Entanglement spectrum in non-Hermitian systems

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Princeton University

1. Introduction to non-Hermitian systems
2. Topology in nH systems
3. Entanglement spectrum in nH systems

Non-Hermiticity as a simple model for dissipation

- To represent dissipation, typically use master equations

Lindblad, Commun. Math. Phys. 48 (1976)

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$$\frac{d\rho}{dt} = -i[\mathcal{H}, \rho] + \sum_n 2L_n \rho L_n^\dagger - L_n^\dagger L_n \rho - \rho L_n^\dagger L_n$$

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- Lindblad mechanism + postselection of no events trajectories
  $\frac{d\rho}{dt} = -i[\mathcal{H}, \rho] + \sum_n 2L_n\rho L_n^\dagger - L_n^\dagger L_n\rho - \rho L_n^\dagger L_n$
  $\frac{d\rho}{dt} = -i(\mathcal{H}_{\text{eff}}\rho - \rho\mathcal{H}_{\text{eff}}^\dagger), \quad \mathcal{H}_{\text{eff}} = \mathcal{H} - i\sum_n L_n^\dagger L_n$

Lindblad, Commun. Math. Phys. 48 (1976), Dalibard *et al*, PRL (1992), Dum *et al*, PRA (1992), Lieu, arXiv:1908.08834

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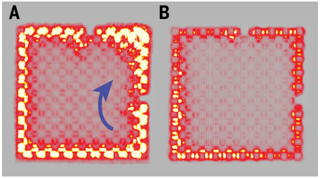
- Many-body Green function

$$G(\omega) = \left(\omega - \mathcal{H} - i\Sigma(0) - i\Sigma'(\omega)\right)^{-1}, \; \mathcal{H}^{\text{eff}} = \mathcal{H} - i\Sigma(0)$$

Kozii *et al*, arXiv:1708.05841, Yoshida *et al*, PRB 98 (2018) Zyuzin *et al*, PRB 97 (2018), PRB 99 (2019)

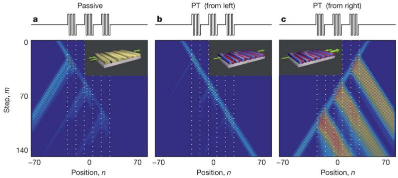
Flashing some experimentals results

Topological lasers



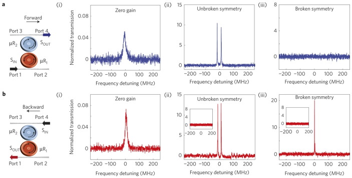
St Jean *et al*, Nat. Phot. 11 (2017)

Unidirectional invisibility



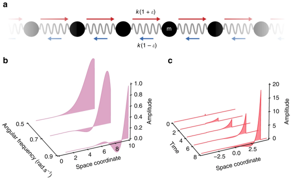
Regensburger *et al*, Nature 488 (2012)

Unidirectional lasing



Peng *et al*, Nat Phys 10 (2014)

Non-Hermitian skin effect



Brandenbourger *et al*, Nat. Comm. 10 (2019)

Some basic mathematical properties

- Matrices are not all diagonalizable but only admit Jordan blocks

$$\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

Non-diagonalizable blocks are called **exceptional points**

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- Eigenvalues can be **complex** and the basis **not orthogonal**

$$\begin{pmatrix} 1+i & i\sqrt{2} \\ 0 & 1-i \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 0 & \sqrt{2} \end{pmatrix} \begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix} \begin{pmatrix} 1 & \frac{\sqrt{2}}{2} \\ 0 & \frac{\sqrt{2}}{2} \end{pmatrix}$$

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- Leading to **different left and right eigenvectors**.

$$E_+ = 1+i,\ \left|\psi_+^R\right\rangle = (1,0),\ \left|\psi_+^L\right\rangle = (1, \frac{\sqrt{2}}{2})$$

$$E_- = 1-i,\ \left|\psi_-^R\right\rangle = (-1,\sqrt{2}),\ \left|\psi_-^L\right\rangle = (0,1)$$

with $H\left|\psi_\pm^R\right\rangle = E_\pm\left|\psi_\pm^R\right\rangle$, $\left\langle\psi_\pm^L\right| H = E_\pm\left\langle\psi_\pm^L\right|$ and $\left\langle\psi_\alpha^L|\psi_\beta^R\right\rangle = \delta_{\alpha,\beta}$

New topological properties

Hatano-Nelson model with periodic boundary conditions

$$H_{HN} = -\sum_j J_L c_j^\dagger c_{j+1} + J_R c_{j+1}^\dagger c_j$$

$$E_k = -(J_L + J_R)\cos k - i(J_L - J_R)\sin k$$

Hatano and Nelson, PRL 77 (1996), Gong *et al*, PRX 8 (2017), Kawabata *et al* PRX 9 (2019), Okuma *et al*, arXiv:1910.02878

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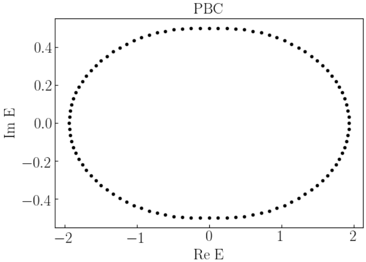
The eigenenergies are winding around the origin

$$\nu = \frac{1}{2\pi}\int_{\mathrm{BZ}} dk \partial_k \ln E_k \neq 0 \text{ iff. } J_L \neq J_R$$



Hatano and Nelson, PRL 77 (1996), Gong *et al*, PRX 8 (2017), Kawabata *et al* PRX 9 (2019), Okuma *et al*, arXiv:1910.02878

And the old ones are also valid

Starting from the Hermitian Chern insulator model

$$h_{\vec{k}} = \Delta_x \sin k_x \sigma^x + \Delta_y \sin k_y \sigma^y + (\mu - 2t\cos k_x - 2t\cos k_y)\sigma^z$$

One can add small non-Hermitian perturbations:

$$\delta h_{\vec{k}} = i\kappa_x \sigma^x + i\kappa_y \sigma^y + i\delta\sigma^z$$

Picture from Shen *et al*, PRL 120 (2018)

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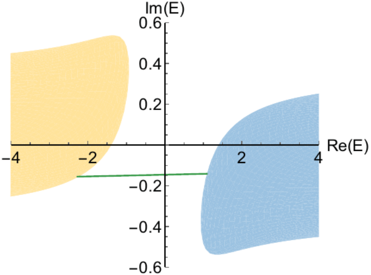

Four equivalent Chern invariants

$$C^{\alpha\beta} = \frac{1}{2\pi} \iint_{\mathrm{BZ}} \varepsilon_{ij} B_{ij}^{\alpha\beta} d\vec{k}$$

$$B_{ij}^{\alpha\beta} = i\langle \partial_i \psi^\alpha | \partial_j \psi^\beta \rangle, \quad \alpha, \beta = L/R \text{ is the generalized Berry connection}$$

Picture from Shen *et al*, PRL 120 (2018)

Chiral non-Hermitian SSH model

Chiral non-Hermitian Su-Schrieffer-Heeger model

$$-(t_1+\gamma)c^\dagger_{j,A}c_{j,B} - (t_1-\gamma)c^\dagger_{j,B}c_{j,A} - t_2(c^\dagger_{j+1,B}c_{j,A} + c^\dagger_{j+1,A}c_{j,B})$$

Momentum space formulation

$$h_k = \begin{pmatrix} 0 & q_1 = -(t_1+\gamma) - t_2 e^{-ik} \\ q_2 = -(t_1-\gamma) - t_2 e^{ik} & 0 \end{pmatrix}$$

Su PRB 92 (1980), Yao *et al* PRL 121 (2018), Kunst *et al* PRL 121 (2018), Yin *et al* PRA 97 (2018)

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- $\gamma=0$: $q_1=q_2^*$. 1 topological invariant: winding of q_1.

$$\nu_-=\frac{1}{2\pi}\int dk\partial_k \ln q_1-\partial_k \ln q_2$$

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$$\nu_- = \frac{1}{2\pi}\int dk \partial_k \ln q_1 - \partial_k \ln q_2$$

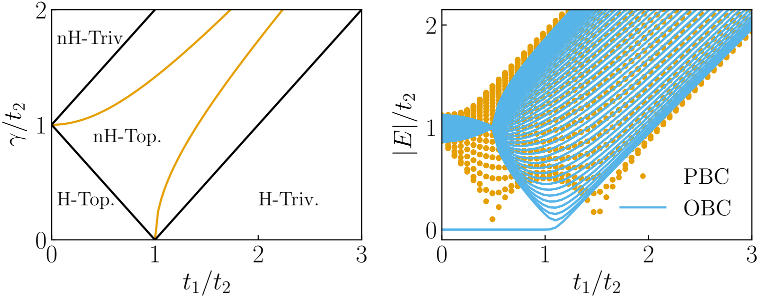
- $\gamma \neq 0$: $q_1 \neq q_2^*$. 2 topological invariants: windings of q_1 and q_2.

$\nu_+ = \dfrac{1}{2\pi}\displaystyle\int dk \partial_k \ln q_1 + \partial_k \ln q_2$, equivalent to winding of the energies

Su PRB 92 (1980), Yao *et al* PRL 121 (2018), Kunst *et al* PRL 121 (2018), Yin *et al* PRA 97 (2018)

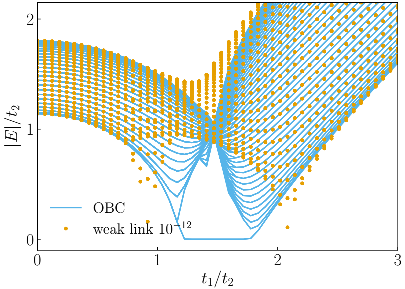
Breakdown of the bulk-boundary correspondence

Phase diagram actually strongly depends on boundary conditions.



Su PRB 92 (1980), Yao *et al* PRL 121 (2018), Kunst *et al* PRL 121 (2018), Yin *et al* PRA 97 (2018)

Exponentially small perturbations change the complete spectrum



Non-unitary gauge change: $H_{OBC}(t_1, t_2, \gamma) \Rightarrow H_{OBC}(t_1^{\text{eff}}, t_2, 0)$

$$c_1^\dagger c_L + c_L^\dagger c_1 \Rightarrow e^{aL} c_1^\dagger c_L + e^{-aL} c_L^\dagger c_1$$

Boundary conditions are not the only factor of instability.

Xiong *et al*, JoP Comm 2 (2018), Kunst *et al*, PRL 121 (2018), Lee *et al*, PRB 99 (2014), Herviou *et al*, PRA 99 (2019)

Singular value decomposition is stable and verifies bulk-boundary correspondence

Solution: singular value decomposition $\Leftrightarrow$ eigenvalues of $H^\dagger H$ or $HH^\dagger$.

$$H = USV^\dagger$$

U, V are unitary matrices. S is real positive diagonal.
For Hermitian matrices, SVD and eigenvalue decomposition coincide.

Gong *et al* PRX 8 (2018), Herviou *et al*, PRA 99 (2019), Kawabata *et al* PRX 9 (2019) Zhou *et al*, PRB 99 (2019)

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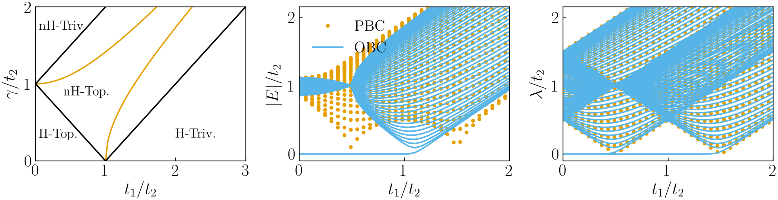

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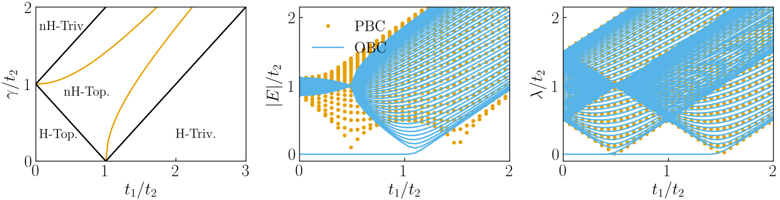

How to go to many-body physics as singular values do not sum?

Gong *et al* PRX 8 (2018), Herviou *et al*, PRA 99 (2019), Kawabata *et al* PRX 9 (2019) Zhou *et al*, PRB 99 (2019)

Entanglement spectrum $\approx$ edge of a topological model

Another form of bulk-boundary correspondence
Density matrix $\rho = |\Psi\rangle \langle\Psi|$

Reduced density matrix $\rho_{\mathcal{A}} = \mathrm{Tr}_{\overline{\mathcal{A}}}\rho$

Entanglement Hamiltonian $\rho_{\mathcal{A}} = e^{-H_E} \Leftrightarrow H_E = -\log \rho_{\mathcal{A}}$

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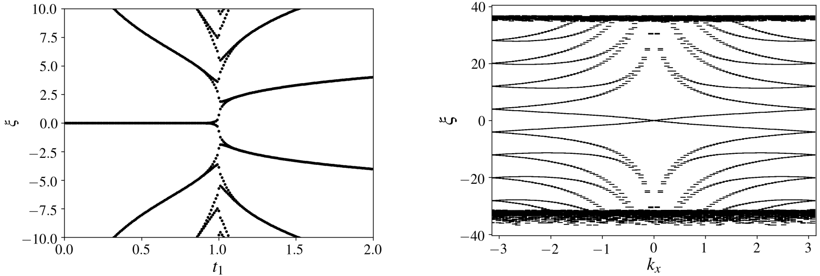
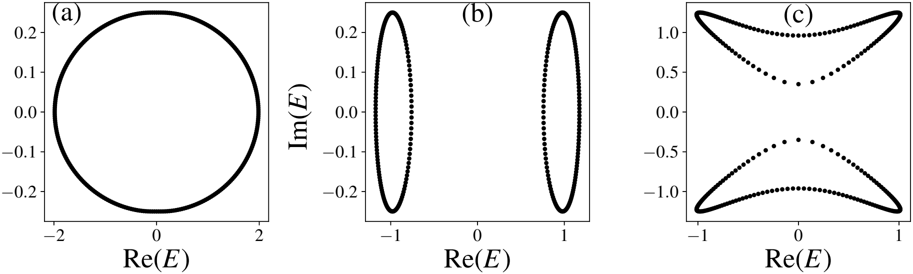
In topological systems, H_E has similar physics than the edge theory of the model: e.g. existence of (chiral) zero modes (with some caveats)



Gaps and eigenstate selection

Two types of gap: separable vs non-separable bands



Choice of many-body eigenstate: either bands, or symmetries

Kawabata *et al* arXiv:1812.09133

Two possible density matrices in non-Hermitian systems

Non-Hermitian Hamiltonians have different left and right eigenstates

Biorthogonal DM

$$\rho^{RL} = \left|\psi^R\right\rangle \left\langle\psi^L\right|$$

- Non-Hermitian
- Standard Heisenberg time evolution
- $\langle O \rangle = \left\langle \psi^L | O | \psi^R \right\rangle$
- Natural in Green function approaches

Right DM

$$\rho^{R} = \left|\psi^R\right\rangle \left\langle\psi^R\right|$$

- Hermitian
- Modified non-linear Heisenberg time evolution
- $\langle O \rangle = \left\langle \psi^R | O | \psi^R \right\rangle$
- Natural in Lindblad with post-selection

Sergi *et al*, JoMP B 27 (2013)

Wick theorem and Peschel trick

In both cases, Wick theorem is still valid
A modified version of the Peschel trick to compute the ES.
Entanglement spectrum $\mathcal{H}_E^\alpha = \vec{c}^\dagger H_E^\alpha \vec{c}$

$$H_E^\alpha = \ln\left[(C_{\mathcal{A}}^\alpha)^{-1} - 1\right]$$

Peschel, JoP A 36 (2003), Herviou *et al*, SciPost Phys. 7 (2019)

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$$H = \sum_n E_n \left|R_n\right\rangle \left\langle L_n\right|$$

Biorthogonal correlation matrix: $C_{i,j}^{RL} = \langle \psi^L | c_j^\dagger c_i | \psi^R \rangle = \sum_n s_n \left|R_n\right\rangle \left\langle L_n\right|$

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Right correlation matrix $C_{i,j}^{R} = \langle\psi^R|c_j^\dagger c_i|\psi^R\rangle = \sum_n \left|Q_n\right\rangle \left\langle Q_n\right|$ for $\left|Q_n\right\rangle$ an orthogonal basis of the occupied subspace $\mathrm{Span}(\left|R_{i_1}\right\rangle, \ldots \left|R_{i_m}\right\rangle)$.

Peschel, JoP A 36 (2003), Herviou *et al*, SciPost Phys. 7 (2019)

Symmetries in non-Hermitian systems

Four types of symmetries to classify non-Hermitian Hamiltonians through the Bernard-LeClair symmetry classes

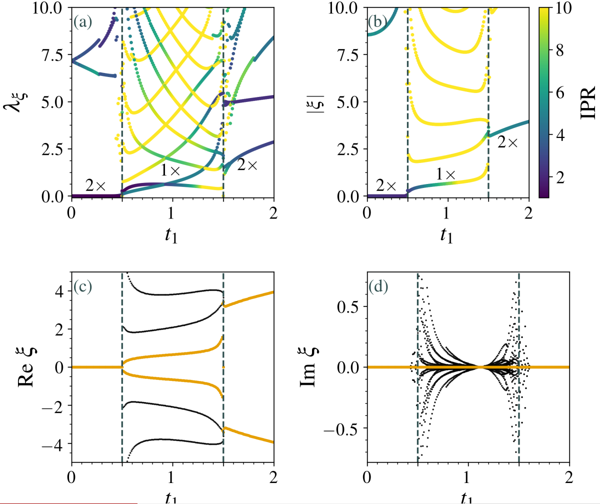
$$Ch : H = -u_c H u_c^\dagger, \text{ with } u_c u_c^\dagger = I, \; u_c^2 = I$$
$$T_{\varepsilon_t} : H = \varepsilon_t u_t H^* u_t^\dagger, \text{ with } u_t u_t^\dagger = I, \; u_t u_t^* = \eta_t I$$
$$P_{\varepsilon_p} : H = \varepsilon_p u_p H^T u_p^\dagger, \text{ with } u_p u_p^\dagger = I, \; u_p u_p^* = \eta_p I$$
$$PH_{\varepsilon_{ph}} : H = \varepsilon_{ph} u_{ph} H^\dagger u_{ph}^\dagger, \text{ with } u_{ph} u_{ph}^\dagger = I, \; u_{ph}^2 = I$$

Ch is a chiral symmetry, T and P are two flavors of particle-hole ($\varepsilon = -1$) or time-reversal ($\varepsilon = 1$) symmetries and PH is pseudo-hermiticity.

Lieu PRB 98 (2018)

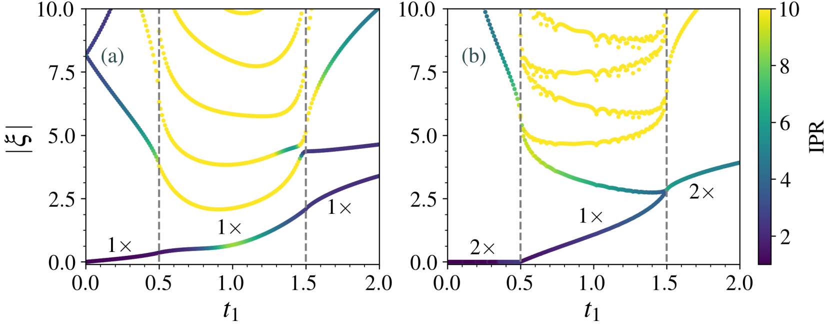
Biorthogonal entanglement spectrum

Captures the physics of separable bands, but fails for pure non-Hermitian topology



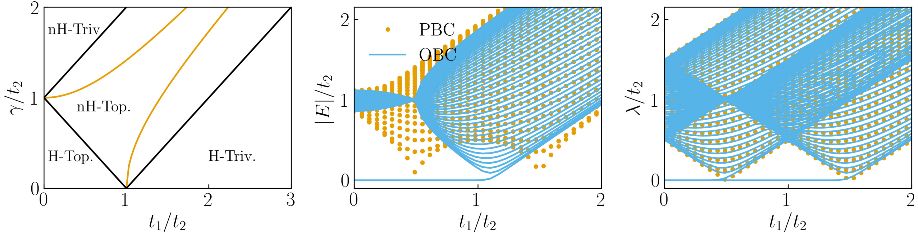
Right entanglement spectrum

Some topological phases of the biorthogonal Hamiltonian do not lead to topological properties of the right density matrix.
And conversely.

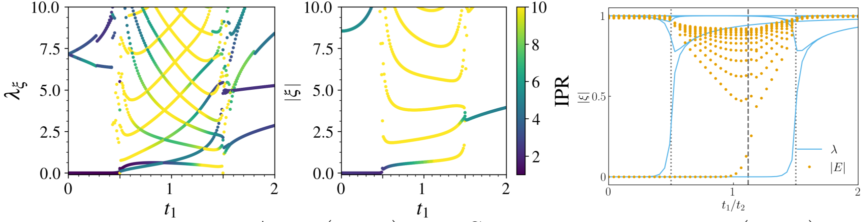


Discussions

Bulk-boundary correspondence from SVD



Entanglement spectrum



More results on PRA 99 (2019) and SciPost Phys. 7, 069 (2019).