Supersymmetric Lattice Models

Kareljan Schoutens
University of Amsterdam

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QuSoft
Research Center for Quantum Software

Lecture 1/3

The name of the game

N=2 lattice supersymmetry

$$\left\{Q, Q^{+}\right\} = H$$

this is supersymmetric (susy) quantum mechanics
– space time susy can emerge in scaling limit

We propose and investigate cases susy models for (spinless) fermions on a lattice or (graph) and connect tot susy QFT in a continuum limit

Thanks to susy friends and collaborators ...

Paul Fendley,

Jan de Boer, Bernard Nienhuis,

Hendrik van Eerten,

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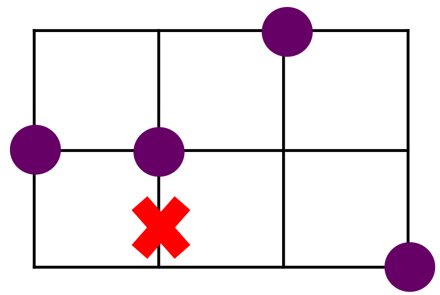
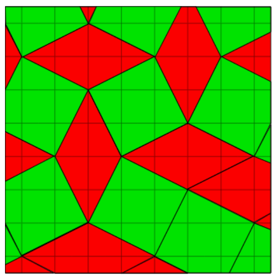
Tristan Kuen, Ruben La, Sergey Shadrin,

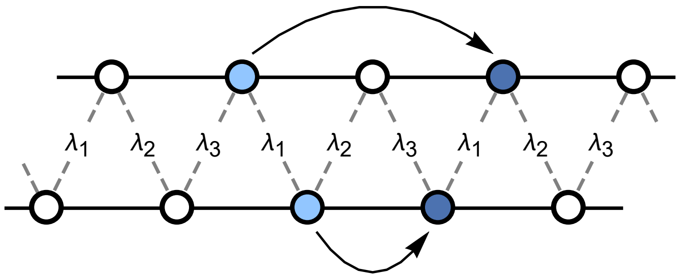
Jasper Dingerink,

Bart van Voorden, Jiri Minar

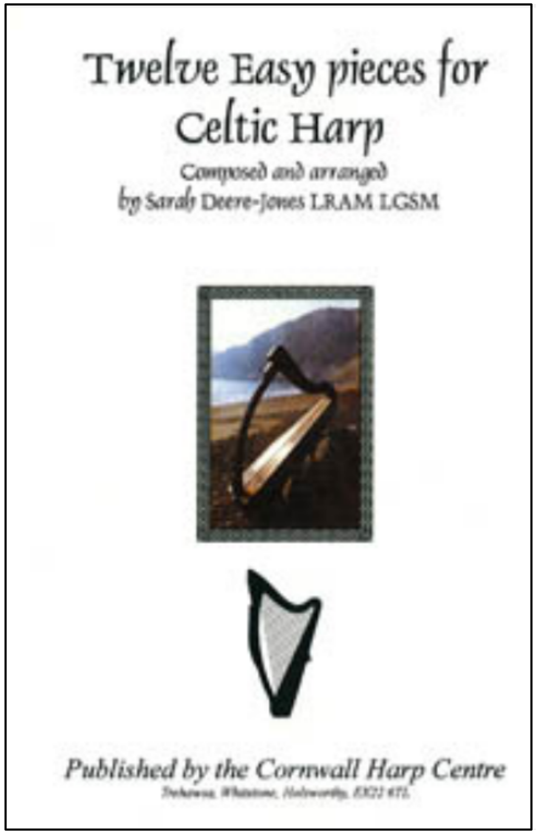
Hosho Katsura, Natalia Chepiga

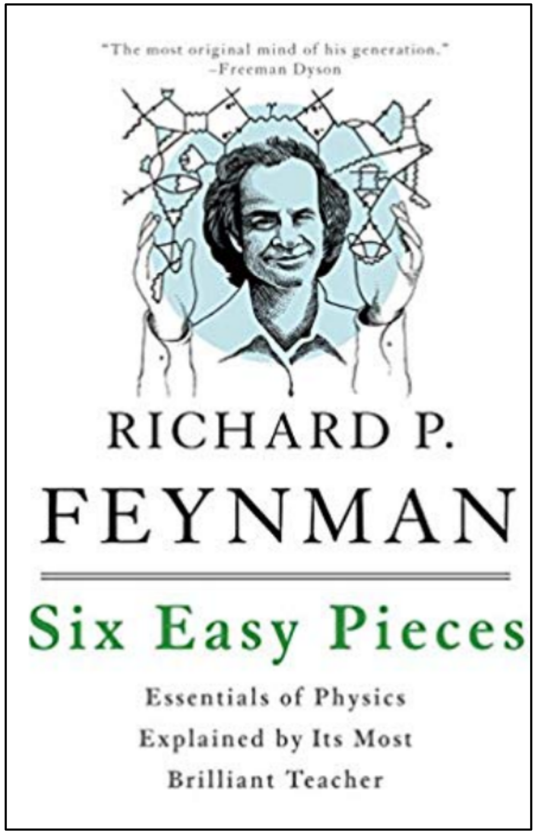


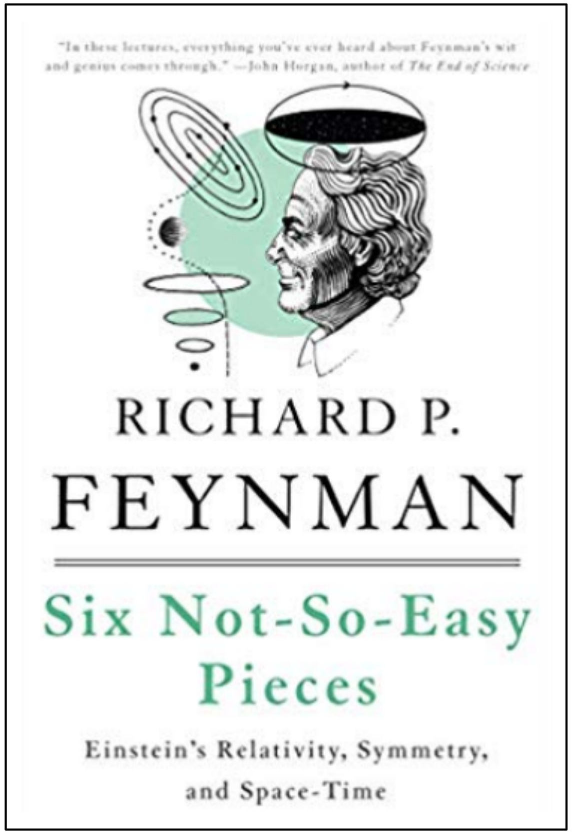




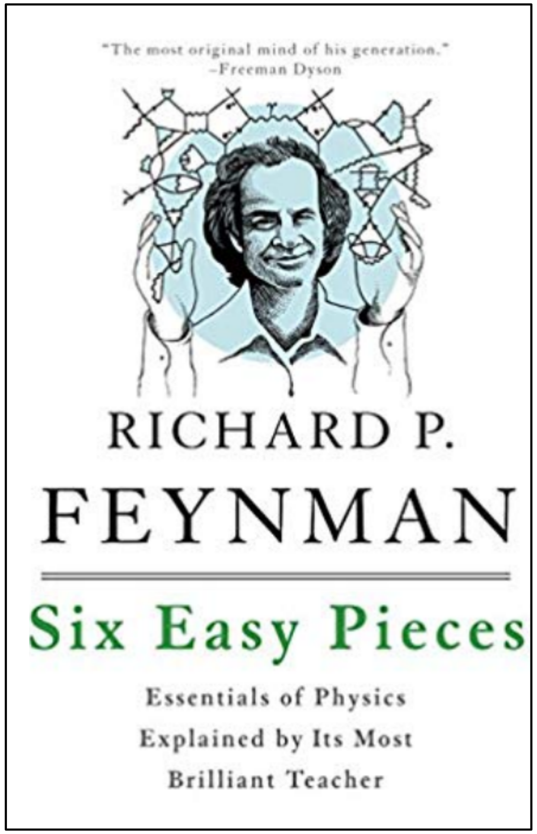
Twelve Easy Pieces







Twelve Easy Pieces



Easy Piece

+ 15 minutes

+ single topic

+ up to 10 slides

+ key idea on board

Twelve Easy Pieces

1. **Teaser**
2. ***N*=2 susy – Witten index**
3. **M_1 model in 1D: Witten index, spectra, CFT connection**
4. **M_1 model: scaling form of 1-pt functions from CFT**
5. **M_k models: Witten index, CFT**
6. **M_k models off criticality → massive (integrable) QFT**
7. **M_1 model on square ladder: CFT, 1-pt functions, $<\sigma\sigma\sigma\sigma>$**
8. **PH symmetric model and coupled fermion chains**
9. **Superfrustration on ladder: zig-zag, Nicolai, Z_2 Nicolai**
10. **Superfrustration on 2D grids**
11. **Back to 1D M_1 model: kink dynamics**
12. **Towards realization with Rydberg-dressed cold atoms**

Twelve easy pieces

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N=2 supersymmetry

QM with *N*=2 supersymmetry

$$Q^2 = 0, \qquad (Q^\dagger)^2 = 0$$

$$[Q,H] = 0, \qquad H = \{Q,Q^\dagger\}, \qquad [Q^\dagger,H] = 0$$

[not to be confused with graded Lie algebra
symmetries such as `supersymmetric *tJ*-model']

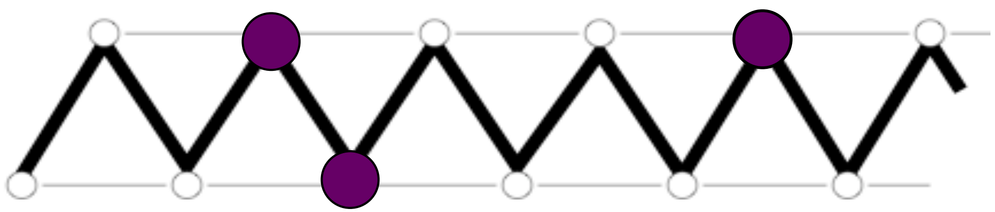
Supersymmetric lattice models

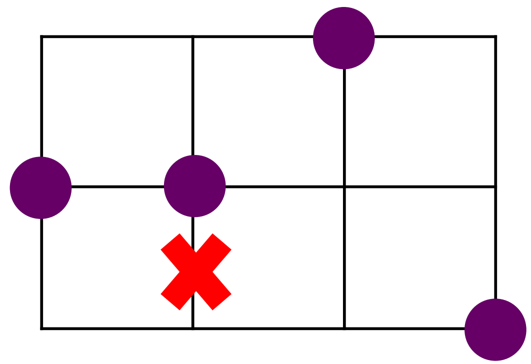
susy QM for lattice fermions

$$\{c_i, c_j^\dagger\} = \delta_{ij}, \qquad i,j \in \Lambda$$

supercharges expressed in fermion operators

➽ Hamiltonians with kinetic (hopping) terms and strong interactions

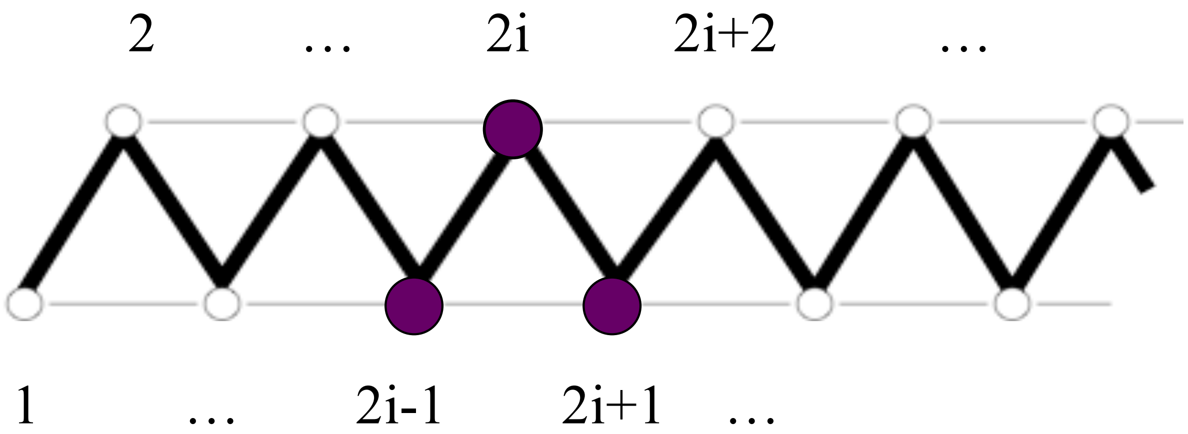




Nicolai model

supercharge

$$Q^{\mathrm{Nic}} = \sum_i c_{2i-1} c^{\dagger}_{2i} c_{2i+1}$$

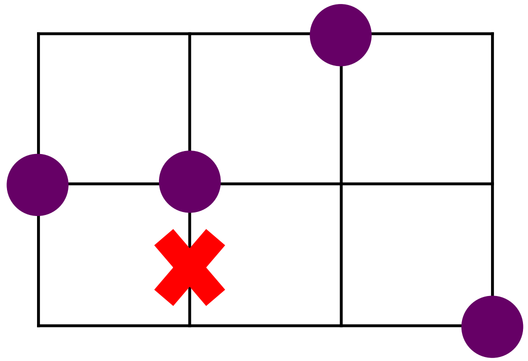


Nicolai 1976

M_1 model

configurations:

lattice fermions with nearest neighbor exclusion



supercharge

takes out particle where possible

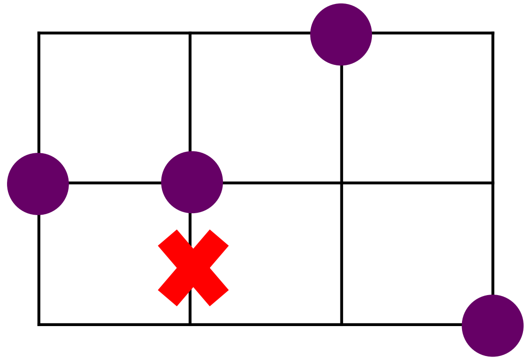
$$Q^{\mathrm{M}_1} = \sum_i c_i P_i, \qquad P_i = \prod_{<ij>} (1 - c_j^\dagger c_j)$$

Fendley-KjS-de Boer 2003

Features

features of

susy lattice models



- integrability
- critical behaviour ➟ supersymmetric CFT
- off-critical ➟ kink picture of integrable susy QFT
- superfrustration ➟ proliferation of susy ground states
- dynamics, I ➟ adiabatic driving in susy gs manifold
- dynamics, II ➟ out-of-equilibrium transport, MBL?

Susy groundstates in M_1 model

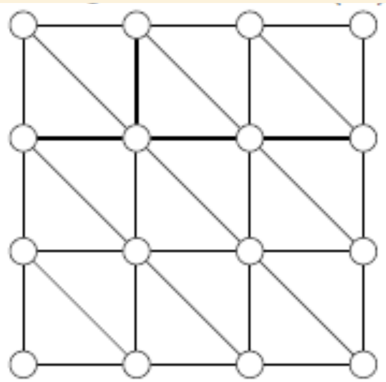
susy groundstates (M_1 model):

- 1D closed chain, $L=3l$: $\boldsymbol{G = 2}$

- 2D square lattice, periodic BC, $N \times N$ sites
  $\boldsymbol{G = 1, 1, 4, 7, 9, 14, 1, 7, 40, 9, 1, 166, \ldots}$

- 2D triangular lattice, periodic BC, $N \times N$ sites
  $\boldsymbol{G_{6\times 6} \geq 102}$, $\boldsymbol{G_{8\times 8} \geq 881}$, $\boldsymbol{G_{10\times 10} \geq 950592}$, etc.

M_1 model, 2D triangular lattice

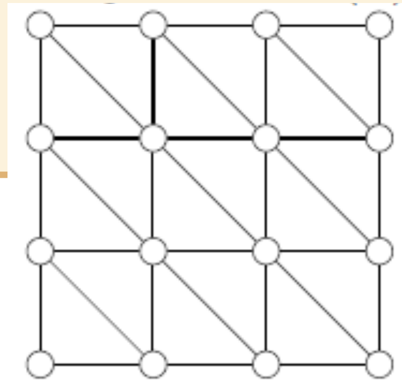


Witten index for $N \times M$ sites with periodic BC

	1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1	1
2	1	-3	-5	1	11	9	-13	-31	-5	57
3	1	-5	-2	7	1	-14	1	31	-2	-65
4	1	1	7	-23	11	25	-69	193	-29	-279
5	1	11	1	11	36	-49	211	-349	811	-1064
6	1	9	-14	25	-49	-102	-13	-415	1462	-4911
7	1	-13	1	-69	211	-13	-797	3403	-7055	5237
8	1	-31	31	193	-349	-415	3403	881	-28517	50849
9	1	-5	-2	-29	881	1462	-7055	-28517	31399	313315
10	1	57	-65	-279	-1064	-4911	5237	50849	313315	950592
11	1	67	1	859	1651	12607	32418	159083	499060	2011307
12	1	-47	130	-1295	-589	-26006	-152697	-535895	-2573258	-3973827
13	1	-181	1	-77	-1949	67523	330331	-595373	-10989458	-49705161
14	1	-87	-257	3641	12611	-139935	-235717	5651377	4765189	-232675057
15	1	275	-2	-8053	-32664	272486	-1184714	-1867189	134858383	-702709340

van Eerten 2005

M_1 model, 2D triangular lattice



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15	1	275	-2	-8053	-32664	134858383	-702709340

\`superfrustration'

van Eerten 2005

Twelve easy pieces

1. Teaser

2. **N=2 susy – Witten index**

3. M_1 model in 1D: Witten index, spectra, CFT connection

4. M_1 model: scaling form of 1-pt functions from CFT

Basic structure of susy spectra

- $\boldsymbol{E \geq 0}$ for all states

- $\boldsymbol{E > 0}$ states are paired into **doublets**

$$\{|\psi\rangle, Q^\dagger|\psi\rangle\}, \quad Q|\psi\rangle = 0$$

- $\boldsymbol{E = 0}$ iff a state is a **singlet** under supersymmetry

$$Q|\psi_{\text{gs}}\rangle = 0, \quad Q^\dagger|\psi_{\text{gs}}\rangle = 0$$

Fermion number and Witten index

Supercharges change fermion number F by ± 1

$$[F, Q] = -Q, \quad [F, Q^\dagger] = Q^\dagger$$

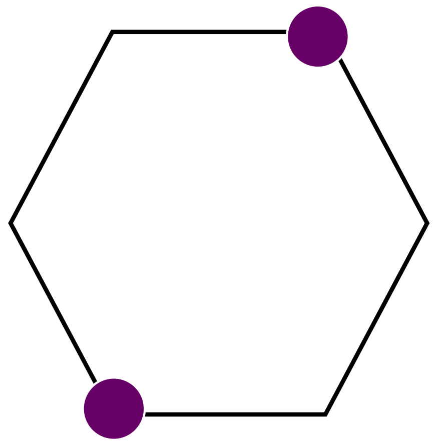
Witten index

$$W = \mathrm{Tr}\left[(-1)^F\right]$$

- W easily evaluated by computing trace over all states
- $E > 0$ doublets cancel in W, only $E = 0$ singlets contribute
- $W \neq 0$ implies existence of **at least** $|\boldsymbol{W}|$ $E = 0$ singlets

M_1 model on 6 site chain

$$W = \mathrm{Tr}\left[(-1)^F\right]$$



$F = 0$: 1 state

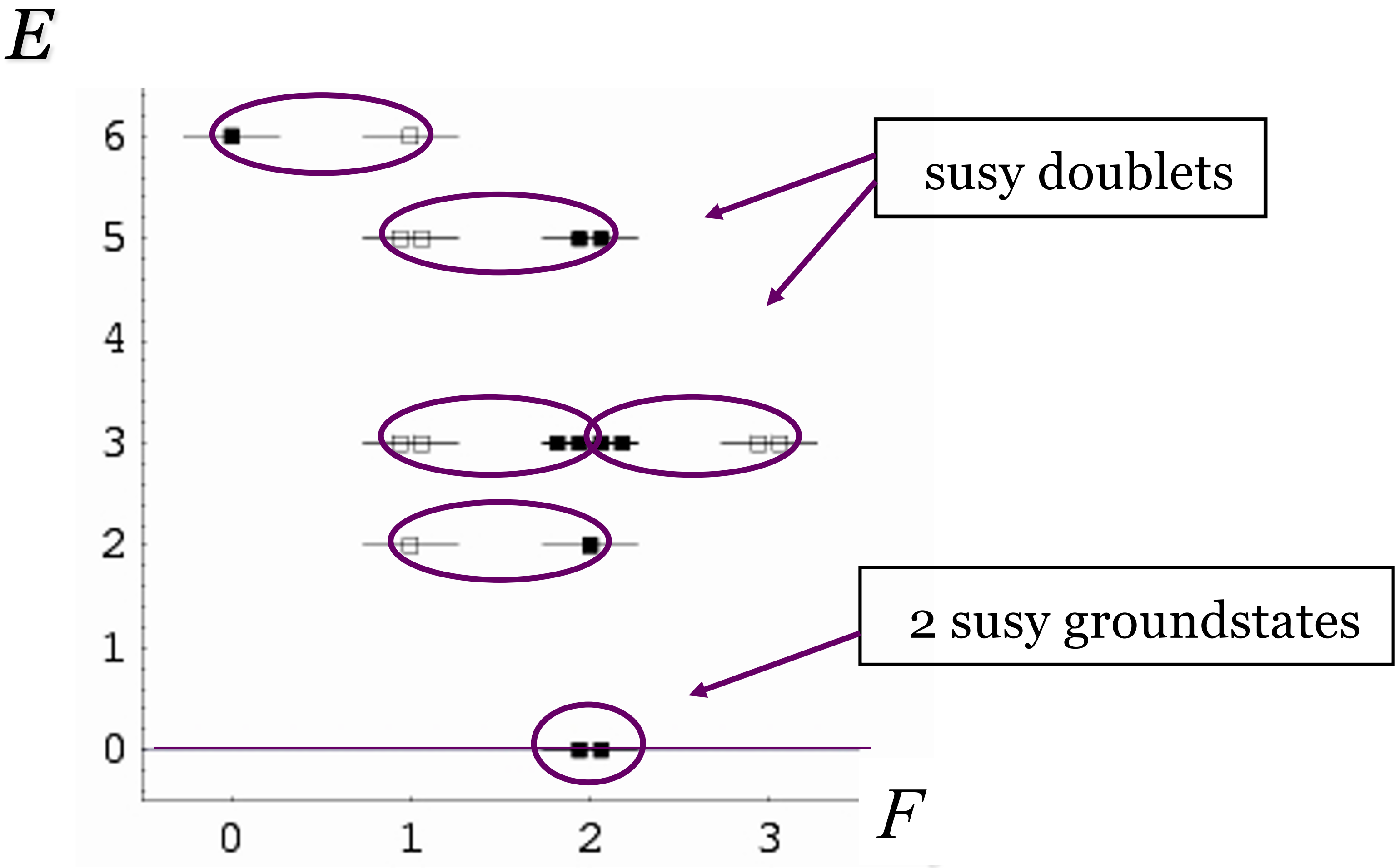
$F = 1$: 6 states

$F = 2$: 9 states

$F = 3$: 2 states

$\Rightarrow W = 1 - 6 + 9 - 2 = 2$

M_1 model, 6 site chain, W=2



Finding groundstates: cohomology

+ supercharges define complex

$$\mathcal{H} = \oplus C_n;$$

$$C_0 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_1 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_2 \underset{Q^\dagger}{\overset{Q}{\rightleftarrows}} C_3 \ldots$$

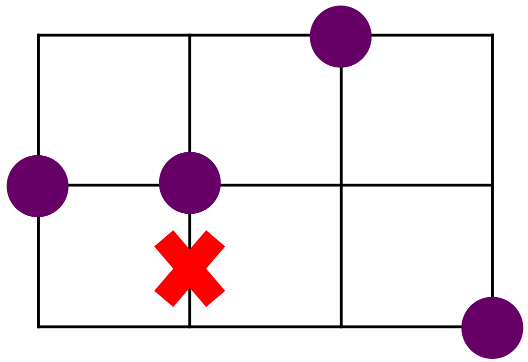
+ can show that susy groundstates are 1-1 with cohomology classes

→ advanced math techniques available

M_1 model on general graph - challenge

configurations:

lattice fermions with nearest neighbor exclusion



supercharge

takes out particle where possible

$$Q^{M_1} = \sum_i c_i P_i , \qquad P_i = \prod_{<ij>} (1 - c_j^\dagger c_j)$$

Fendley-KjS-de Boer 2003

M_1 model on general graph - challenge

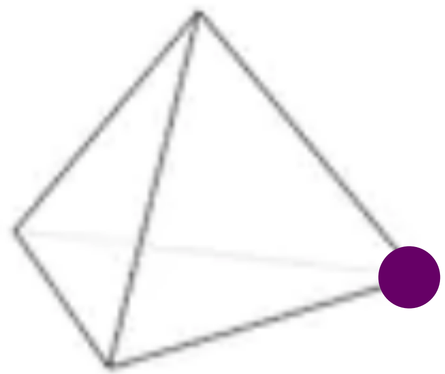
Example of tetrahedron:

+ Witten index : W=1-4=-3

+ back-of-the envelope:

  + one $E=4$ susy doublet

  + three $E=0$ susy singlets



M_1 model on general graph - challenge

Challenge:

For each of the other platonic solid graphs, find the number of *E=0* susy groundstates and, for each of them, the number of fermions

Reward

A chocolate bar for the first correct result @ c.j.m.schoutens@uva.nl

Twelve easy pieces

1. Teaser
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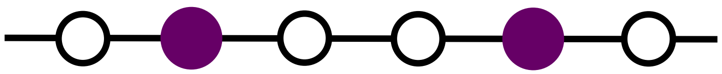
4. M_1 model: scaling form of 1-pt functions from CFT

M_1 model on 1D lattice

configurations

lattice fermions with nearest neighbor exclusion



supercharge and Hamiltonian

$$Q^{\mathrm{M}_1} = \sum_i (1-n_{i-1})c_i(1-n_{i+1}), \qquad n_i = c_i^\dagger c_i$$

n.n. exclusion

$$H^{\mathrm{M}_1} = \sum_i \left[(1-n_{i-1})c_i^\dagger c_{i+1}(1-n_{i+2}) + \mathrm{h.c.}\right] + \sum_i n_{i-1}n_{i+1} - 2F + L$$

hopping

n.n.n. repulsion

Susy lattice model M_1 , $D=1$

Closed chain

$W=2$ for $L=3l$ sites → find 2 groundstates at filling $f/L=1/3$

Open chain

$W=\pm 1$ for $L=3l, 3l-1$; $W=0$ for $L=3l+1$

Analysis: Bethe Ansatz, mapping to XXZ @ $\Delta=-1/2$

Numerics: ED, finite size spectra

Find that M_1 model is **critical** → CFT

Susy lattice model M_1 , *D=1*

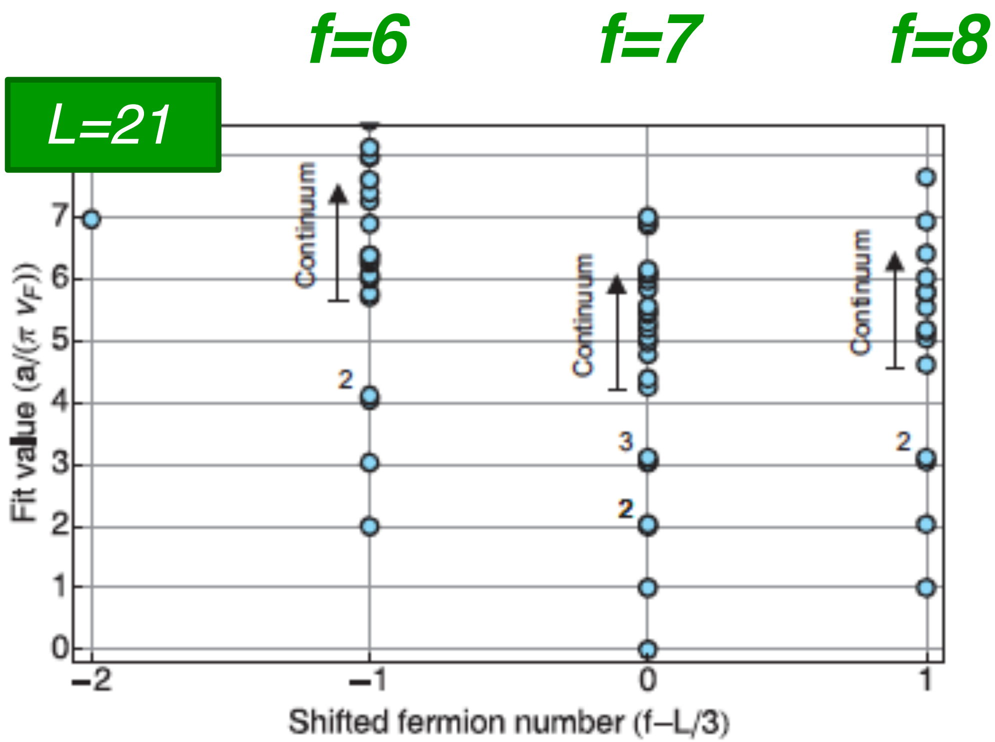
CFT for open chain spectra

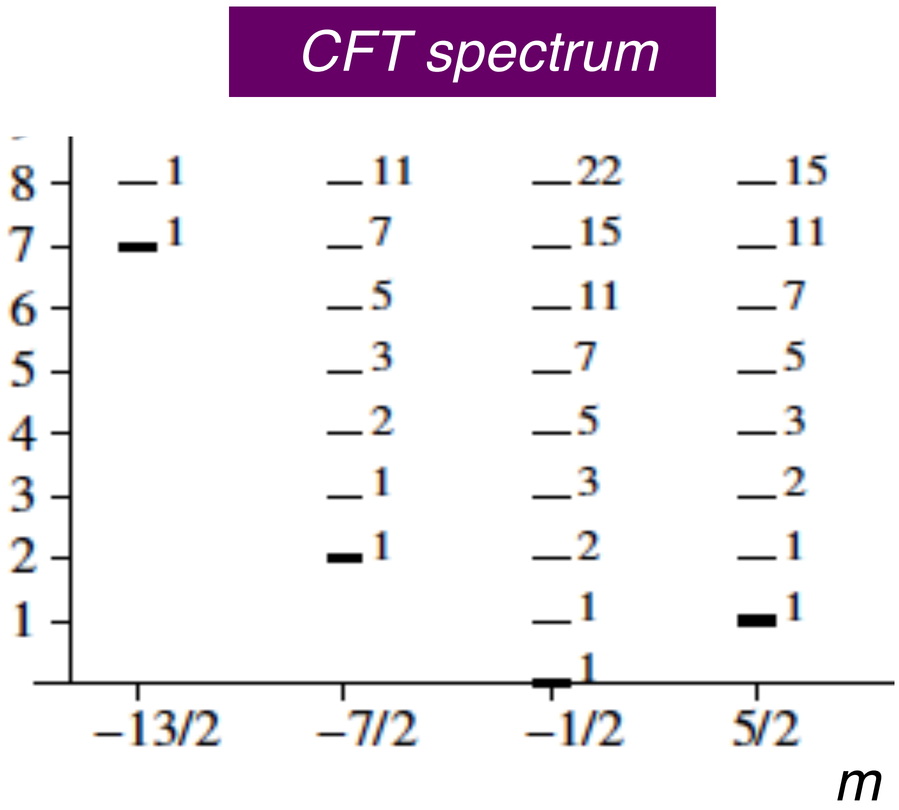
what CFT? → *k=1* minimal model of
N=2 superconformal field theory

c=1, write using single scalar field Waterson, 1986

Susy lattice model M_1 , open chain

Huijse, 2010

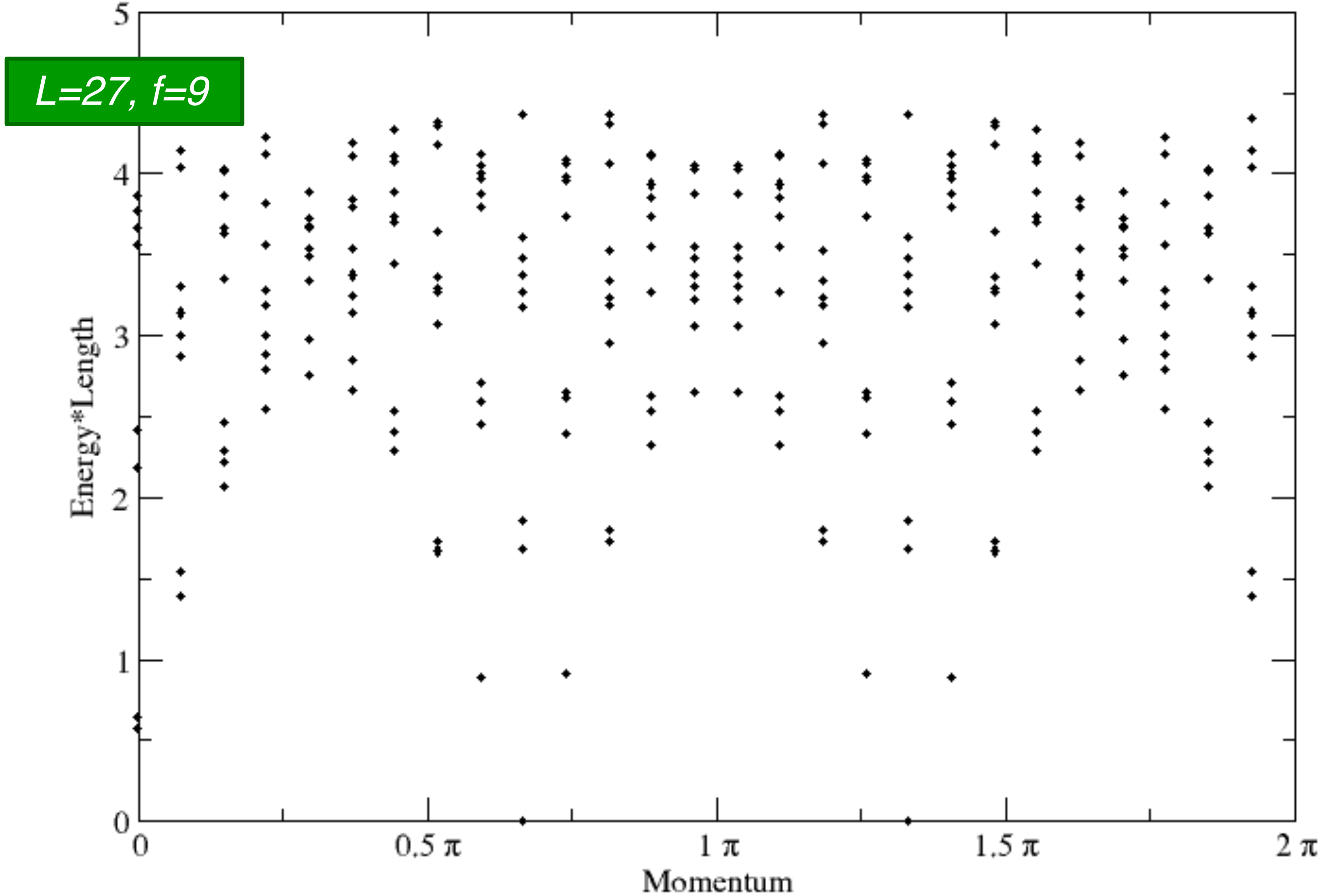




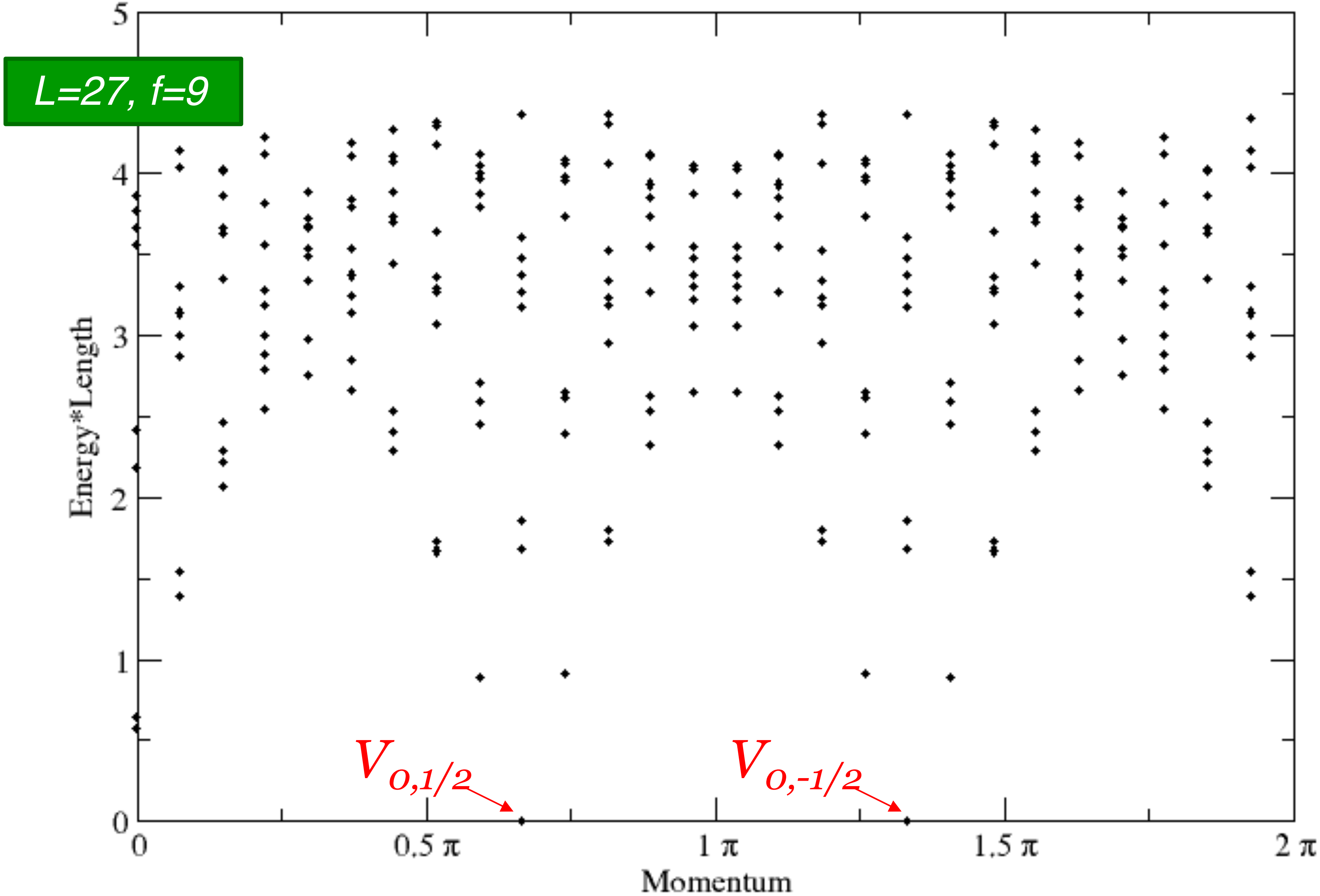
Ramond-sector affine *U(1)* modules built on charge *m* vertex operator V_m

$m = 3f - L - \frac{1}{2}$

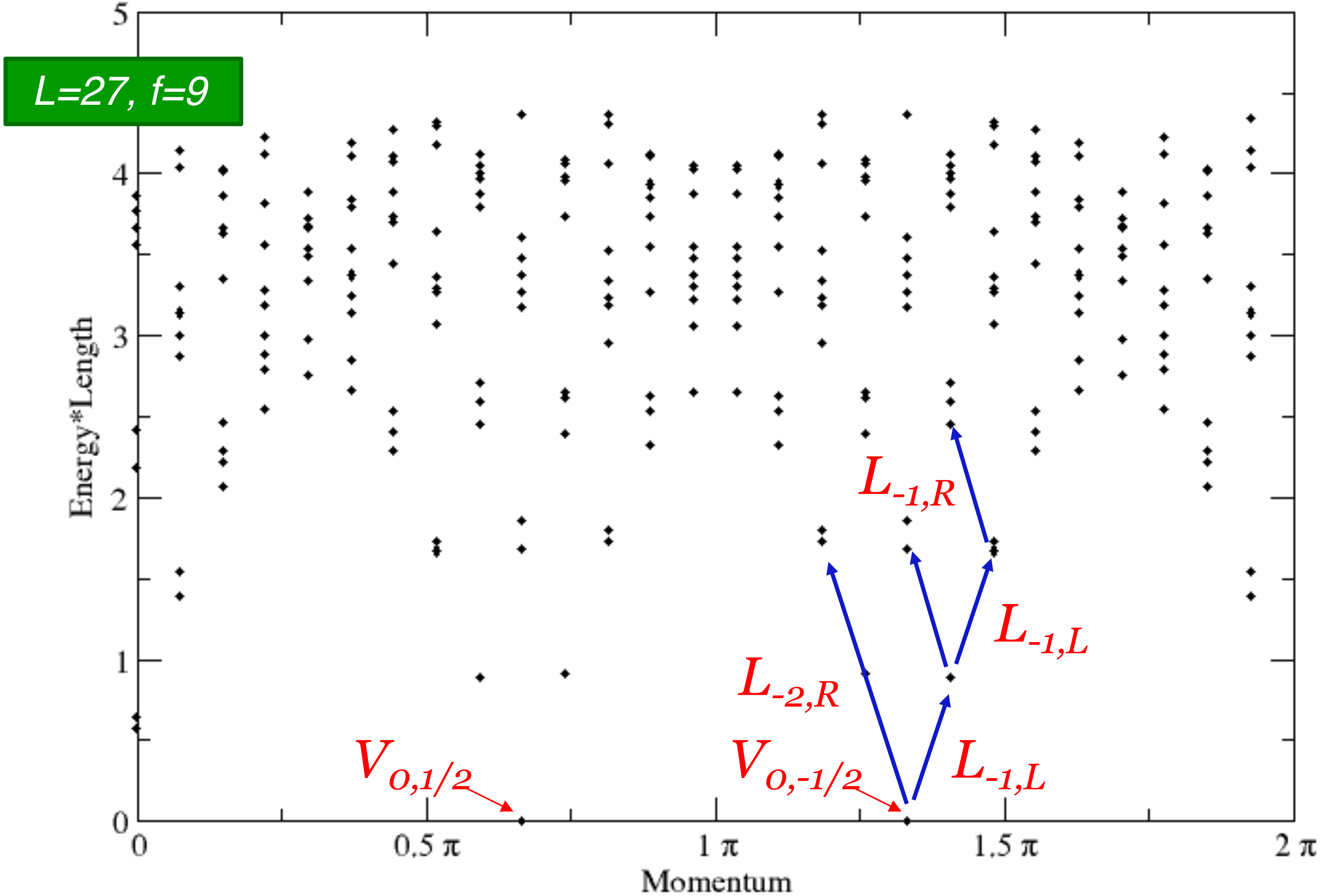
Susy lattice model M_1 , closed chain



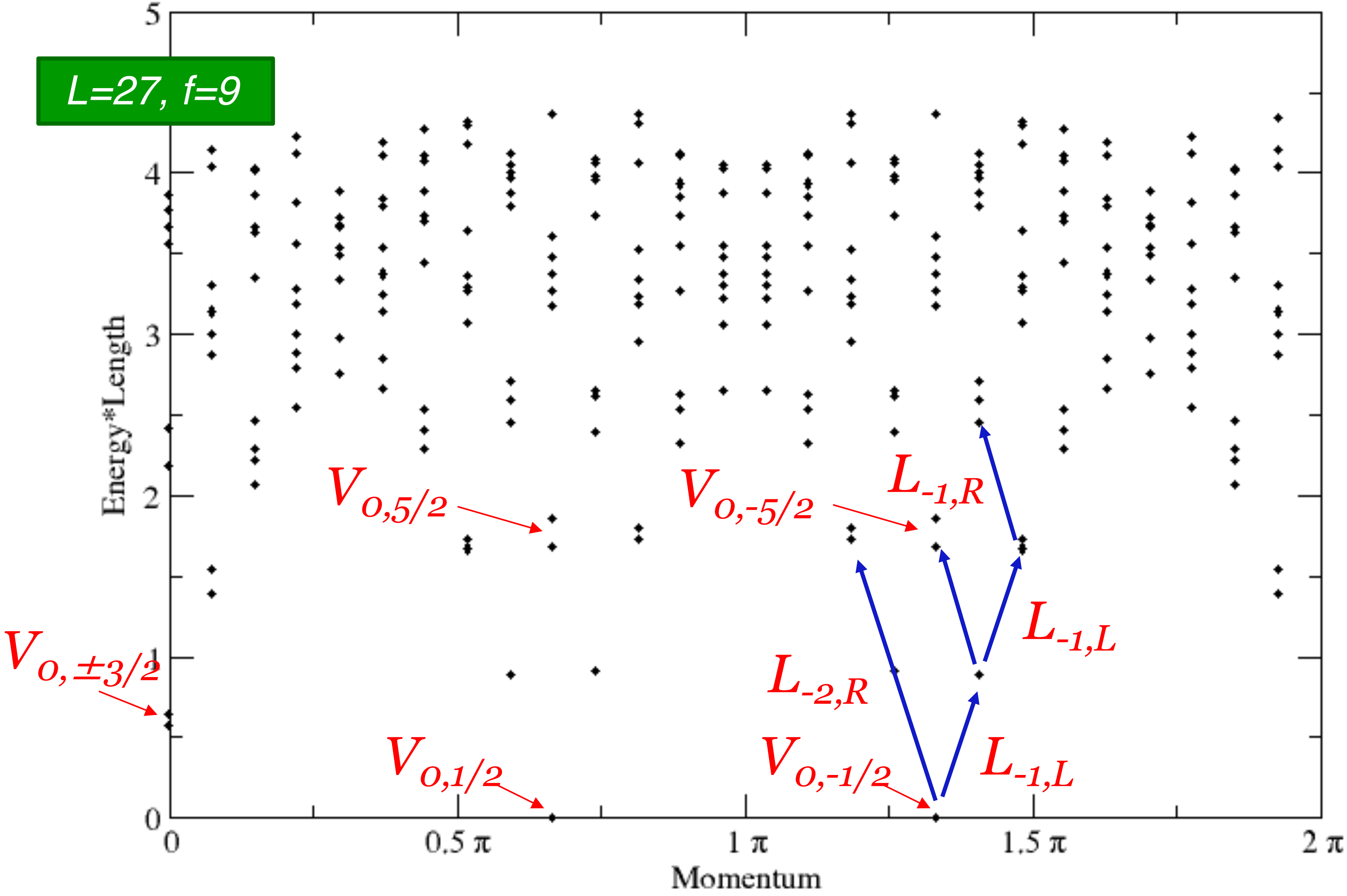
Susy lattice model M_1 , closed chain



Susy lattice model M_1 , closed chain

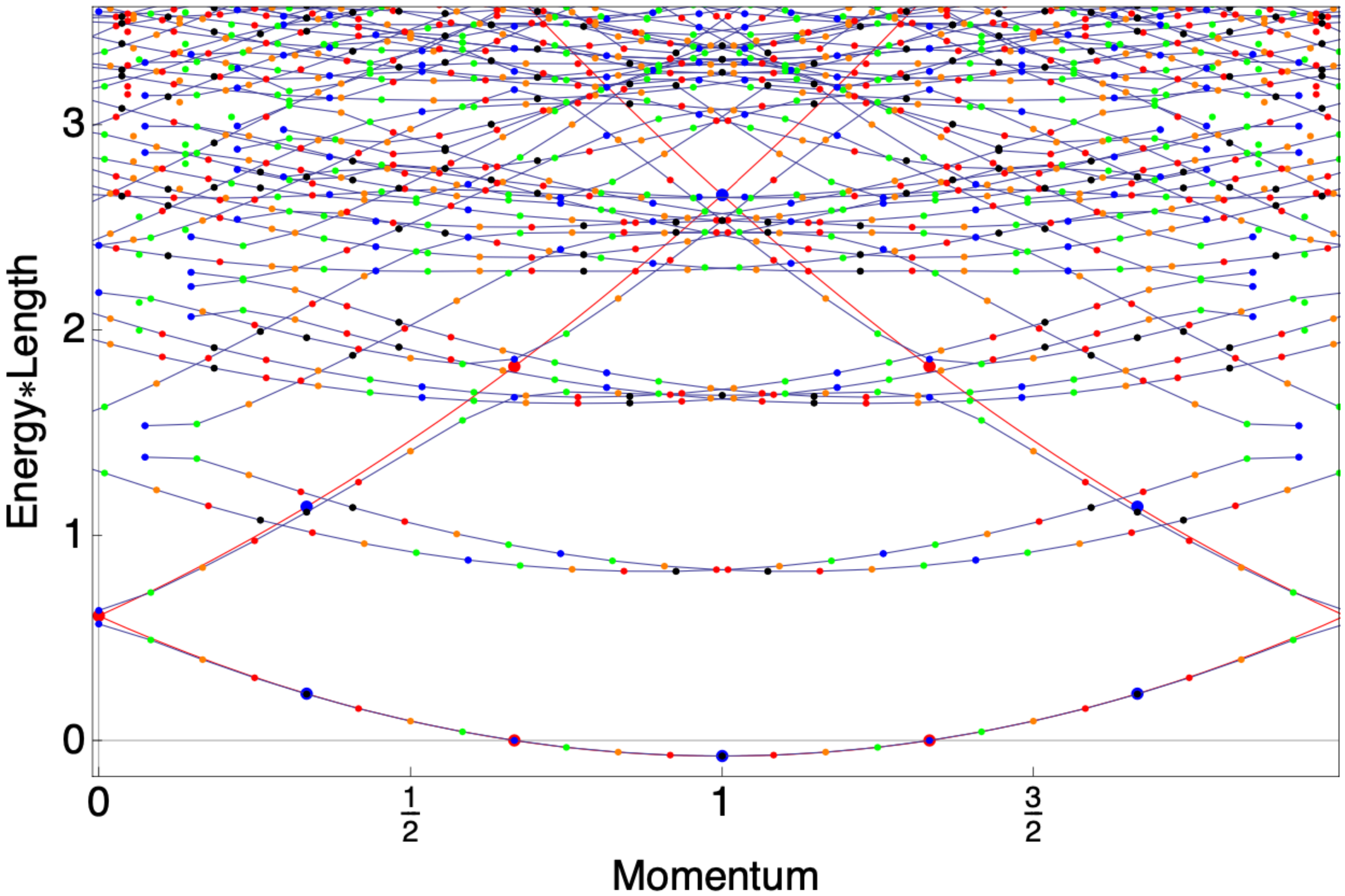


Susy lattice model M_1 , closed chain



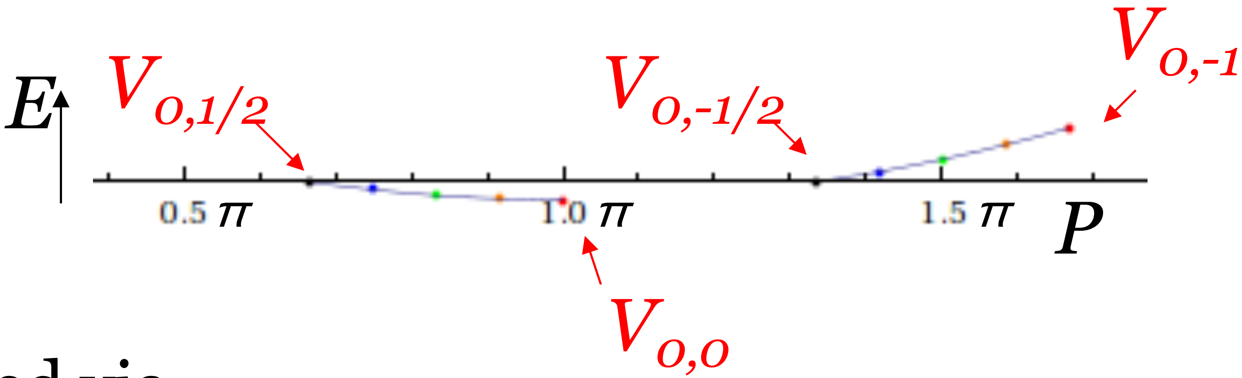
Susy lattice model M_1 , closed chain

L=27, f=9



Susy lattice model M_1 , closed chain

- twist bc
- trace out (E,P)
- SCFT data extracted via parabolic fits to (E,P)



L=27, f=9

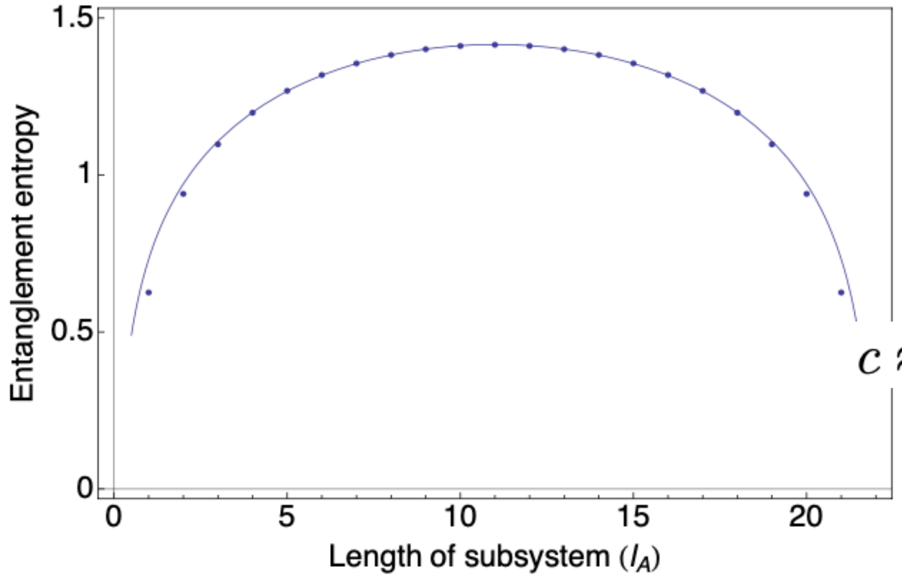
numerics

sector	E/c	Q_0/c	c*v_F
R	0	-0.334	3.92
NS	-0.083	0	3.92
R	0	0.342	3.89
NS	0.254	0.675	3.89

SCFT (c=1)

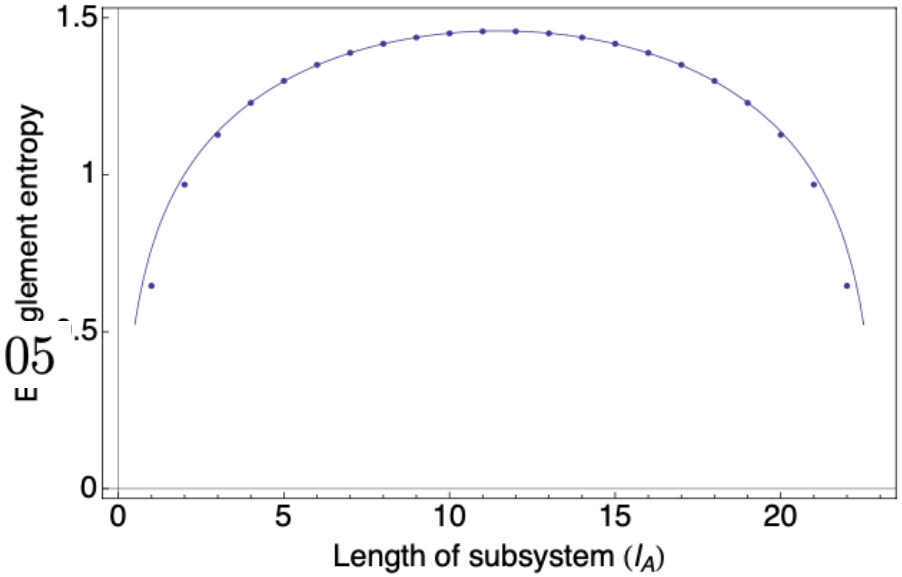
state	E	Q_0
$V_{0,1/2}$	0	-1/3
$V_{0,0}$	-1/12	0
$V_{0,-1/2}$	0	1/3
$V_{0,-1}$	1/4	2/3

Susy lattice model M_1 , closed chain, S_E



(a) $L = 22$ and $f = 7$.

$c \approx 1.05$



(b) $L = 23$ and $f = 8$.

$$S(l_A) = \frac{c}{3}\ln\left(\frac{L}{\pi}\sin\left(\frac{l_A\pi}{L}\right)\right) + b,$$

$c \approx 1.05$

Susy lattice model M_1 , staggering

+ to break criticality, one can stagger the amplitudes in the supercharge with position dependent factors λ_i

+ for λ_i periodic with period 3 the staggered model is still integrable, eg

$$\ldots\ \lambda\ 1\ 1\ \lambda\ 1\ 1\ \lambda\ 1\ 1\ \ldots$$

+ ground states at λ close to 0 are (generalized) product states → kinks as leading low energy excitations

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4. M_1 model: scaling form of 1-pt functions from CFT

Susy lattice model M_1 , open chain

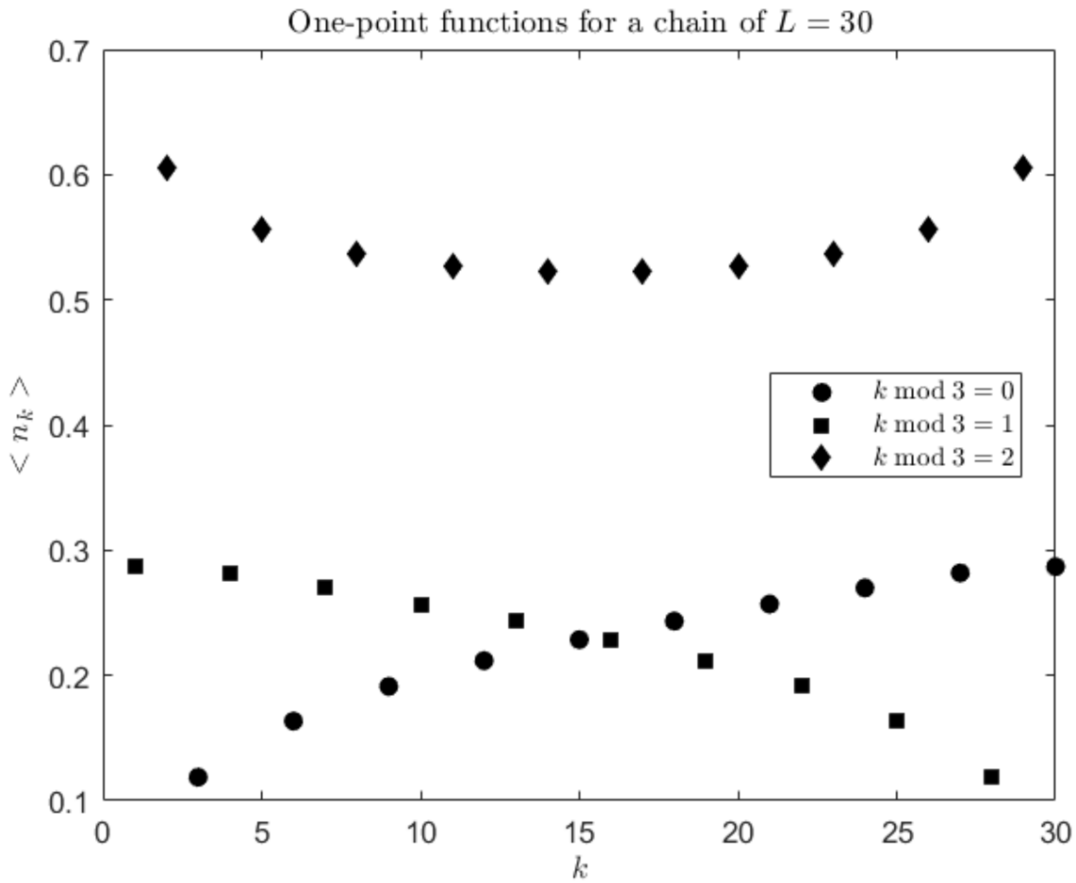


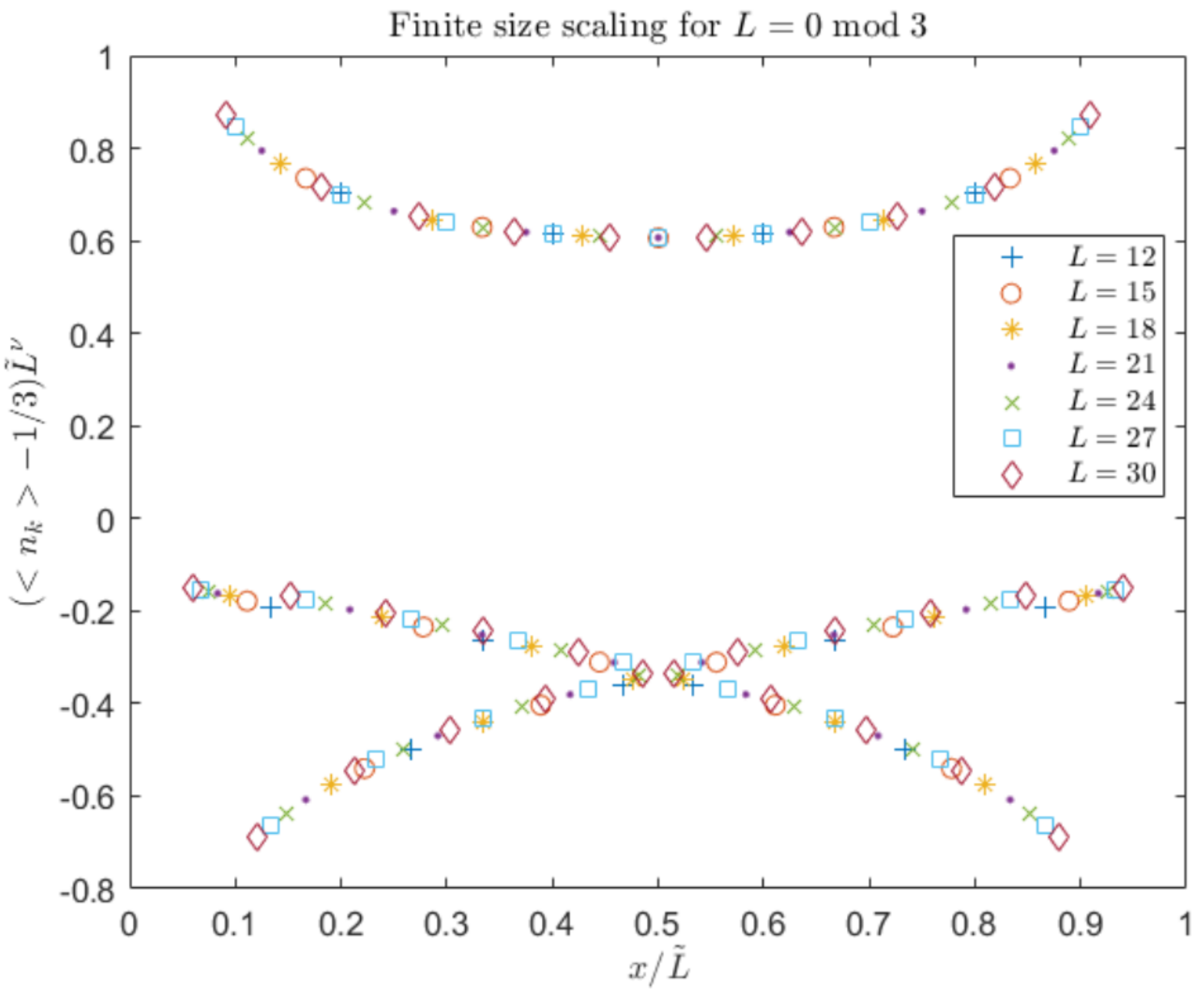
Figure 5.2: Example of the $\mathbb{Z}_3$ behaviour for the one-point functions $\langle n_k \rangle$ for a chain of $L = 30$. There are three branches, one for each value of k mod 3.

numerics: MPS

Dingerink,

MSc thesis 2019

Susy lattice model M_1 , open chain



(a) Finite size scaling for $L \bmod 3 = 0$, with a scaling of $\nu = 1/3$.

numerics: MPS

Dingerink,
MSc thesis 2019

Susy lattice model M$_1$, open chain

Step I

express density operator in CFT fields

$$
\begin{aligned}
n_k &= \frac{1}{3} + \frac{1}{3}\left(A_1 V_1 + A_{-1} V_{-1}\right), \\
n_{k+1} &= \frac{1}{3} + \frac{1}{3}\left(e^{2\pi i/3} A_1 V_1 + e^{-2\pi i/3} A_{-1} V_{-1}\right), \\
n_{k-1} &= \frac{1}{3} + \frac{1}{3}\left(e^{-2\pi i/3} A_1 V_1 + e^{2\pi i/3} A_{-1} V_{-1}\right).
\end{aligned}
$$

Huijse, 2010

Susy lattice model M$_1$, open chain

Step II

determine density operator 1-pt functions in CFT

$$
\begin{aligned}
\langle V_1(x)\rangle_{\text{strip}} &= \lim_{z_1\to\infty, z_4\to 0} \left(\frac{i\pi}{\tilde{L}}\right)^{1/3} (-z_2 z_3)^{1/6} \langle 0|V_{1/2}(z_1) z_1^{1/12} V_1(z_2) V_{-1}(z_3) V_{-1/2}(z_4)|0\rangle \\
&= (-1)^{1/6} \left(\frac{i\pi}{\tilde{L}}\right)^{1/3} z_3^{1/3} (z_2 - z_3)^{-1/3} \\
&= (-1)^{1/6} \left(\frac{\pi}{2\tilde{L}}\right)^{1/3} \frac{e^{-i\pi x/(3\tilde{L})}}{\sin^{1/3}(\pi x/\tilde{L})} \\
&= \left(\frac{\pi}{2\tilde{L}}\right)^{1/3} \frac{e^{-i\pi (x-\tilde{L}/2)/(3\tilde{L})}}{\sin^{1/3}(\pi x/\tilde{L})}.
\end{aligned}
$$

Huijse, 2010

Susy lattice model M_1 , open chain

Step III

wrap up

$$\langle n_k \rangle = \frac{1}{3} + \frac{2A_1}{3}\left(\frac{\pi}{2\tilde{L}}\right)^{1/3} \frac{\cos\left[\pi(x-\tilde{L}/2)/(3\tilde{L})\right]}{\sin^{1/3}(\pi x/\tilde{L})},$$

$$\langle n_{k+1} \rangle = \frac{1}{3} + \frac{2A_1}{3}\left(\frac{\pi}{2\tilde{L}}\right)^{1/3} \frac{\sin\left[\pi(x-\tilde{L})/(3\tilde{L})\right]}{\sin^{1/3}(\pi x/\tilde{L})},$$

$$\langle n_{k-1} \rangle = \frac{1}{3} - \frac{2A_1}{3}\left(\frac{\pi}{2\tilde{L}}\right)^{1/3} \frac{\sin\left[\pi x/(3\tilde{L})\right]}{\sin^{1/3}(\pi x/\tilde{L})}.$$

Huijse, 2010

Susy lattice model M_1, open chain

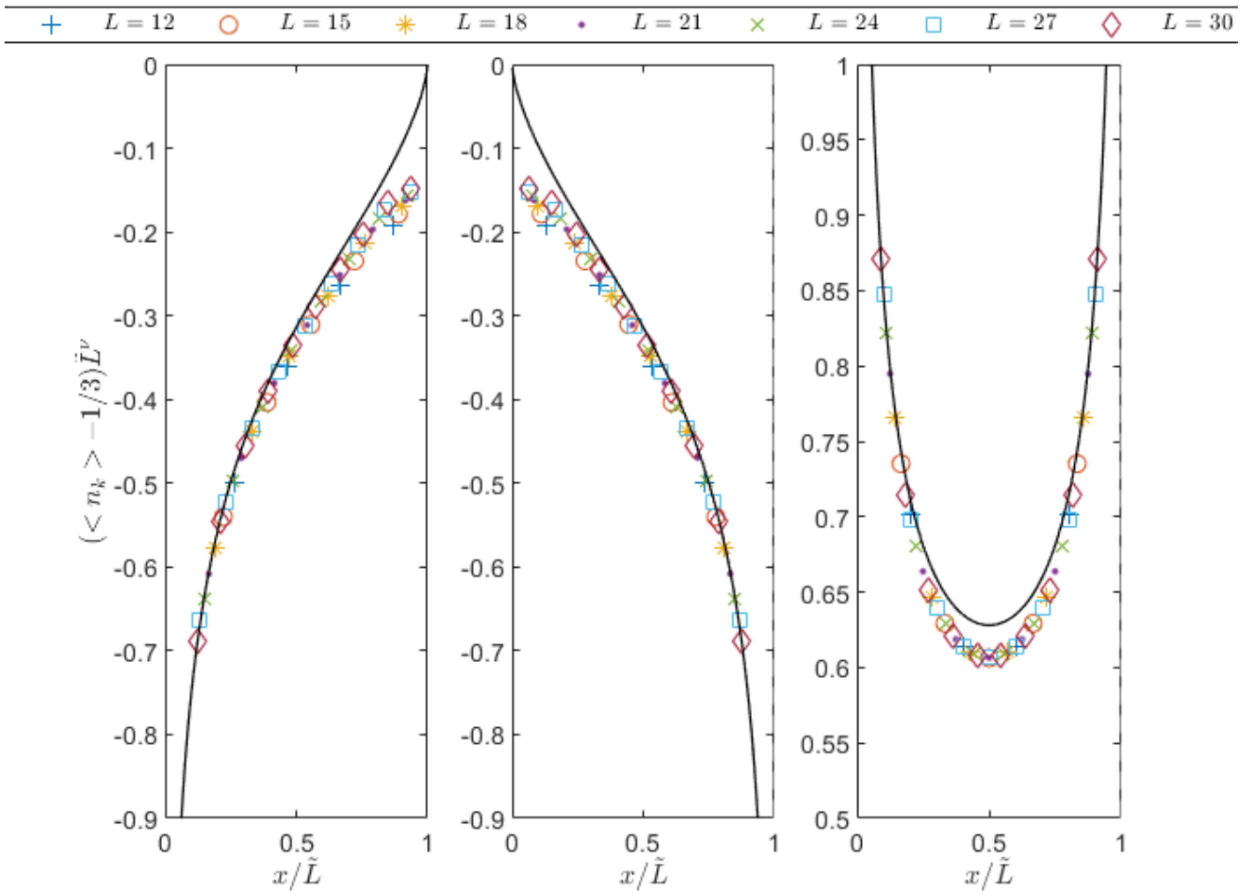


Figure 5.4: The one-point functions $\langle n_k \rangle$ of chains of length $L \bmod 3 = 0$ for $k \bmod 3 = 0$ (left), $k \bmod 3 = 1$ (middle) and $k \bmod 3 = 2$ (right). For each of the three cases the corresponding equation (5.23) was fitted (solid line) with a fit parameter of $A_1 = 0.810$. The x on the x-axis is related to lattice position k by $x = k + 1$.

Dingerink,

MSc thesis 2019

Twelve Easy Pieces

1. **Teaser**
2. ***N*=2 susy – Witten index**
3. **M_1 model in 1D: Witten index, spectra, CFT connection**
4. **M_1 model: scaling form of 1-pt functions from CFT**
5. M_k models: Witten index, CFT
6. M_k models off criticality → massive (integrable) QFT
7. M_1 model on square ladder: CFT, 1-pt functions, $\langle\sigma\sigma\sigma\sigma\rangle$
8. PH symmetric model and coupled fermion chains
9. Superfrustration on ladder: zig-zag, Nicolai, Z_2 Nicolai
10. Superfrustration on 2D grids
11. Back to 1D M_1 model: kink dynamics
12. Towards realization with Rydberg-dressed cold atoms