QCD
Taller de Altas Energías - TAE 2019

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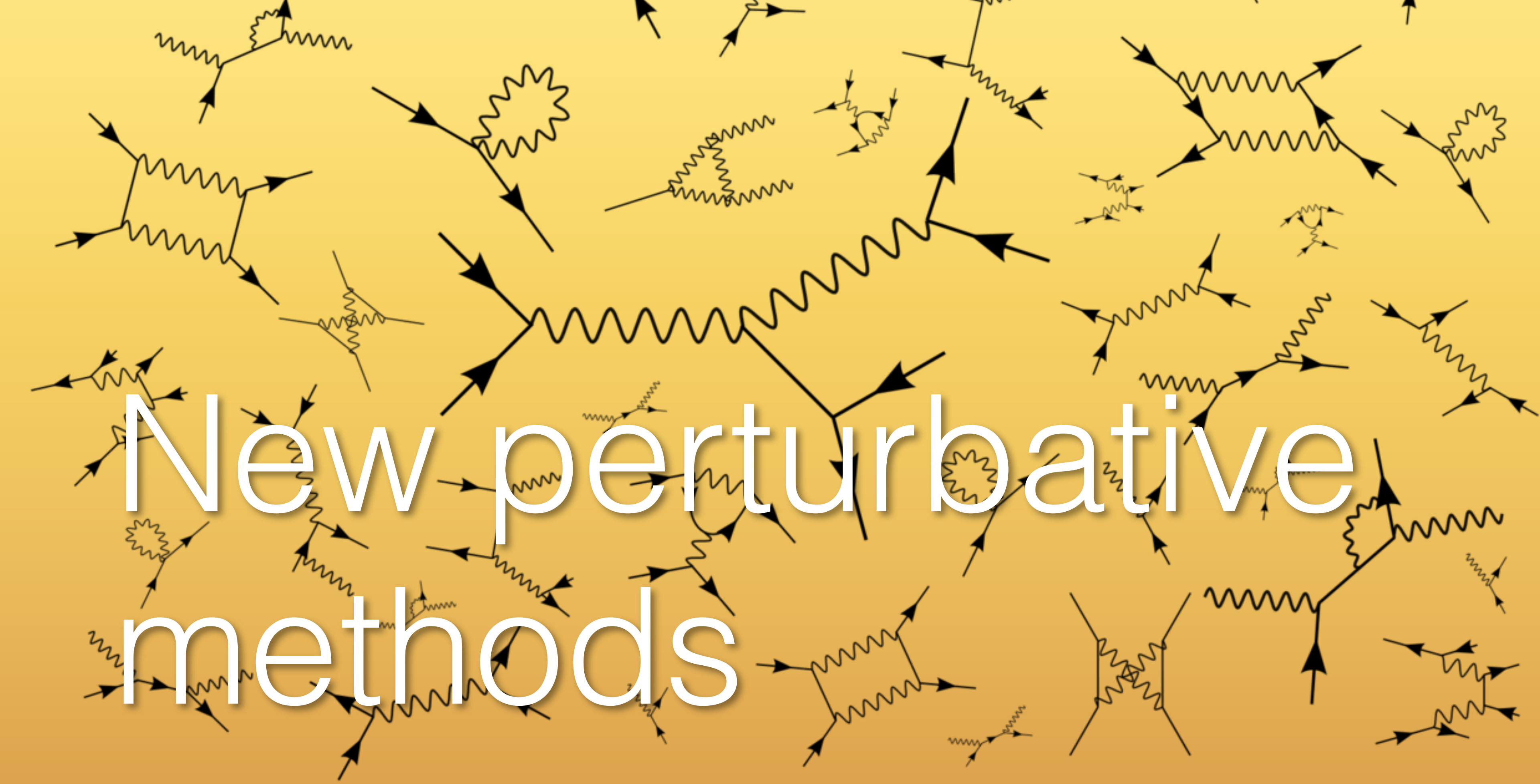
Lecture 2: new perturbative methods

Centro de Ciencias de Benasque Pedro Pascual

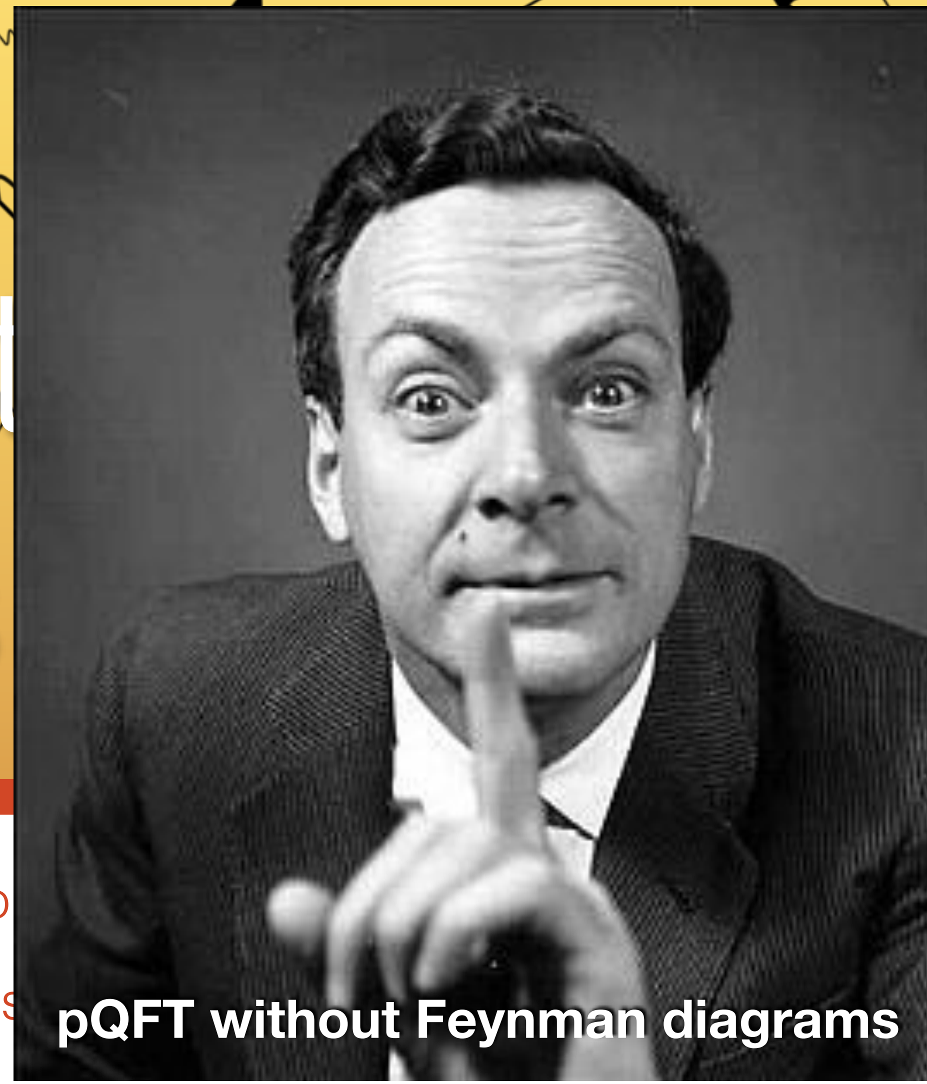
particleface · cost European Cooperation in Science & Technology



New perturbative methods

- To reach a new frontier in higher order calculations
- But also to better understand the structure of Quantum Field Theory

New pert
methods



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- But also to better understand the s

One-loop amplitudes

- The classical paradigm for the calculation of one-loop diagrams was established in 1979

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Scalar one-loop integrals
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Reduction of tensor one-loop integrals to scalar integrals

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Reduction of tensor one-loop integrals to scalar integrals

- Not adequate for processes beyond 2→2 (Gramm determinants+large number of Feynman diagrams)

Properties of the S-Matrix

- **Analyticity**: scattering amplitudes are determined by their singularities

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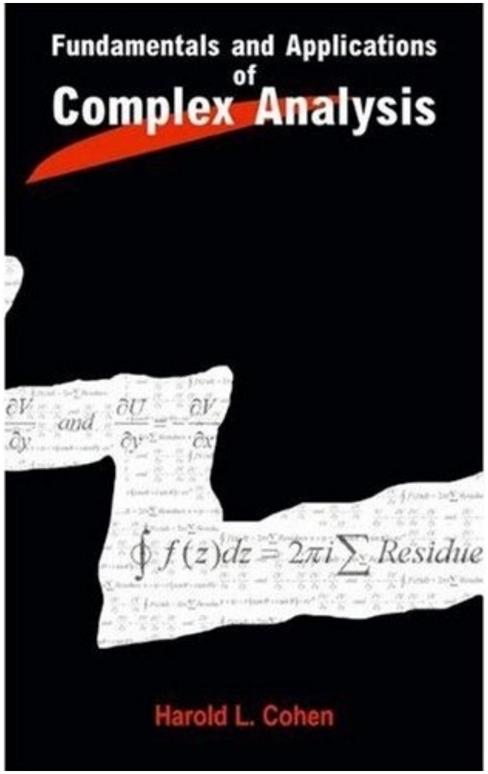
- **recycling**: using scattering amplitudes to calculate other scattering amplitudes

Recursion relations and unitarity methods



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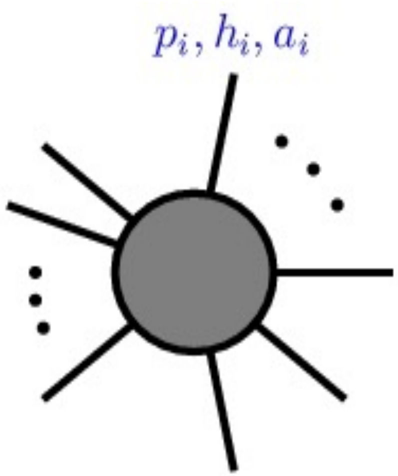
Here are the words of some enthusiast: “One of the most remarkable discoveries in elementary particle physics has been that of the existence of the complex plane”, “... the theory of functions of complex variables plays the role not of a mathematical tool, but of a fundamental description of nature inseparable from physics … .”

J. Schwinger, *Particles, Sources, and Fields*, Vol.1, p.36

- **recycling**: using scattering amplitudes to calculate other scattering amplitudes

Helicity basis + colour decomposition



Expressions simplify by using “right variables”
for N-gluons at tree level

$$\mathcal{M}_N(\{p_i, h_i, a_i\}) = \sum_{P(1,\ldots,N)} \mathrm{Tr}(\mathbf{t}^{a_1}\mathbf{t}^{a_2}\cdots\mathbf{t}^{a_N})\, \mathcal{A}_N(\{p_i, h_i\})$$

sum over permutations

color ordered factor

colour ordered subamplitude:

- Depends on the momenta and helicities
- gauge-invariant
- fixed cyclic order of external legs

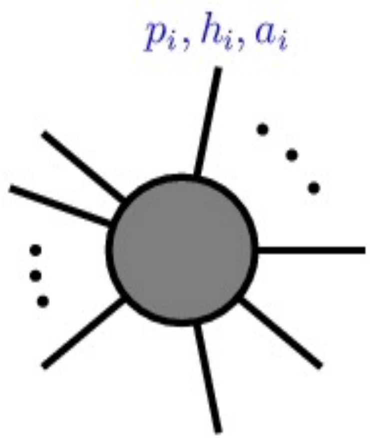
N-gluon amplitude

n	# diagrams	# colour-ord diagrams
4	4	3
5	25	10
6	220	36
7	2485	133
8	34300	501
9	559405	1991
10	10525900	7225

[Cvitanovic, Lauwers, Scharbach, Berends, Giele, Mangano, Parke, Xu,Bern,Kosower, Lee, Nair]

Spinors

[Xu,Zhang,Chang, Berends, Kleiss, De Causmaeker, Gastmans, Wu,Gunion, Kunzst]



Four-dimensional spinors of definite helicity

$$|i^\pm\rangle = \frac{1}{2}(1 \pm \gamma_5) u(p_i) = v_\mp(p_i) \ , \qquad \langle i^\pm| = \bar{u}_\pm(p_i) = \bar{v}_\mp(p_i)$$

$$p_i^2 = 0 \ , \qquad p_i^{a\dot{a}} = k_i^\mu \sigma_\mu^{a\dot{a}} = \lambda_i^a \tilde{\lambda}_i^{\dot{a}}$$

- spinor inner products and other useful identities

$$\langle ij\rangle = \langle i^-|j^+\rangle = \varepsilon_{ab}\lambda_i^a \lambda_j^b = \sqrt{|s_{ij}|} e^{i\phi_{ij}} = -\langle ji\rangle \qquad \text{holomorphic}$$

$$[ij] = [i^+|j^-] = \varepsilon_{\dot{a}\dot{b}} \tilde{\lambda}_i^{\dot{a}} \tilde{\lambda}_j^{\dot{b}} = -\langle ij\rangle^* = -[ji] \qquad \text{antiholomorphic}$$

$$[i|\gamma^\mu|j\rangle = \langle j|\gamma^\mu|i]$$

$$s_{ij} = (p_i + p_j)^2 = \langle ij\rangle [ji]$$

$$\not{p}_i = |i\rangle[i| + |i]\langle i| \qquad \text{sum over polarizations}$$

$$\not{p}_i |i^\pm\rangle = 0 \qquad \text{equation of motion} \qquad \langle ij] = 0 = \langle ii\rangle$$

- polarization vector $\quad \epsilon^2 = 0 \ , \quad \epsilon^+ \cdot \epsilon^- = 0 \ , \quad k \cdot \epsilon^\pm(k) = 0$

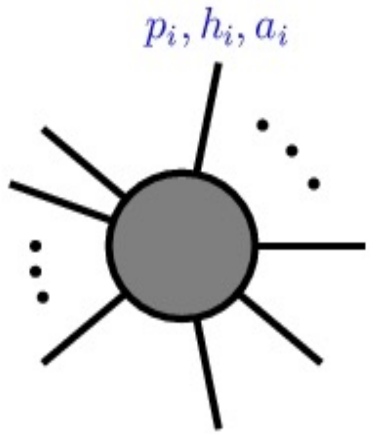
$$\epsilon_\mu^+(k,\xi) = \frac{\langle \xi|\gamma_\mu|k]}{\sqrt{2}\langle \xi k\rangle}$$

$$\epsilon_\mu^-(k,\xi) = \frac{[\xi|\gamma_\mu|k\rangle}{\sqrt{2}[k\xi]}$$

- equivalent to axial gauge $\boldsymbol{\xi = n}$
- a clever choice of the gauge momentum can simplify calculations

Spinors

[Xu,Zhang,Chang, Berends, Kleiss, De Causmaeker, Gastmans, Wu,Gunion, Kunzst]



- spinor identities

$$\langle 1|\gamma^\mu|2][3|\gamma_\mu|4\rangle = 2\langle 14\rangle[32] \qquad \text{Fierz}$$

$$\langle 12\rangle\langle 34\rangle = \langle 14\rangle\langle 32\rangle + \langle 13\rangle\langle 24\rangle \qquad \text{Shouten}$$

Exercise: proof the Fierz and Shouten identities

Hint: divide and multiply by **⟨23⟩** and apply the Dirac identity

$$\gamma^\mu\gamma^\nu\gamma^\sigma\gamma_\mu = 4g^{\nu\sigma}$$

Exercises:

Calculate the scattering amplitudes and square amplitude for $e^+e^- \to q\bar{q}$ by using the helicity method, and compare with the traditional calculation

How many independent helicity amplitudes there are ?

$$M_{e^+e^-\to q\bar{q}} \sim [\bar{u}(p_1)\gamma^\mu v(p_2)]\,[\bar{v}(p_3)\gamma^\nu u(p_4)]\,d_{\mu\nu}(p_{12},n)$$

$$|M|^2 \sim \mathrm{Tr}(\not{p}_1\gamma^\mu\not{p}_2\gamma^\sigma)\mathrm{Tr}(\not{p}_3\gamma^\nu\not{p}_4\gamma^\rho)d_{\mu\sigma}(p_{12},n)d_{\nu\rho}(p_{12},n)$$

MHV amplitudes

Multi-gluonic amplitudes at tree level: Amplitude for all gluons of positive helicity or one single gluon of negative helicity vanishes

- two negative helicities (Maximal Helicity Violating Amplitude) rather simple [Parke-Taylor, 1986]

$$A_n(1^+,\ldots,i^\pm,\ldots,n) = 0$$

$$A_n^{\mathrm{MHV}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+) = i\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}$$

proven via recursion relations [Berends-Giele, Mangano-Parke-Xu,1988]

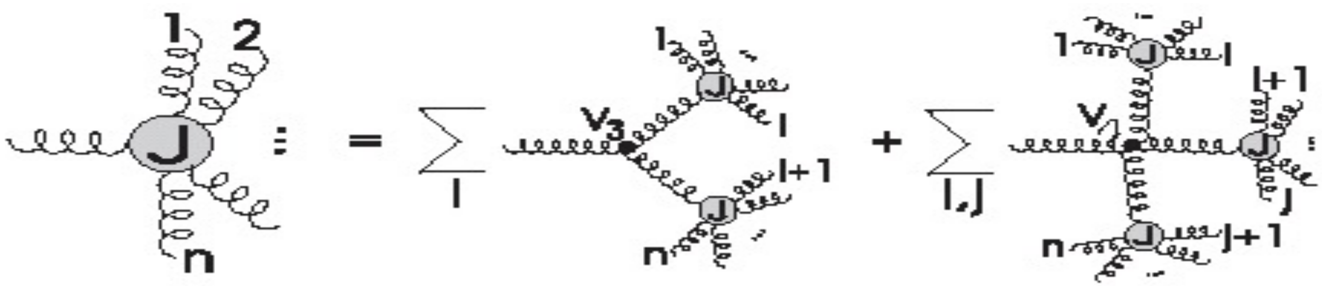
next-to-MHV $\quad A_n^{\mathrm{NMHV}}(1^+,\cdots,i^-,\cdots,j^-,\cdots,k^-,\cdots,n^+)$

does contain both $\langle ij\rangle$ and $[ij]$ [Kosower,1990]

Off-shell recursion relations

[Berends, Giele]

- Define Off-shell current: amplitude with one off-shell leg, building block for the off-shell current with higher multiplicity



- the gluonic current particularly simple for some helicity configurations

$$J^\mu(i^+, \ldots, j^+) = \frac{\langle \xi \,|\, \gamma^\mu \not{p}_{i,j} \,|\, \xi \rangle}{\sqrt{2}\langle \xi i \rangle \langle i(i+1) \rangle \cdots \langle j \xi \rangle}$$

- on-shell amplitude by setting on-shell the off-shell leg

On-shell recursion relations at tree-level: BCFW

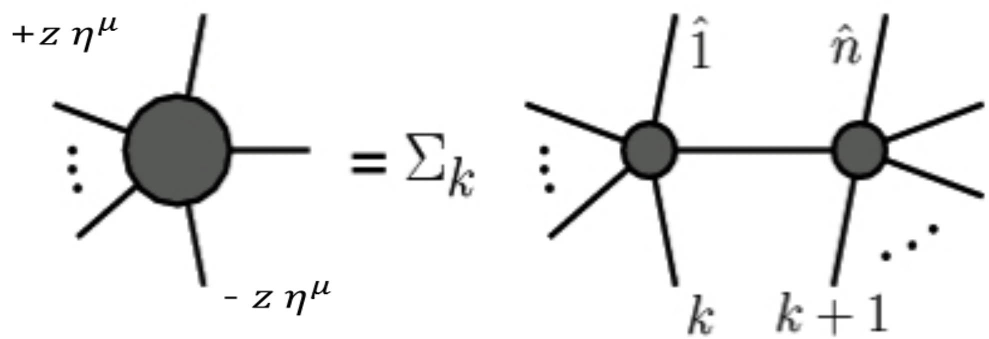
[Britto, Cachazo, Feng, Witten]

How to reconstruct scattering amplitude from its singularities

Add $z\,\eta^\mu$ (z complex) to the four-momentum of one external particle and subtract it on another such that the shift leaves them on-shell

$$0 = \frac{1}{2\pi i}\oint_{C\,\mathrm{at}\,\infty} \frac{A_n(z)}{z} = A_n(0) - \sum_{z_i} \frac{\mathrm{Res}_{z_i}(A_n(z))}{z_i}$$

has the correct residue at any multi-particle pole



$$A_n(1,2,\ldots,n) = \sum A_L(\hat{1},2,\ldots,-\hat{p}_{1,k})\,\frac{i}{s_{1,k}}\,A_R(\hat{p}_{1,k},k+1,\ldots,\hat{n})$$

- Diagrammatic proof [Draggiotis, Kleiss, Lazopoulos, Papadopoulos]
- Compact analytical results, although colour dressed Berends-Giele (off-shell recursion) might be more efficient numerically [Duhr, Höche, Maltoni]

in practice

holomorphic shift ((-,+) is not a safe shift)

$$\hat{p}_i^\mu = p_i^\mu + \frac{z}{2}[i|\gamma^\mu|j\rangle \qquad |\hat{i}\rangle = |i\rangle + z|j\rangle \qquad |\hat{i}] = |i]$$

$$\hat{p}_j^\mu = p_j^\mu - \frac{z}{2}[i|\gamma^\mu|j\rangle \qquad |\hat{j}\rangle = |j\rangle \qquad |\hat{j}] = |j] - z|i]$$

anti-holomorphic shift (i ↔ j)

z determined setting on-shell the intermediate momenta

$$\hat{p}_{1,k}^\mu = p_{1,k}^\mu + \frac{z}{2}[i|\gamma^\mu|j\rangle \,, \qquad \hat{p}_{1,k}^2 = 0 \,, \qquad z = -\frac{s_{1,k}}{[i|p_{1,k}|j\rangle}$$

☑ use only on-shell amplitudes

☑ rather compact expressions

☒ generates spurious poles at $[i|p_{1,k}|j\rangle$

while physical IR divergences at $s_{i,j} = (p_i + p_{i+1} + \ldots + p_j)^2$

Exercises:

Proof by induction that the Maximal Helicity Violating (MHV) amplitude for gluons is given by the expression

$$A_n(1^+,\ldots,i^\pm,\ldots,n)=0$$

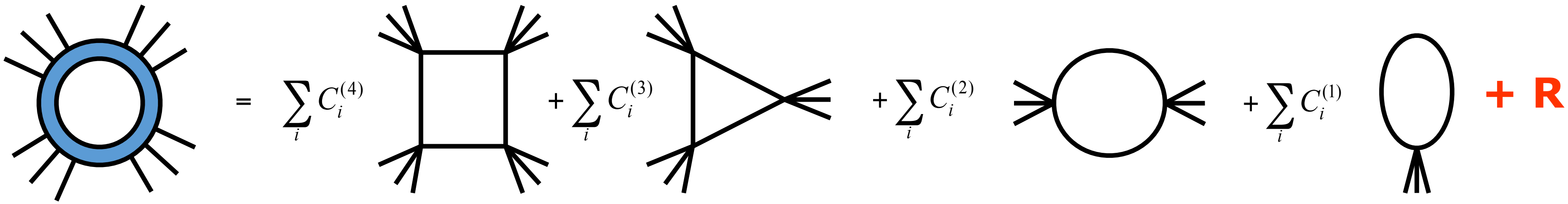
$$A_n^{\mathrm{MHV}}(1^+,\ldots,i^-,\ldots,j^-,\ldots,n^+)=i\frac{\langle ij\rangle^4}{\langle 12\rangle\langle 23\rangle\cdots\langle (n-1)n\rangle\langle n1\rangle}$$

Calculate by using BCFW the six-gluon amplitude

$$A_6(1^+,2^+,3^+,4^-,5^-,6^-)=$$
$$\frac{i}{\langle 2|1+6|5]}\left(\frac{\langle 6|1+2|3]^3}{\langle 61\rangle\langle 12\rangle[34][45]s_{126}}+\frac{\langle 4|5+6|1]^3}{\langle 23\rangle\langle 34\rangle[56][61]s_{561}}\right)$$

Generalized Unitarity: the one-loop basis

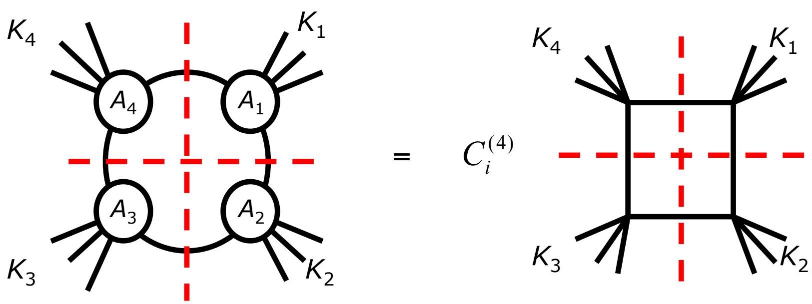
A dimensionally regulated n-point one-loop integral (scattering amplitude) is a linear combination of boxes, triangles, bubbles and tadpoles with rational coefficients



- Pentagons and higher n-point functions can be reduced to lower point integrals and higher dimensional polygons that only contribute at O(ε) [Bern, Dixon, Kosower]
- The task is reduced to determining the coefficients: by applying multiple cuts at both sides of the equation [Brito, Cachazo, Feng]
- R is a finite piece that is entirely rational: can not be detected by four-dimensional cuts

Generalized Unitarity

Quadruple cut



The discontinuity across the leading singularity is unique

$$C_i^{(4)} = A_1 \times A_2 \times A_3 \times A_4$$

Four on-shell constrains

➡ freeze the loop momenta

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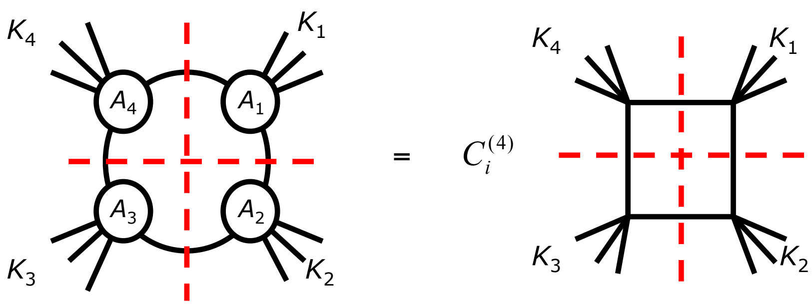

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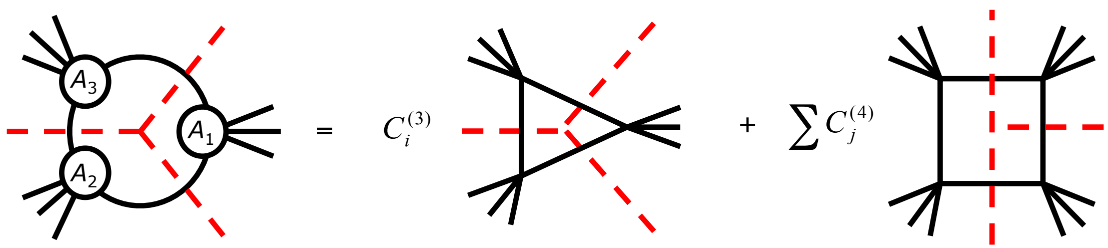


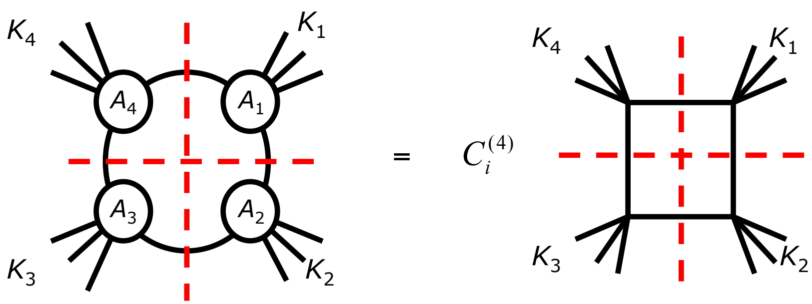
Only three on-shell constrains ➡ one free component of the loop momentum

And so on for double and single cuts

- OPP [Ossola, Pittau, Papadopoulos]: a systematic way to extract the coefficients

Generalized Unitarity

Quadruple cut



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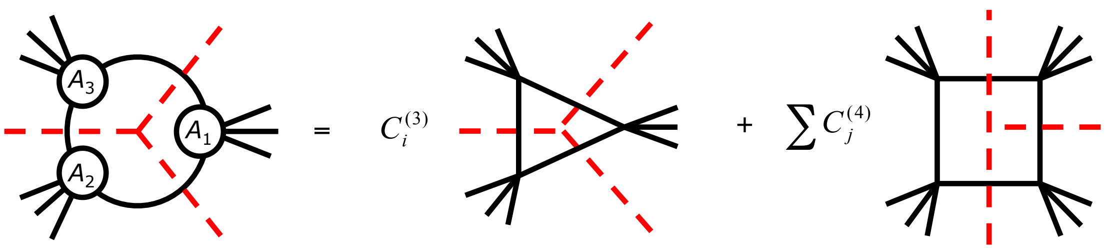

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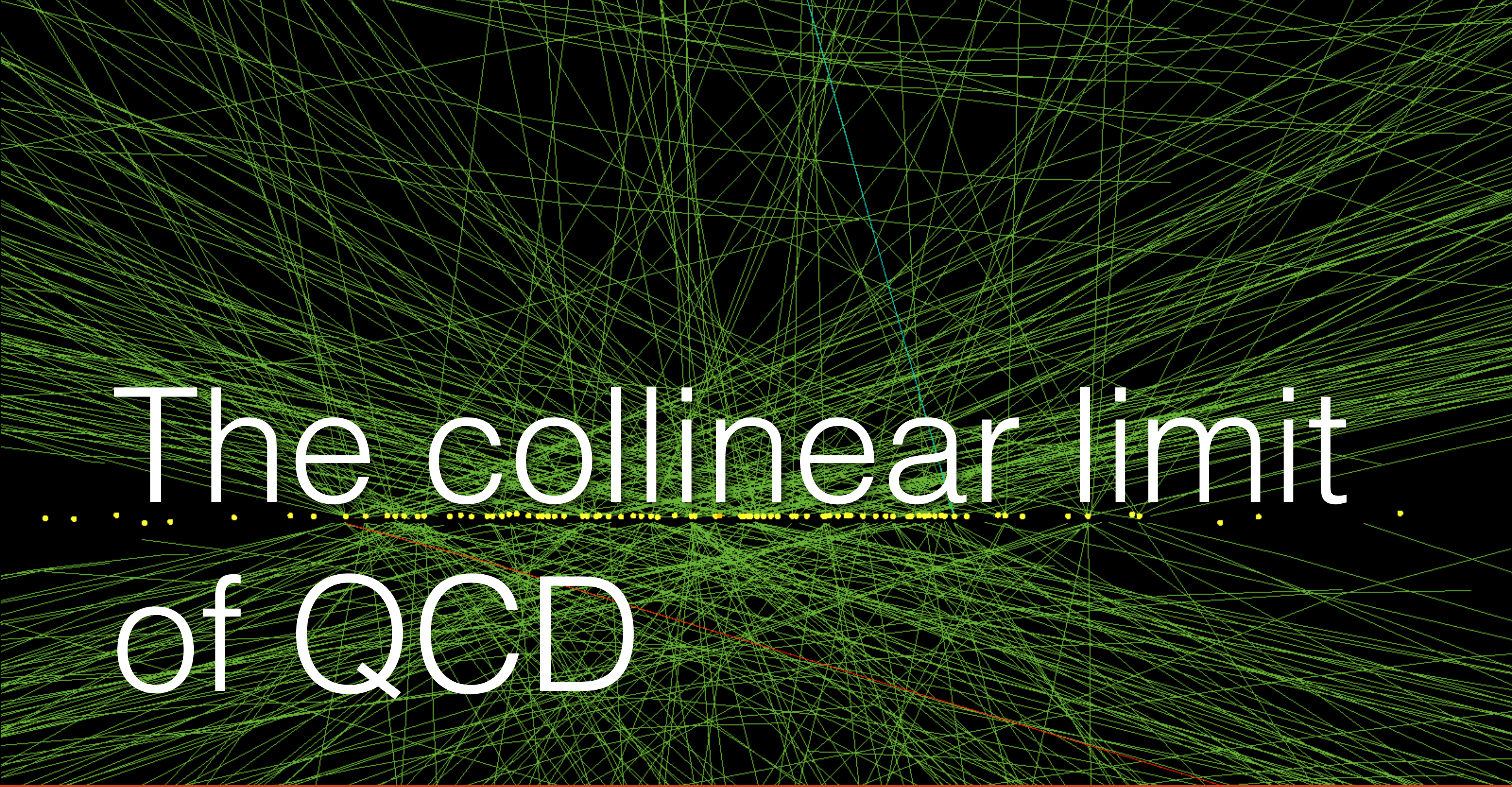
Rational terms

d-dimensional cuts, recursion relations (BCFW), Feynman rules …

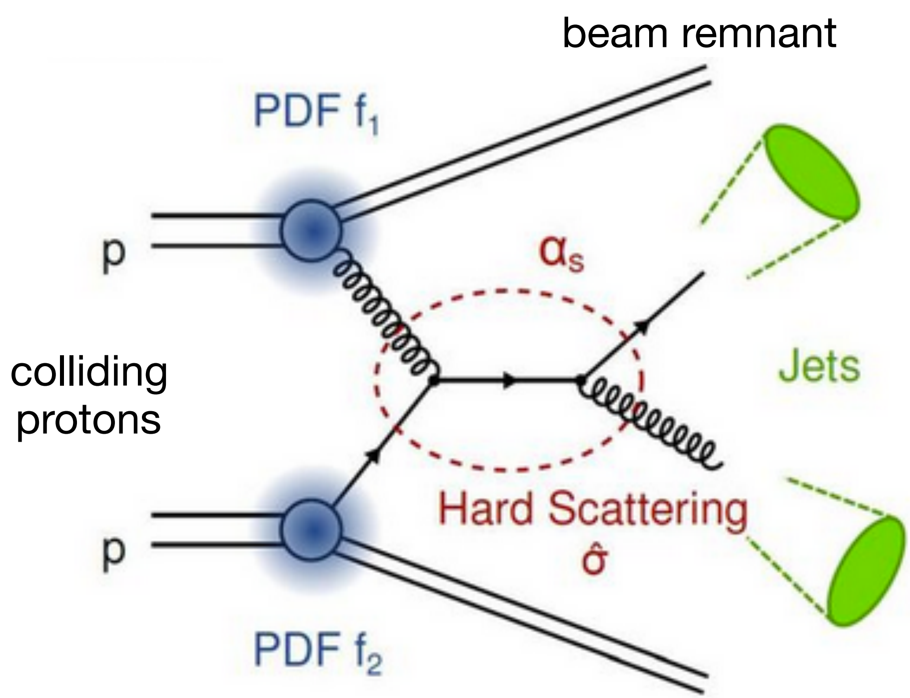


The collinear limit of QCD

Relevance of the collinear limit in QCD

- evaluate IR finite cross-sections ▶ subtraction terms
- IR properties of amplitudes exploited to compute logarithmic enhanced perturbative terms ▶ resummations
- improve physics content of Monte Carlo event generators ▶ parton showers
- Evolution of PDF´s and fragmentation functions
- beyond QCD: hints on the structure of highly symmetric gauge theories (e.g. N=4 super-Yang-Mills)
- Factorization theorems: pQCD for hard processes

Factorisation in hadronic collisions



- Factorise physics into **long distance** (hadronic $\sim M_{\rm had}$) and **short distance** (partonic $Q \gg M_{\rm had}$)
- factorisation violation is power suppressed $\sim \mathcal{O}(M_{\rm had}/Q)^q$

Parton densities PDF

Hard scattering cross-section

$$\sigma = \sum_{a,b} \int dx_1\, dx_2\, f_a(x_1, \mu_F) f_b(x_2, \mu_F)\, \hat{\sigma}_{ab}(x_1 p_A, x_2 p_B; \mu_F, \mu_R) + \mathcal{O}\left(\frac{1}{Q}\right)$$

Factorization and renormalization scales

Higher twist

Partonic cms energy $\hat{s} = x_1 x_2 s$

Collinear factorisation theorem proven for sufficiently inclusive observables in the final state of the scattering of colorless hadrons

[Collins, Soper, Sterman]

- Offen assumed that partonic scattering amplitudes factorize: fixed order and resummations
- Monte Carlo event generators are based on factorization
- In neither of these cases factorization is guaranteed.

pQCD for hard-scattering processes based on universality:

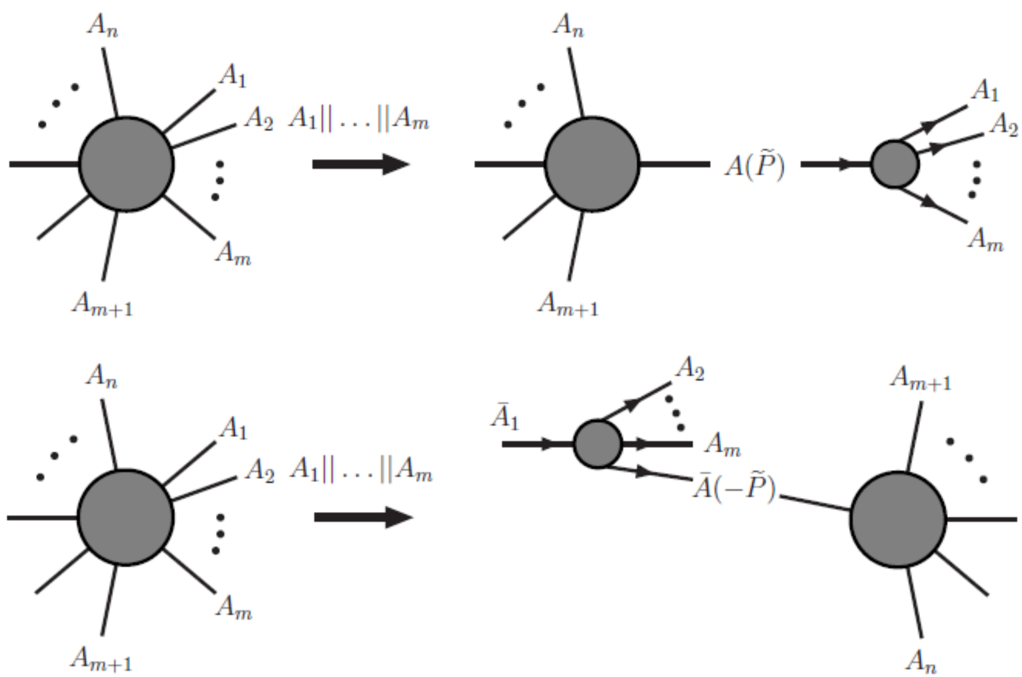
- the sole uncancelled IR divergences are due to partonic states whose momenta are collinear to the collider partons
- removed by redefinition of bare parton densities

Collinear factorisation at tree-level

- Momenta $p_1, \ldots, p_m$ of m **partons become collinear**
- Sub-energies $s_{ij} = (p_i + p_j)^2$ of the same order and vanish simultaneously
- leading singular behaviour $(\sqrt{s_{1,m}})^{1-m}$ with $p_{1,m} = p_1 + \ldots + p_m$

time - like fragmentation

space - like parton densities



$$
|M^{(0)}(p_1, \ldots, p_n)\rangle = \boldsymbol{Sp}^{(0)}(p_1, \cdots, p_m; \tilde{P}) \, |\overline{M}^{(0)}(\tilde{P}; p_{m+1}, \ldots, p_n)\rangle + \mathcal{O}((\sqrt{s_{1,m}})^{3-m})
$$

Collinear limit

- Most singular behaviour captured by **universal** (process independent) splitting amplitudes: the same for e+e-, DIS and hadron collisions
- The **splitting amplitude** depends on the collinear partons only
- Space-like and time-like related by crossing
- Process dependence in the **reduced matrix element**

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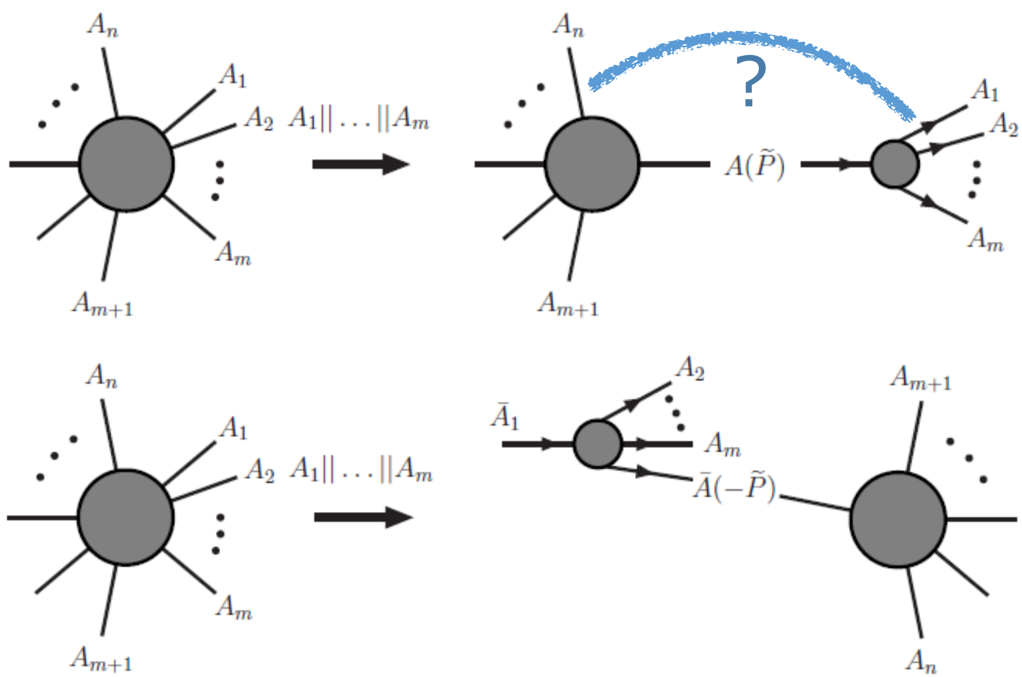

$$
\begin{aligned}
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At two loops

$$|M^{(2)}(p_1,\dots,p_n)\rangle$$
$$\simeq \mathbf{Sp}^{(0)}(p_1,\dots,p_m;\widetilde{P})\,|\overline{M}^{(2)}(\widetilde{P};p_{m+1},\dots,p_n)\rangle$$
$$+\mathbf{Sp}^{(1)}(p_1,\dots,p_m;\widetilde{P})\,|\overline{M}^{(1)}(\widetilde{P};p_{m+1},\dots,p_n)\rangle$$
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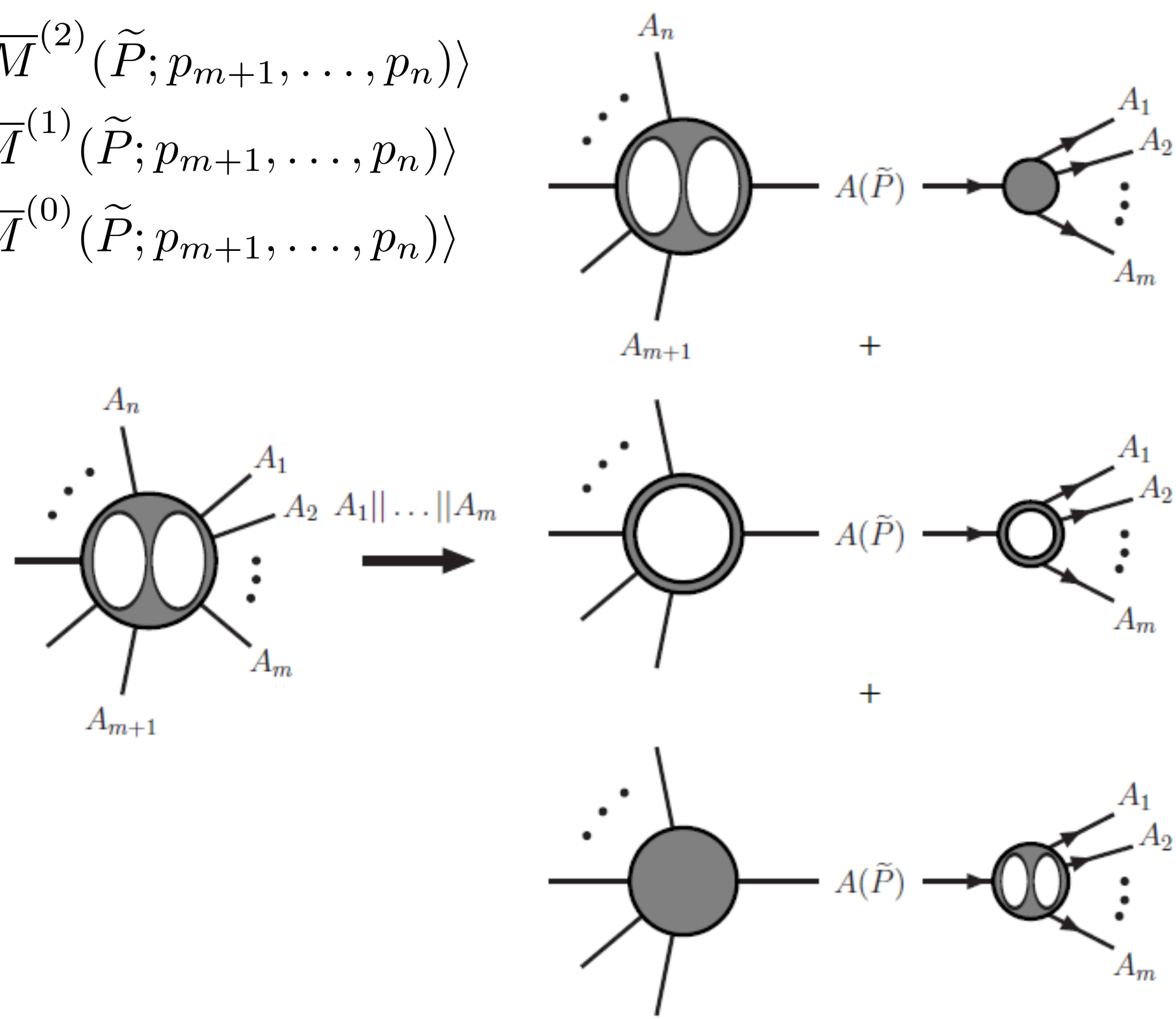

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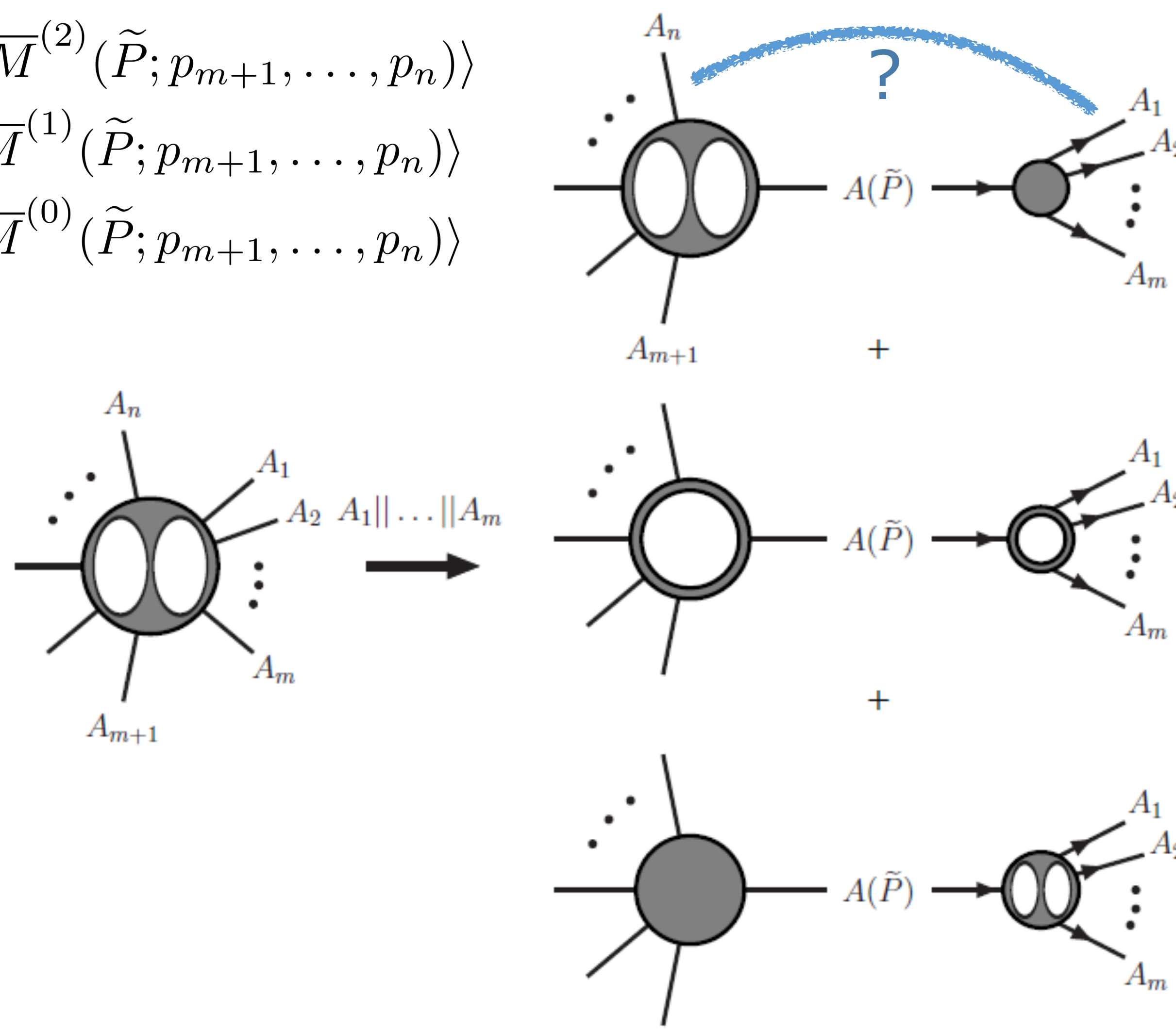

Qualitative interpretation: two collinear partons

- **Tree level:**
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  - Interactions separately spoil factorisation, but $\theta_{j1} \simeq \theta_{j2} \simeq \theta_{j\tilde{P}}$ and $\mathbf{T}_j \cdot (\mathbf{T}_1 + \mathbf{T}_2) = \mathbf{T}_j \cdot \mathbf{T}_{\tilde{P}}$: **colour coherence** restores factorisation, the parton j sees the two collinear partons as a single one.

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  - Both collinear partons in the final- or initial-state, otherwise colour coherence is limited by ***causality***

The collinear projection

- The **projection over the collinear limit** is obtained by setting the parent parton at on-shell momenta

$$\tilde{P}^\mu = p_{1,m}^\mu - \frac{s_{1,m}\, n^\mu}{2n\cdot\tilde{P}}$$

$\tilde{P}^\mu$: collinear direction

n^μ: describes how the collinear limit is approached $\tilde{P}^2 = 0,\ n^2 = 0$

$z_i = \dfrac{n\cdot p_i}{n\cdot\widetilde{P}}$: longitudinal momentu fraction $\sum z_i = 1$

- Factorisation holds in any arbitrary gauge, however, it is more evident in the **axial gauge** (physical polarisations): only diagrams where the parent parton emitted and absorbed collinear radiation

$$\frac{1}{\not{p}_{12}} = \frac{1}{s_{12}}\not{p}_{12} = \frac{1}{s_{12}}\left(\tilde{\not{P}} + \frac{s_{12}}{2n\cdot\tilde{P}}\not{n}\right) \simeq \frac{1}{s_{12}}\, u(\tilde{P})\bar{u}(\tilde{P}) + \ldots$$

$$d_{\mu\nu}(k,n) = d_{\mu\nu}(\tilde{P},n) + \ldots \simeq \epsilon_\mu(\tilde{P})\epsilon^*_\nu(\tilde{P}) + \ldots$$

Splitting functions

The square of the splitting amplitude, summed over final-state colours and spins, and averaged over colours and spins of the parent parton, defines the m-parton (unpolarised) splitting function

$$\langle P \rangle = \left(\frac{s_{1,m}}{2\mu^{2\epsilon}} \right)^{m-1} \overline{|\mathbf{Sp}|^2}$$

which is a generalisation of the customary (i.e. with $m = 2$) **Altarelli-Parisi** splitting function

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- Perturbarive expansion $P = P^{(0)} + P^{(1)} + P^{(2)} + \ldots$
- Probability to emit futher radiation with a given longitudinal momenta, from the leading singular behaviour
- Universal (process independent): the same fro e+e-, DIS or hadron collisions

Exercise:

- Proof that $\mathbf{T}_j \cdot (\mathbf{T}_q + \mathbf{T}_{\bar{q}}) = \mathbf{T}_j \cdot \mathbf{T}_g$, and test other flavour combinations
- Calculate the splitting functions for the collinear processes $q \to qg$, $g \to q\bar{q}$ and $g \to gg$ by using the helicity method

Hint:

$$\mathbf{Sp}^{(0)}_{q \to q_1 g_2} = \mathbf{T}^{a_2} \frac{1}{s_{12}} \bar{u}(p_1) \not{\varepsilon}(p_2) v(\tilde{P})$$

$$P^{(0)}_{q \to q_1 g_2} = C_F \frac{1+z^2}{1-z} \qquad z = z_1 = \frac{n \cdot p_1}{n \cdot \tilde{P}} \qquad z_2 = 1 - z$$

- Compare with

$$\mathcal{M}^{(0)}_{q\bar{q}g} \simeq (-\imath e_q)\,(\imath g_S)\,\mathbf{T}^a\,\bar{u}(p_1)\,\gamma^\mu\,v(p_2)\left(\frac{p_1 \cdot \varepsilon}{p_1 \cdot k} - \frac{p_2 \cdot \varepsilon}{p_2 \cdot k}\right)$$