

Lecture 2/3

Twelve Easy Pieces

1. Teaser
2. N=2 susy – Witten index
3. M_1 model in 1D: Witten index, spectra, CFT connection
4. M_1 model: scaling form of 1-pt functions from CFT
5. **M_k models: Witten index, CFT**
6. **M_k models off criticality → massive (integrable) QFT**
7. **M_1 model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$**
8. **PH symmetric model and coupled fermion chains**
9. Superfrustration on ladder: zig-zag, Nicolai, Z_2 Nicolai
10. Superfrustration on 2D grids
11. Back to 1D M_1 model: kink dynamics
12. Towards realization with Rydberg-dressed cold atoms

Twelve Easy Pieces



Easy Piece

+ 15 minutes

+ single topic

+ up to ~~10 slides~~ **7 slides**

+ key idea on board

Twelve Easy Pieces

5. M_k models: Witten index, CFT

6. M_k models off criticality → massive (integrable) QFT

7. M_1 model on square ladder: CFT, 1-pt functions, <$\sigma\sigma\sigma\sigma$>

8. PH symmetric model and coupled fermion chains

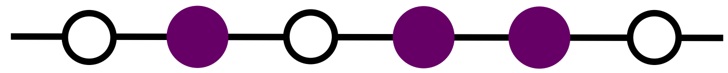
M_k model in 1D

Fendley-Nienhuis-KjS 2003

configurations

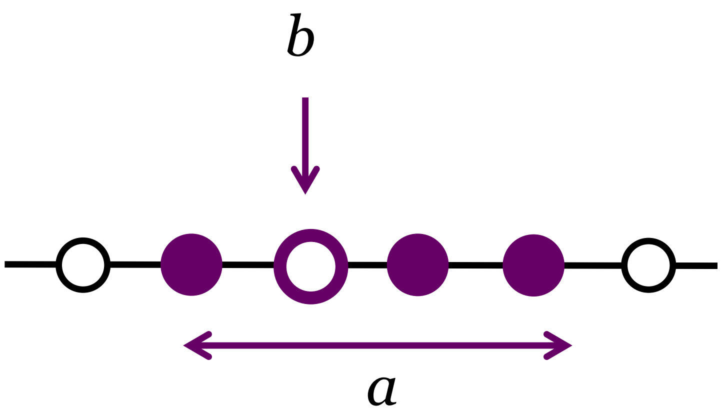
lattice fermions **up to *k***

nearest neighbors occupied



supercharge

$$Q^{M_k} = \sum_i \sum_{a,b} \lambda_{[a,b],i} d_{[a,b],i}$$



annihilates particle at position b from string of length a

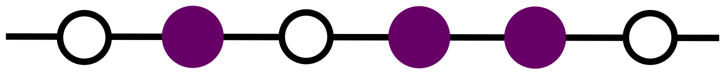
M_2 model

Fendley-Nienhuis-KjS 2003

configurations

lattice fermions **up to 2**

nearest neighbors occupied



M_2[x] :
amplitude x if Q annihilates particle w/o neighbor, else 1

$x=0$: maps to ferromagnetic integrable tJ model, $J=-2t$

$x=\sqrt{2}$: maps to integrable spin-1 XXZ model; critical

General x not well studied

M_k model, properties

Fendley-Nienhuis-KjS 2003, Hagendorf-Huijse 2015

Closed chain:

$W=k+1$ for $L=l(k+2)$,

→ $k+1$ groundstates at filling $f/L = k/(k+2)$

Open chain:

$W=(-1)^k$ for $L=k+1$, $k+2$ modulo $k+2$

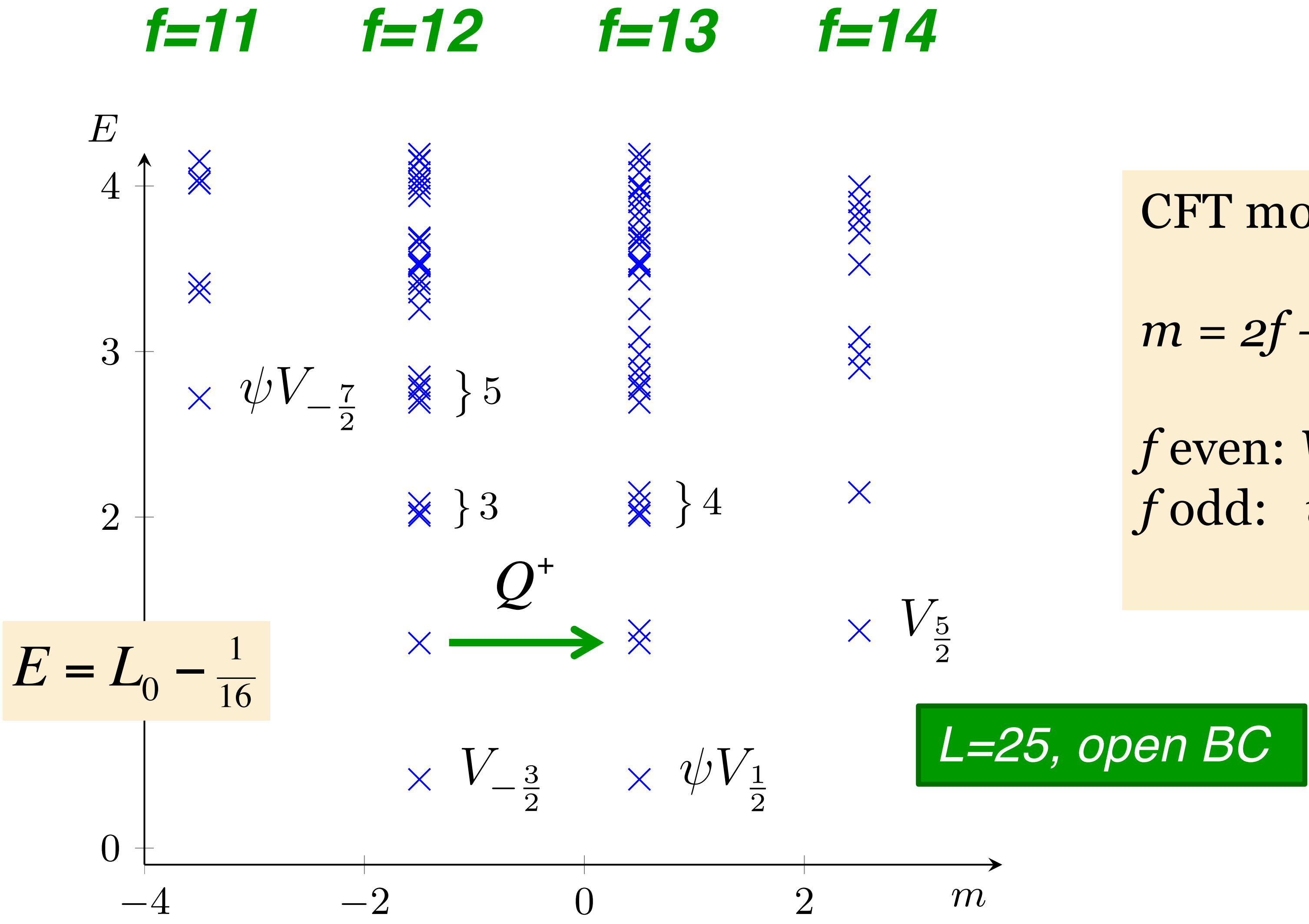
0 else

Critical behavior:

CFT_k, k^{th} minimal model of $N=2$ SCFT,

$U(1) \times Z_k$ parafermions

M_2 model, open chain

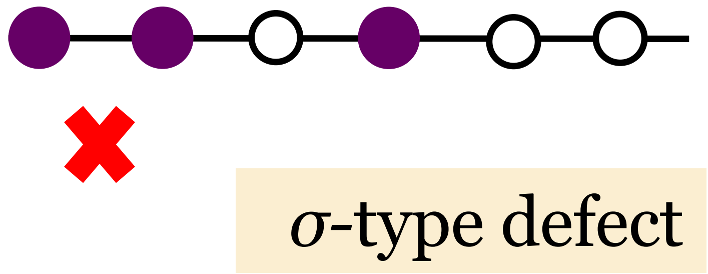


CFT modules:

$m = 2f - L - ½$

f even: V_m
f odd: ψV_m

M_2 model, open chain, CFT spectra

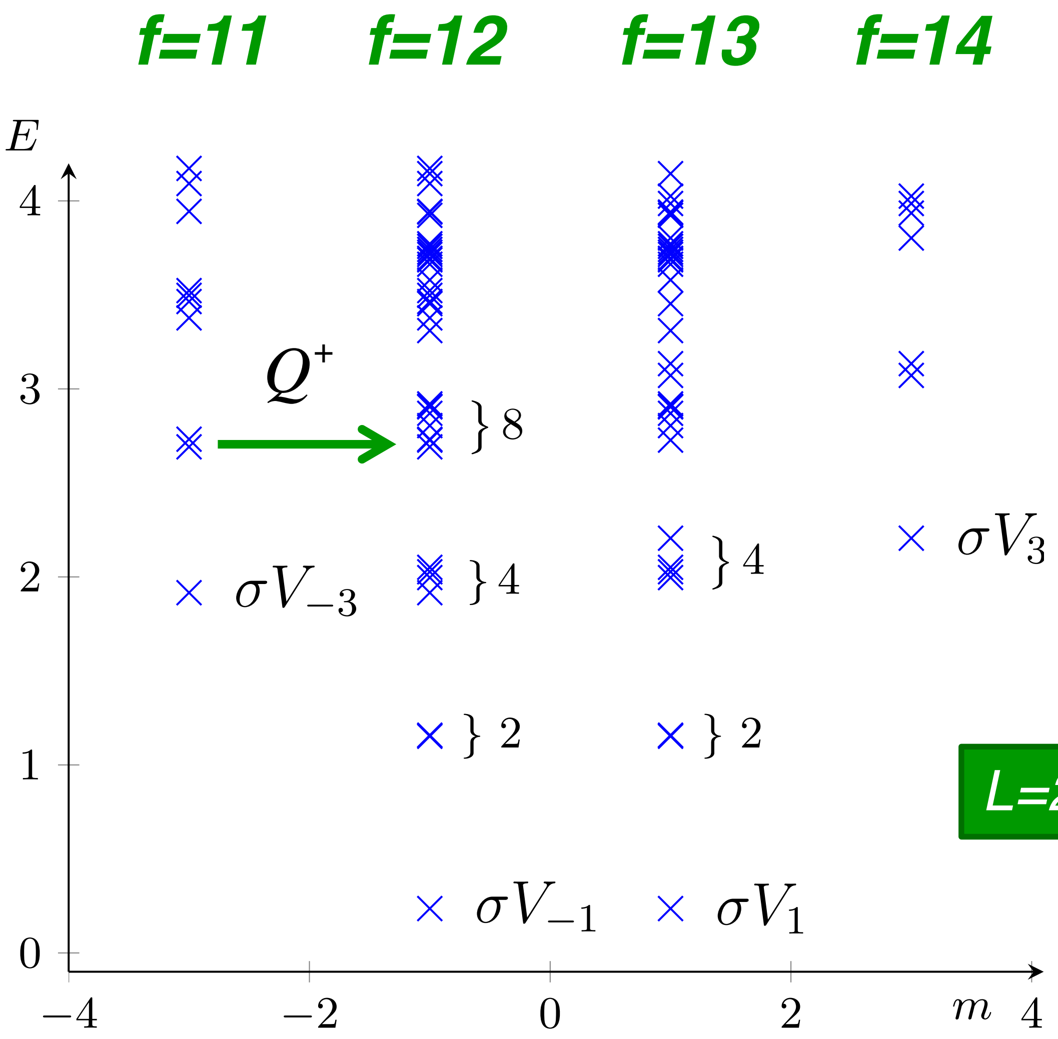


σ-type defects and BC

•locally forbid simultaneous occupancy of n.n. sites

•in the CFT these correspond to the injection of spin fields σ

M_2 model, σ /open boundary conditions



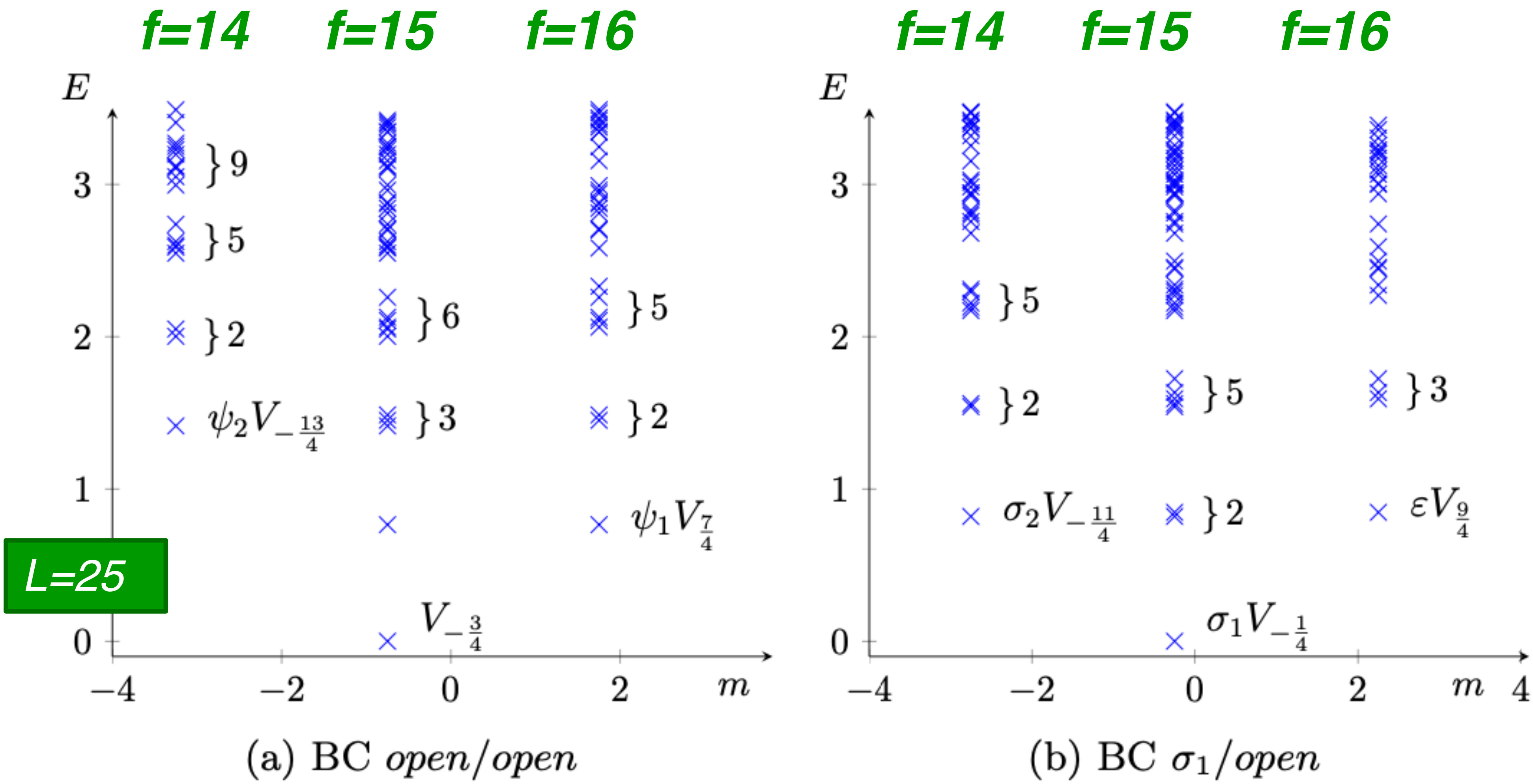
CFT modules:

σV_m with

$m = 2f - L$

L=25, σ / open BC

M_3 model, various boundary conditions



k=3 minimal model of *N*=2 SCFT → Z_3 parafermions

Twelve Easy Pieces

5. M_k models: Witten index, CFT

6. M_k models off criticality → massive (integrable) QFT

7. M_1 model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$

8. PH symmetric model and coupled fermion chains

M_1 model with staggering

+ to break criticality, one can stagger the amplitudes in the supercharge with position dependent factors λ_i

+ for λ_i periodic with period 3 the staggered model is still integrable, eg

$$\ldots \lambda \; 1 \; 1 \; \lambda \; 1 \; 1 \; \lambda \; 1 \; 1 \ldots$$

Integrability

Hagendorf, Huijze and Fokkema found 1-parameter family of models $M_k[\lambda]$ with couplings $\lambda_{[a,b],j}$ such that

- all couplings repeat for j modulo $(k+2)$
- $M_k[\lambda]$ model integrable by nested Bethe Ansatz for all λ
- $M_k[\lambda]$ is critical for $\lambda=1$

Hagendorf-Fokkema-Huijse 2014

Hagendorf-Huijse 2015

$M_k[\lambda]$, definitions

M_3 : staggering mod 5, $\lambda << 1$

M_2 : $x = \sqrt{2}$,
staggering modulo 4

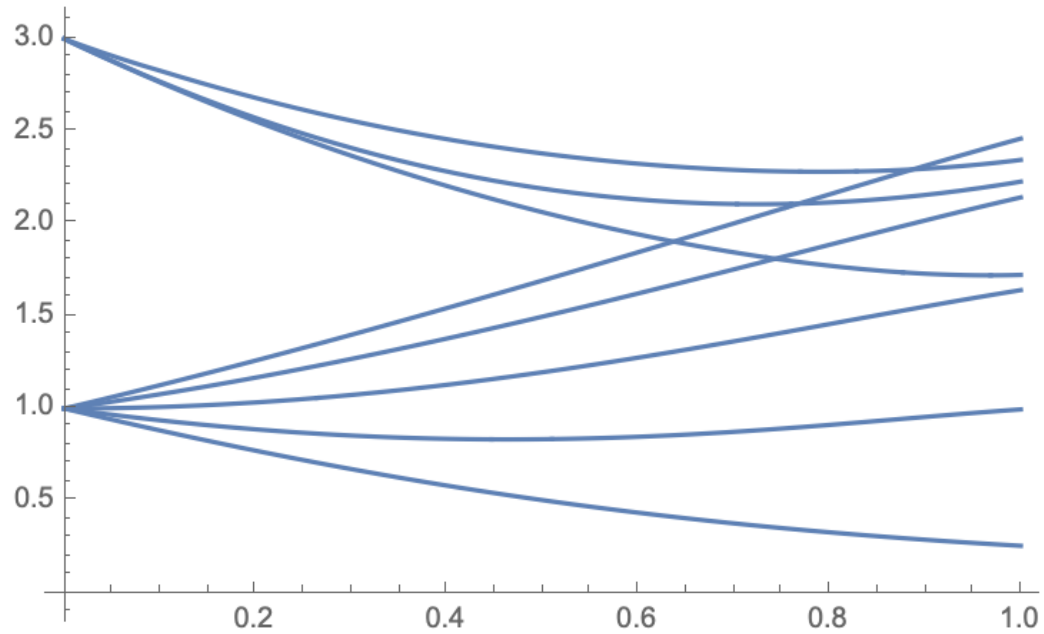
$$\begin{array}{llllll}
\lambda_{[1,1],j}: & \dots\sqrt{2} & \sqrt{2}\lambda & \sqrt{2} & \sqrt{2}\lambda\dots \\
\lambda_{[2,1],j}: & \dots\ 1 & \lambda & 1 & \lambda\ \dots \\
\lambda_{[2,2],j}: & \dots\ 1 & \lambda & 1 & \lambda\ \dots
\end{array}$$

$$\begin{array}{llllllll}
\lambda_{[1,1],j}: & \dots & 1 & \sqrt{2} & \sqrt{2}\lambda & \sqrt{2} & 1 & \dots \\
\lambda_{[2,1],j}: & \dots & 1 & \sqrt{2} & \lambda & \sqrt{2} & \lambda & \dots \\
\lambda_{[2,2],j}: & \dots & \lambda & \sqrt{2} & \lambda & 1 & 1 & \dots \\
\lambda_{[3,1],j}: & \dots & 1 & \lambda & \lambda & 1 & \frac{\lambda}{\sqrt{2}} & \dots \\
\lambda_{[3,2],j}: & \dots & \frac{\lambda}{\sqrt{2}} & 1 & \frac{\lambda^2}{\sqrt{2}} & 1 & \frac{\lambda}{\sqrt{2}} & \dots \\
\lambda_{[3,3],j}: & \dots & \frac{\lambda}{\sqrt{2}} & 1 & \lambda & \lambda & 1 & \dots \\
 & & & & \uparrow & & &
\end{array}$$

M_1 model with staggering

+ plotting the lowest energies of the model on *L=13* sites, OBC, staggering

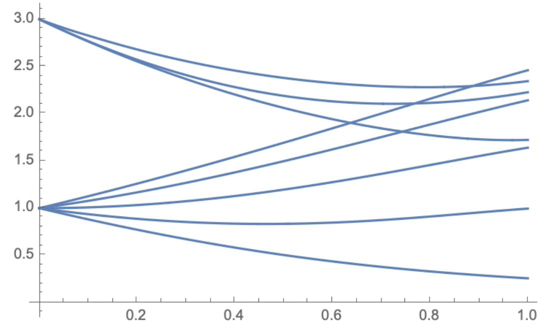
1 1 λ 1 1 λ 1 1 λ 1 1 λ 1



λ =0 **λ =1**

M_1 model with staggering

λ =0 $\qquad$ λ =1



k=1 N=2 SCFT

kink states

$|K_1\rangle, |K_4\rangle, \ldots |K_{13}\rangle$

energy *E=1*

scaling limit for $\lambda \uparrow 1$:

supersymmetric *N=2* QFT:

sine-Gordon theory at $\beta^2=4/3$

Integrable massive QFT

$M_k[\lambda<1]$ connects to $N=2$ supersymmetric integrable massive QFT, with superpotentials of Chebyshev form

$k=1$: sine-Gordon at $N=2$ susy point $\beta^2=4/3$
$k=2$: $N=1$ supersymmetric sine-Gordon at $N=2$ susy point

lattice model excitations at $\lambda \ll 1$ are kinks between $W=k+1$ possible $E=0$ states – they are in 1-1 correspondence with kinks in the $N=2$ integrable QFT

Fokkema-KjS 2017

Integrable massive QFT

susy lattice model M_3 [λ]

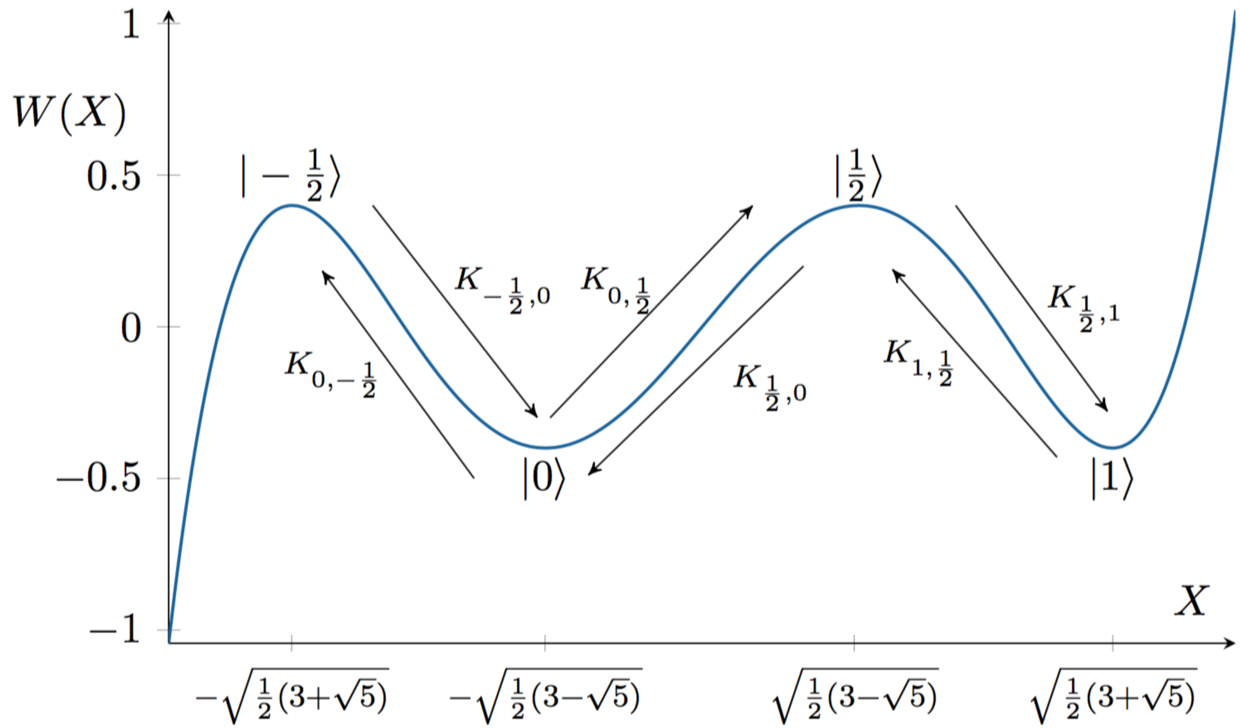
susy ground-states at $\lambda \ll 1$

$$|1\rangle = \ldots 1\underset{\wedge}{1}10011\underset{\wedge}{1}100\ldots, \quad |\tfrac{1}{2}\rangle = \ldots(\cdot\underset{\wedge}{1}\cdots)(\cdot\underset{\wedge}{1}\cdots)\ldots,$$

$$|0\rangle = \ldots 0\underset{\wedge}{1}01110\underset{\wedge}{1}011\ldots, \quad |-\tfrac{1}{2}\rangle = \ldots\underset{\wedge}{0}(\cdot 11\cdot)\underset{\wedge}{0}(\cdot 11\cdot)\ldots,$$

massive N=2 integrable QFT

groundstates as extrema of k=3 Chebyshev superpotential



Twelve Easy Pieces

5. M_k models: Witten index, CFT

6. M_k models off criticality → massive (integrable) QFT

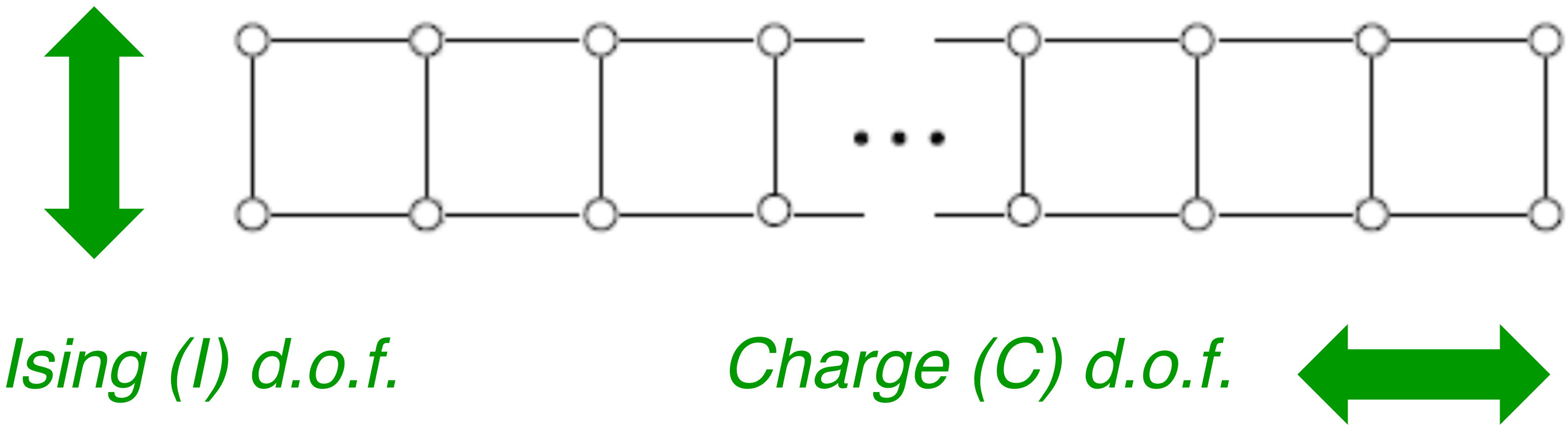
7. **M_1 model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$**

8. PH symmetric model and coupled fermion chains

M_1 model on square ladder

+ find that for PBC, $L=4l$, Witten index $W=3$

+ intuition: charge order (along legs) combined with Ising order (along rungs)



+ expect $k=2$ $N=2$ SCFT, $c=3/2$, as for M_2 model

+ however, finite size spectra puzzling

M_1 model on square ladder

A supersymmetric multicritical point in a model of lattice fermions

Bela Bauer,[1] Liza Huijse,[2] Erez Berg,[3] Matthias Troyer,[4] and Kareljan Schoutens[5]

[1] *Station Q, Microsoft Research, Santa Barbara, CA 93106-6105, USA*
[2] *Department of Physics, Harvard University, Cambridge MA 02138*
[3] *Department of Condensed Matter Physics, Weizmann Institute of Science, Rehovot 76100, Israel*
[4] *Theoretische Physik, ETH Zurich, 8093 Zurich, Switzerland*
[5] *Institute for Theoretical Physics, University of Amsterdam, Science Park 904, P.O.Box 94485, 1090 GL Amsterdam, The Netherlands*

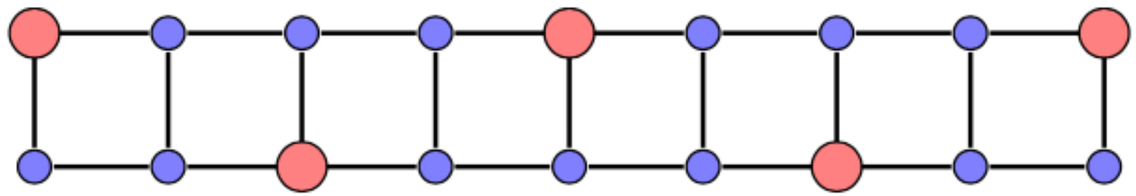
We study a model of spinless fermions with infinite nearest-neighbor repulsion on the square ladder which has microscopic supersymmetry. It has been conjectured that in the continuum the model is described by the superconformal minimal model with central charge $c = 3/2$. Thus far it has not been possible to confirm this conjecture due to strong finite-size corrections in numerical data. We trace the origin of these corrections to the presence of unusual marginal operators that break Lorentz invariance, but preserve part of the supersymmetry. By relying mostly on entanglement entropy calculations with the density-matrix renormalization group, we are able to reduce finite-size effects significantly. This allows us to unambiguously determine the continuum theory of the model. We also study perturbations of the model and establish that the supersymmetric model is a multicritical point. Our work underlines the power of entanglement entropy as a probe of the phases of quantum many-body systems.

M_1 model on square ladder

Adding terms to H_{susy} to stabilize I or C (or CI) order

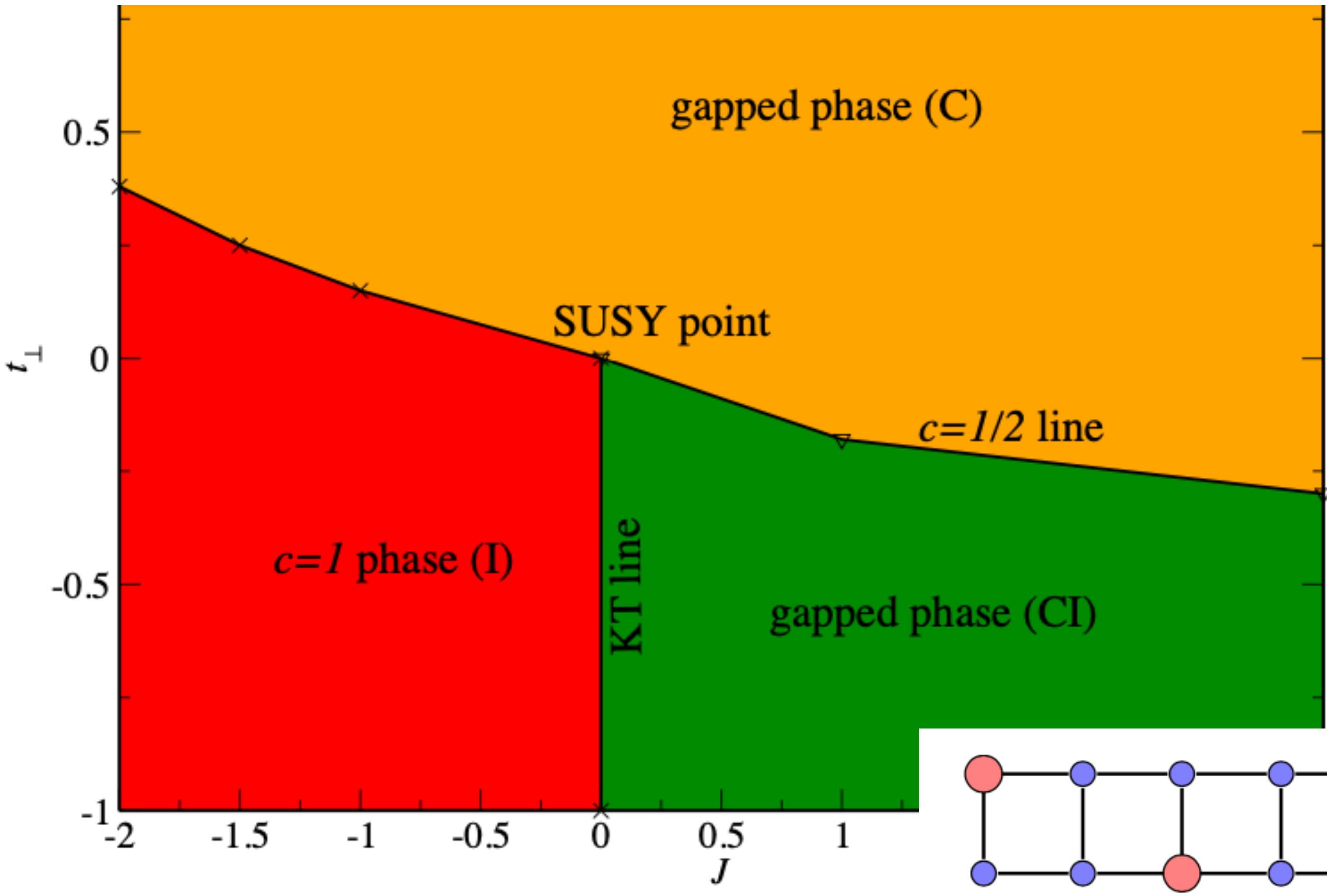
$$H_{\mathrm{pert}} = t_{\perp} \sum_i \left(d^{\dagger}_{i,\substack{\circ\\\bullet}} d_{i,\substack{\bullet\\\circ}} + d^{\dagger}_{i,\substack{\bullet\\\circ}} d_{i,\substack{\circ\\\bullet}} \right)$$
$$+ J \sum_i \left(n_{i,\substack{\circ\\\bullet}} n_{i+1,\substack{\bullet\\\circ}} + n_{i,\substack{\bullet\\\circ}} n_{i+1,\substack{\circ\\\bullet}} \right)$$

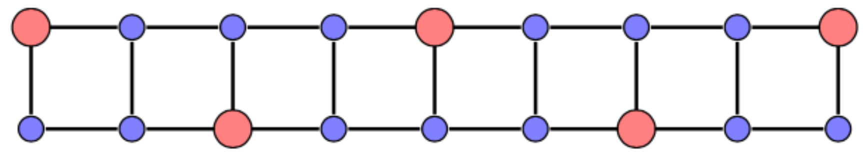
Choosing $J >> 0$, $t_{\perp} << 0$ stabilizes both orders (CI)



M_1 model on square ladder

Susy point is multicritical point

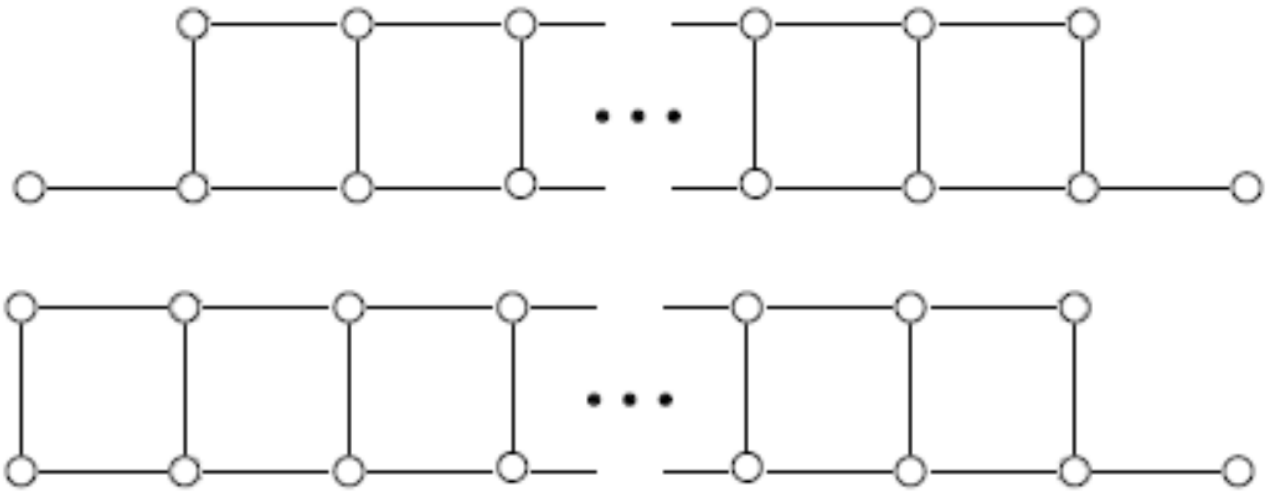




M_1 model on square ladder

Numerical (MPS) measurement of density 1-point functions

$$\langle n_k^+ \rangle = \langle \psi | n_k^+ | \psi \rangle \equiv \langle \psi | c_{k\uparrow}^\dagger c_{k\uparrow} + c_{k\downarrow}^\dagger c_{k\downarrow} | \psi \rangle$$
$$\langle n_k^- \rangle = \langle \psi | n_k^- | \psi \rangle \equiv \langle \psi | c_{k\uparrow}^\dagger c_{k\uparrow} - c_{k\downarrow}^\dagger c_{k\downarrow} | \psi \rangle$$

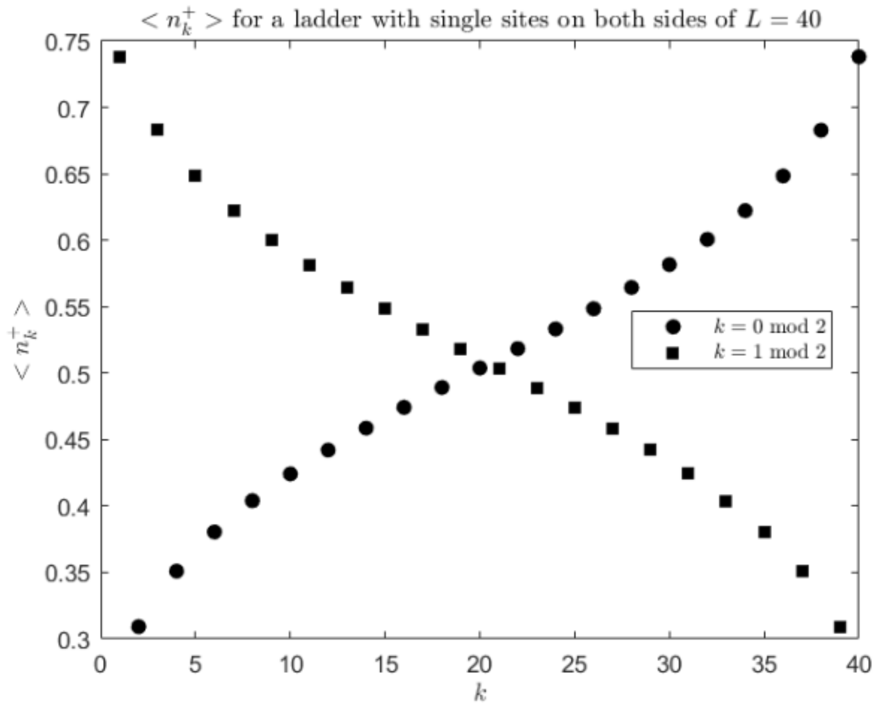


Dingerink,

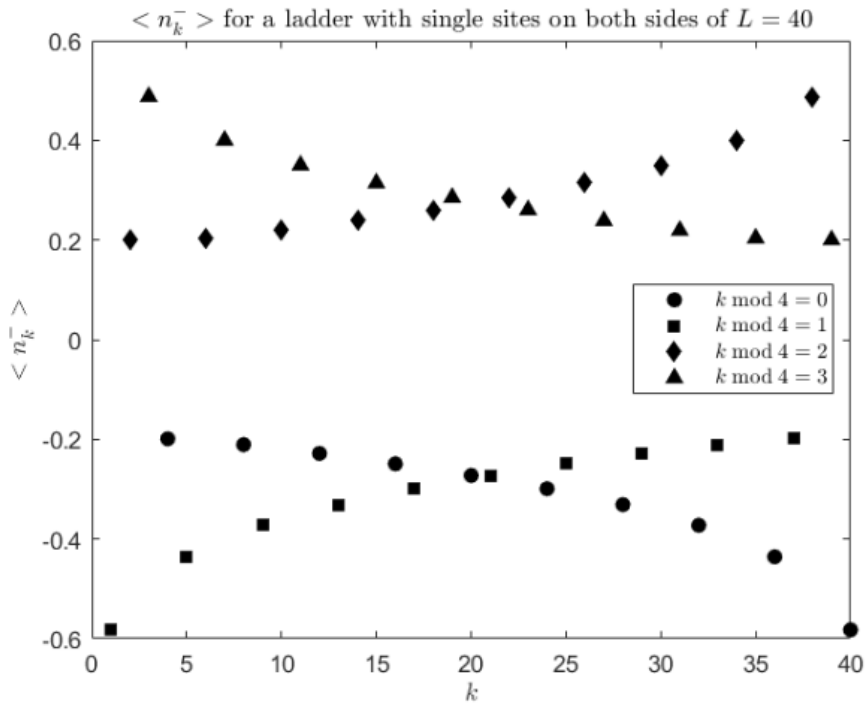
MSc thesis 2019

M_1 model on square ladder

L=40, f=20

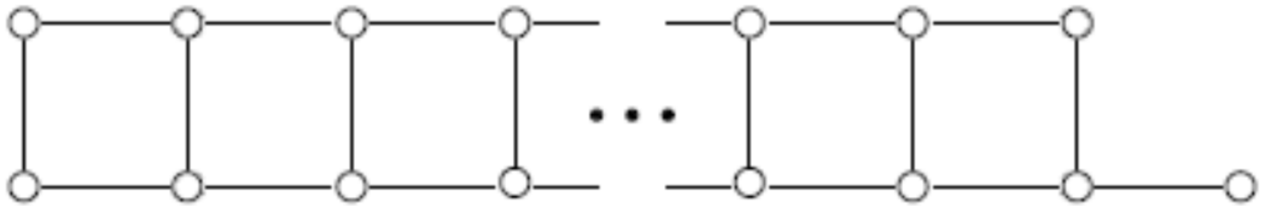


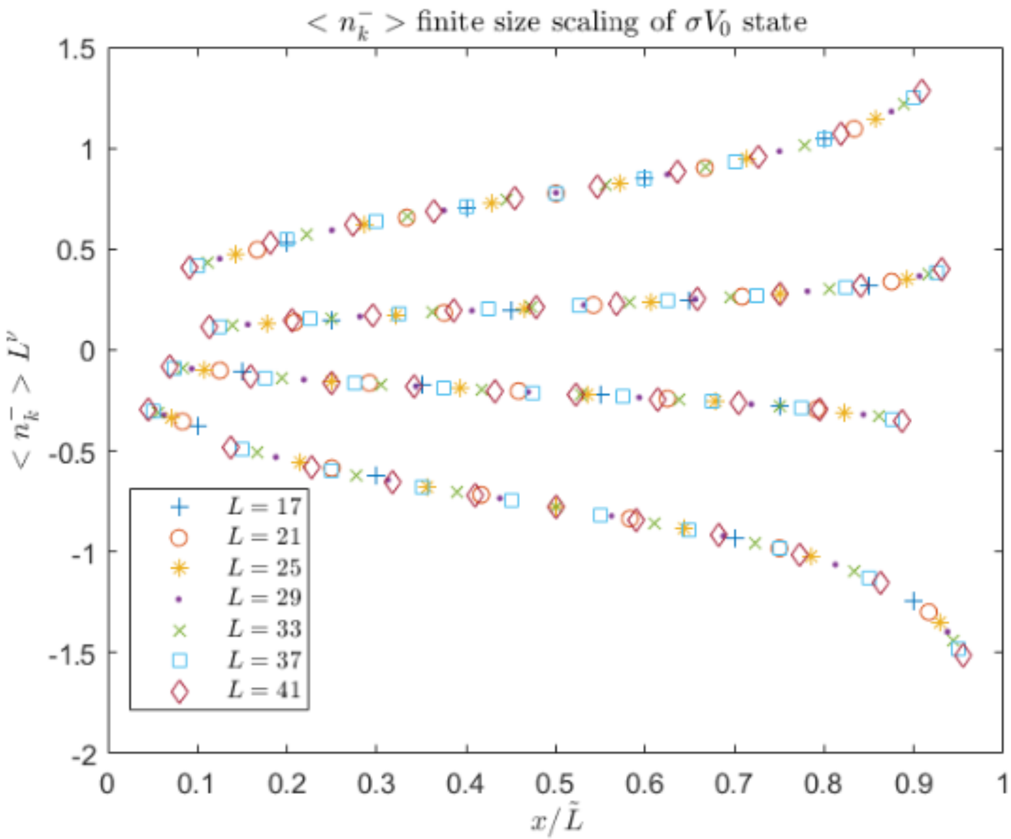
(a) The $\mathbb{Z}_2$ structure of $\langle n_k^+ \rangle$.



(b) The $\mathbb{Z}_4$ structure of $\langle n_k^- \rangle$.

M_1 model on square ladder





$<n_k^->$ data collapse with exponent $\nu=1/4$, CFT expressions for fitting these curves expressed through $<\sigma\sigma\sigma\sigma>$ 4-point function

Twelve Easy Pieces

5. M_k models: Witten index, CFT

6. M_k models off criticality → massive (integrable) QFT

7. M_1 model on square ladder: CFT, 1-pt functions, <$\sigma\sigma\sigma\sigma$>

8. **PH symmetric model and coupled fermion chains**

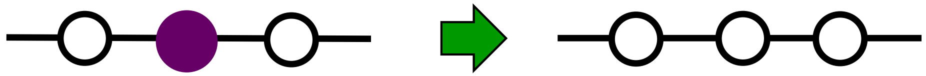
Particle Hole symmetric M_1 model

supercharge

$$Q = \sum_i (d_i + e_i^\dagger)$$

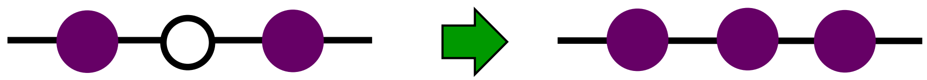
$$d_i = p_{i-1} c_i p_{i+1}$$

$$p_i = 1 - n_i$$



$$e_i^\dagger = n_{i-1} c_i^\dagger n_{i+1}$$

$$n_i = c_i^\dagger c_i$$



de Gier-Feher-Nienhuis-Rusaczonek 2015

Susy groundstates in PH M_1 model

- degenerate groundstates at $E=0$

	periodic		antiperiodic	
L	G	ℓ	G	ℓ
4	8	2	4	3
5	8	6	8	5
6	0	4	16	4
7	16	15	16	14
8	32	7	16	20
9	32	54	32	54
10	0	46	64	94
11	64	204	64	210
12	128	80	64	201

Table 1: The degeneracy of the groundstate, the number of energy levels, and the smallest degree of degeneracy for periodic and antiperiodic boundary conditions

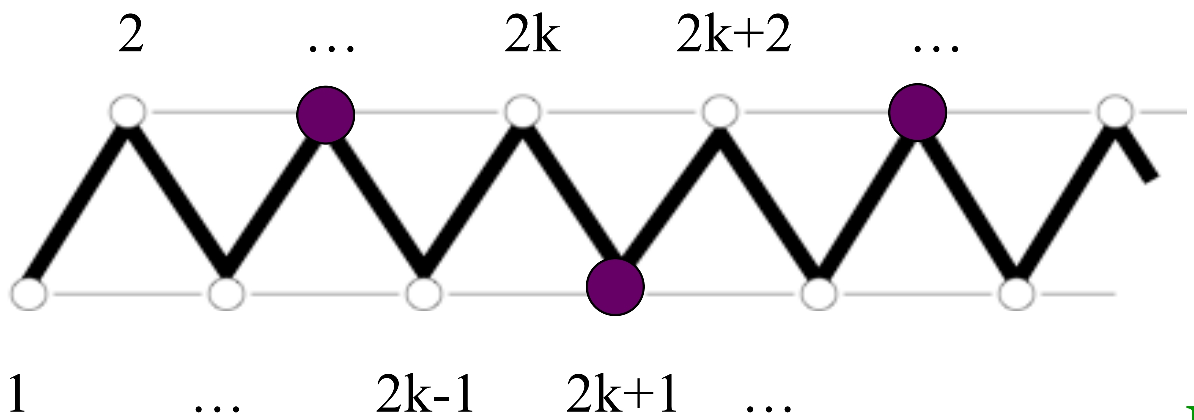
- in addition: extensive degeneracies of excited states

Coupled free fermion chains

supercharge

moves particle from bottom to top chain

$$Q = c_2^\dagger c_1 + \sum_{k=1}^{L-1} \left(e^{i\alpha_{2k-1}\pi/2} c_{2k}^\dagger + e^{i\alpha_{2k}\pi/2} c_{2k+2}^\dagger \right) c_{2k+1}$$

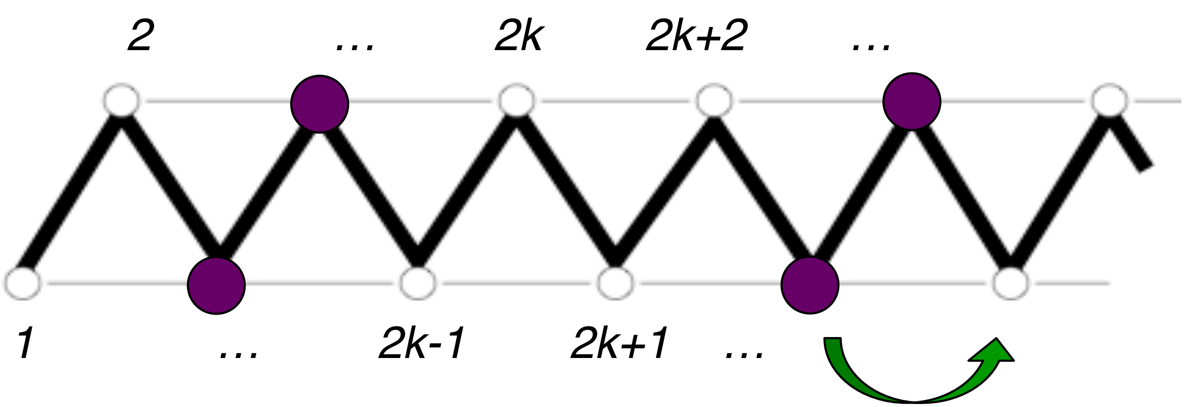


$$\alpha_k \equiv \sum_{j=1}^{k} (-1)^j n_j$$

Fendley-KjS 2007

Coupled free fermion chains

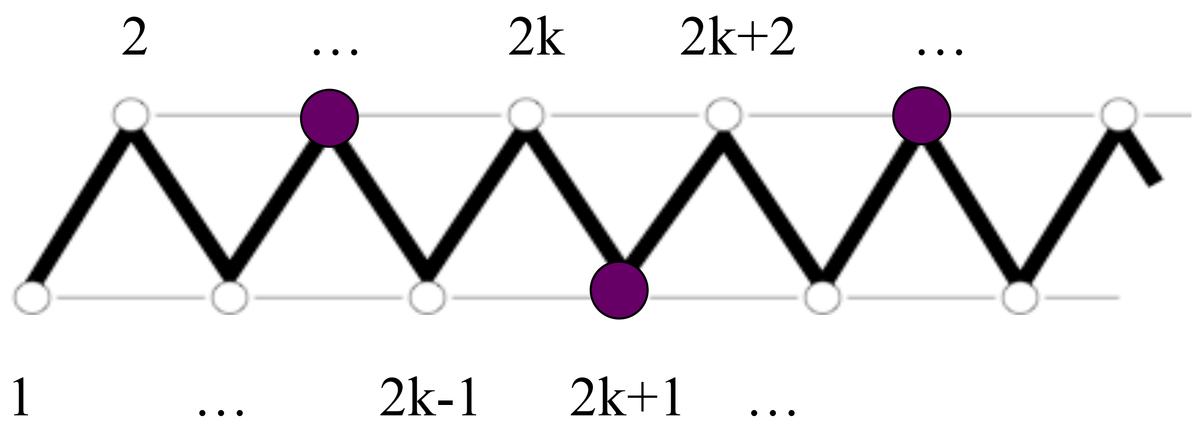
- **coupled free fermion chains**



$$H_c = -\sum_{j=2}^{2L-1}\left[(1-\mathrm{i})c^{\dagger}_{j+1}n_j c_{j-1} + (1+\mathrm{i})c^{\dagger}_{j-1}n_j c_{j+1}\right] - 2\sum_{j=1}^{2L-1} n_j n_{j+1}$$

Fendley-KjS, 2007

Coupled free fermion chains

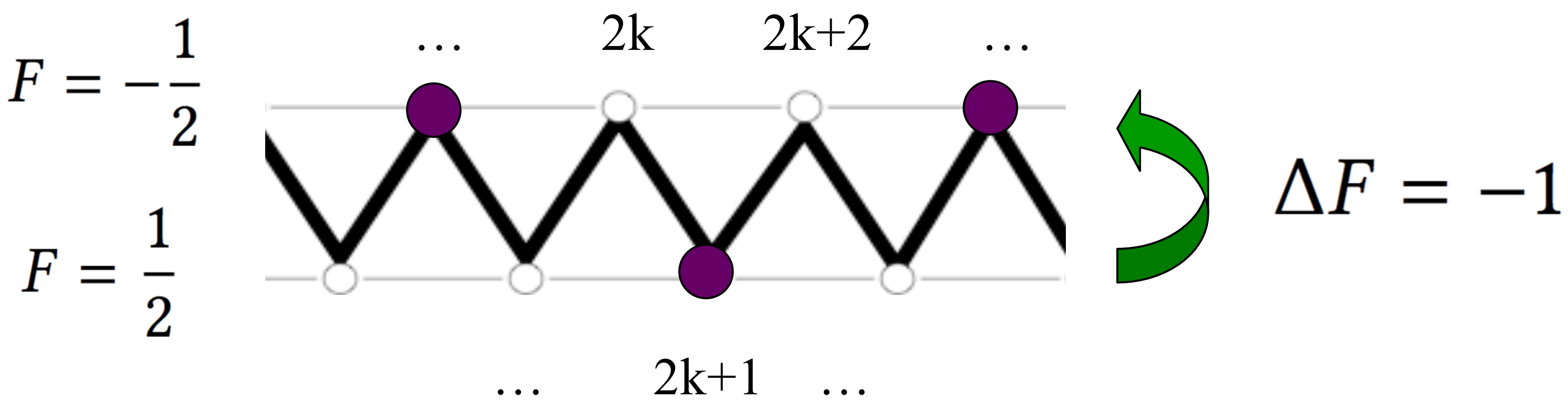


on *L*+*L* sites, OPB :

find 2^L $E=0$ susy groundstates understood via possibility to place up to L tightly bound pairs between top and bottom chains

Coupled free fermion chains

particles on lower (upper) chain have $F = \pm 1/2$ (semions)



Witten index

$$W = \mathrm{Tr}\left[(-1)^F\right] = \prod_1^L (1+i) \prod_1^L (1-i) = 2^L$$

Susy groundstates in `pair models'

- mechanism: condensation of *E=0* `Cooper pairs'
- in Bethe Equations:

$$z_j^L = \pm \mathrm{i}^{-L/2} \prod_{l=1}^{k} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \qquad j = 1, \ldots, m$$

$$1 = \prod_{j=1}^{m} \frac{u_l - (z_j - 1/z_j)^2}{u_l + (z_j - 1/z_j)^2}, \qquad l = 1, \ldots, k.$$

can add pairs of rapidities *(u, -u)* at nested level
without affecting BE at top level

A curious mapping

coupled fermion chain model (FS) and particle-hole symmetric M_1 model (FGNR) turn out to be equivalent

empty ladder :	$\vert 0000\,0000\,0000\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 1100\,1100\,1100\rangle_{FGNR}$
single FS semion :	$\vert 0000\,1000\,0000\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 1100\,0011\,0011\rangle_{FGNR}$
single FS pair :	$\vert 0000\,1100\,0000\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 1100\,0100\,1100\rangle_{FGNR}$
lower leg filled :	$\vert 1010\,1010\,1010\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 0000\,0000\,0000\rangle_{FGNR}$
single FGNR particle :	$\vert 1010\,0110\,1010\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 0000\,1000\,0000\rangle_{FGNR}$
upper leg filled :	$\vert 0101\,0101\,0101\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 1010\,1010\,1010\rangle_{FGNR}$
upper leg plus semion :	$\vert 1101\,0101\,0101\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 0101\,0101\,0101\rangle_{FGNR}$
filled ladder :	$\vert 1111\,1111\,1111\rangle_{FS}$	$\leftrightarrow$	${}_0\vert 0110\,0110\,0110\rangle_{FGNR}$

Feher-Garbali-de Gier-KjS 2017

Twelve Easy Pieces

1. Teaser
2. N=2 susy – Witten index
3. M_1 model in 1D: Witten index, spectra, CFT connection
4. M_1 model: scaling form of 1-pt functions from CFT
5. **M_k models: Witten index, CFT**
6. **M_k models off criticality → massive (integrable) QFT**
7. **M_1 model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$**
8. **PH symmetric model and coupled fermion chains**
9. Superfrustration on ladder: zig-zag, Nicolai, Z_2 Nicolai
10. Superfrustration on 2D grids
11. Back to 1D M_1 model: kink dynamics
12. Towards realization with Rydberg-dressed cold atoms