Exercise 1: Covariant derivative

Prove that the term $\overline{\Psi}\not{D}\Psi$ where the covariant derivative is given by:

$$D_\mu = \partial_\mu - \mathrm{i}g\widetilde{W}_\mu \ , \quad \widetilde{W}_\mu = T_a W^a_\mu$$

is invariant under gauge transformations:

$$\begin{aligned}\Psi &\mapsto U\Psi \ , \quad U = \exp\{-\mathrm{i}T_a\theta^a(x)\} \\ \widetilde{W}_\mu &\mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger\end{aligned}$$

$$\begin{aligned}\Psi &\mapsto U\Psi \\ \widetilde{W}_\mu &\mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \\ D_\mu\Psi = (\partial_\mu - \mathrm{i}g\widetilde{W}_\mu)\Psi &\mapsto (\partial_\mu - \mathrm{i}gU\widetilde{W}_\mu U^\dagger - (\partial_\mu U)U^\dagger)U\Psi \\ &= (\partial_\mu U + U\partial_\mu - \mathrm{i}gU\widetilde{W}_\mu - \partial_\mu U)\Psi \\ &= U(\partial_\mu - \mathrm{i}g\widetilde{W}_\mu)\Psi \\ &= UD_\mu\Psi\end{aligned}$$

$$\Rightarrow \overline{\Psi}\not{D}\Psi \mapsto \overline{\Psi}U^\dagger U\not{D}\Psi = \overline{\Psi}\not{D}\Psi$$

Exercise 2: Non abelian gauge transformations

The Yang-Mills Lagrangian is

$$\mathcal{L}_{\mathrm{YM}} = -\frac{1}{2}\mathrm{Tr}\left\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\right\}$$

where

$$\widetilde{W}_{\mu\nu} \equiv T_a W^a_{\mu\nu} = D_\mu\widetilde{W}_\nu - D_\nu\widetilde{W}_\mu = \partial_\mu\widetilde{W}_\nu - \partial_\nu\widetilde{W}_\mu - \mathrm{i}g[\widetilde{W}_\mu, \widetilde{W}_\nu] \ , \quad \widetilde{W}_\mu \equiv T_a W^a_\mu$$

and T_a are the N generators of a Lie group with algebra $[T_a, T_b] = \mathrm{i}f_{abc}T_c$.

i) Check that under a gauge transformation of the fields:

$$\widetilde{W}_\mu \mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \ , \quad U = \exp\{-\mathrm{i}T_a\omega^a\}$$

the $\widetilde{W}_{\mu\nu}$ transforms as

$$\widetilde{W}_{\mu\nu} \mapsto U\widetilde{W}_{\mu\nu}U^\dagger$$

and therefore $\mathcal{L}_{\mathrm{YM}}$ is gauge invariant.

ii) Check that one may write

$$\mathcal{L}_{\text{YM}} = -\frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu}$$

that contains kinetic terms and cubic and quartic interactions among the gauge fields.

iii) Check that

$$W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu$$

iv) Check that under infinitesimal gauge transformations:

$$W^a_\mu \mapsto W^a_\mu - f_{abc} W^b_\mu \omega^c - \frac{1}{g} \partial_\mu \omega^a$$

---

i)

$$\begin{aligned}
&\widetilde{W}_\mu \mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \,, \quad U = \exp\{-\mathrm{i}T_a\omega^a\} \\
\Rightarrow \quad & \partial_\mu \widetilde{W}_\nu \mapsto (\partial_\mu U)\widetilde{W}_\nu U^\dagger + U(\partial_\mu \widetilde{W}_\nu)U^\dagger + U\widetilde{W}_\nu \partial_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu\partial_\nu U)U^\dagger - \frac{\mathrm{i}}{g}(\partial_\nu U)\partial_\mu U^\dagger \\
& -\mathrm{i}g\widetilde{W}_\mu\widetilde{W}_\nu \mapsto -\mathrm{i}gU\widetilde{W}_\mu\widetilde{W}_\nu U^\dagger - U\widetilde{W}_\mu U^\dagger(\partial_\nu U)U^\dagger - (\partial_\mu U)\widetilde{W}_\nu U^\dagger + \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger(\partial_\nu U)U^\dagger \\
& \qquad = -\mathrm{i}gU\widetilde{W}_\mu\widetilde{W}_\nu U^\dagger + U\widetilde{W}_\mu\partial_\nu U^\dagger - (\partial_\mu U)\widetilde{W}_\nu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)\partial_\nu U^\dagger
\end{aligned}$$

where we have used $UU^\dagger = I \Rightarrow 0 = \partial_\nu(UU^\dagger) \Rightarrow (\partial_\nu U)U^\dagger = -U\partial_\nu U^\dagger$. Therefore

$$\begin{aligned}
& D_\mu \widetilde{W}_\nu \mapsto U(\partial_\mu \widetilde{W}_\nu)U^\dagger + U\widetilde{W}_\nu\partial_\mu U^\dagger - \mathrm{i}gU\widetilde{W}_\mu U^\dagger + U\widetilde{W}_\nu\partial_\mu U^\dagger + U\widetilde{W}_\mu\partial_\nu U^\dagger \\
& \qquad - \frac{\mathrm{i}}{g}(\partial_\nu U)\partial_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)\partial_\nu U^\dagger \\
\Rightarrow \quad & \widetilde{W}_{\mu\nu} \mapsto UW_{\mu\nu}U^\dagger
\end{aligned}$$

ii)

$$-\frac{1}{2}\mathrm{Tr}\{\widetilde{W}_{\mu\nu}\widetilde{W}^{\mu\nu}\} = -\frac{1}{2} W^a_{\mu\nu} W^{b,\mu\nu}\,\mathrm{Tr}\{T_a T_b\} = -\frac{1}{4} W^a_{\mu\nu} W^{a,\mu\nu}$$

where we have used $\mathrm{Tr}\{T_a T_b\} = \frac{1}{2}\delta_{ab}$.

iii)

$$\begin{aligned}
& T_a W^a_{\mu\nu} = T_a(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) - \mathrm{i}g W^b_\mu W^c_\nu\,[T_b, T_c] = T_a(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu) \\
\Rightarrow \quad & W^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g f_{abc} W^b_\mu W^c_\nu
\end{aligned}$$

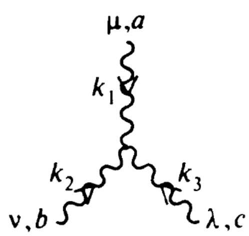
where we have used $[T_b, T_c] = \mathrm{i}f_{bca}T_a = \mathrm{i}f_{abc}T_a$.

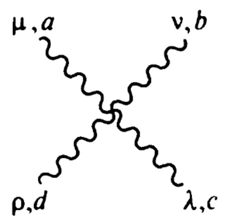
iv) Under an infinitesimal transformation:

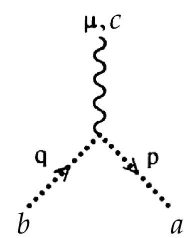
$$
\begin{aligned}
\widetilde{W}_\mu &\mapsto U\widetilde{W}_\mu U^\dagger - \frac{\mathrm{i}}{g}(\partial_\mu U)U^\dagger \\
\Rightarrow \quad T_a W^a_\mu &\mapsto T_a W^a_\mu - \mathrm{i}T_b T_a \omega^b W^a_\mu + \mathrm{i}T_a T_c W^a_\mu \omega^c - \frac{1}{g}\partial_\mu \omega^a T_a (U^\dagger U) \\
&= T_a W^a_\mu + i[T_a, T_b] W^a_\mu \omega^b - \frac{1}{g}\partial_\mu \omega^a T_a \\
&= T_a \left(W^a_\mu - f_{abc} W^b_\mu \omega^c - \frac{1}{g}\partial_\mu \omega^a \right)
\end{aligned}
$$

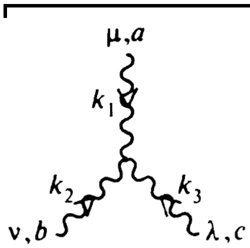
Exercise 3: Feynman rules of general non-Abelian gauge theories

Obtain the Feynman rules for cubic and quartic self-interactions among gauge fields in a general non-Abelian gauge theory, as well as those for the interactions of Faddeev-Popov ghosts with gauge fields:









$$
\mathcal{L}_{\text{cubic}} = -\frac{1}{2}g\sum_{abc} f_{abc}\ (\partial_\mu W^a_\nu - \partial_\nu W^a_\mu) W^{b,\mu} W^{c,\nu}
$$

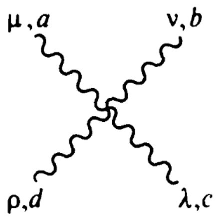
$$
\Rightarrow \quad g f_{abc}\ [g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu]
$$

Proof:
Note that there is a summation over repeated indices abc and assume totally antisymmetric structure constants f_{abc}. Then, fixing abc the summation is over all possible permutations of three indices 123 labeling the momenta and polarization vectors. Write $-\mathrm{i}k$ for every incoming momentum and a polarization vector $\epsilon(k)$ for every incoming vector boson. The polarization vectors factor out of the vertex definition:

$$
\begin{aligned}
\Gamma_{\mu\nu\lambda}[V^a(k_1), V^b(k_2), V^c(k_3)]\epsilon_1^\mu \epsilon_2^\nu \epsilon_3^\lambda &= -\frac{\mathrm{i}}{2} g f_{abc}\ [-\mathrm{i}k_{a\mu}\epsilon_{a\nu} + \mathrm{i}k_{a\nu}\epsilon_{a\mu})\epsilon_b^\mu \epsilon_c^\nu \\
&\qquad - (ab) - (ac) - (bc) + (abc) + (acb)] \\
&= -\frac{1}{2} g f_{abc}\ [(k_1\epsilon_2)(\epsilon_1\epsilon_3) - (k_1\epsilon_3)(\epsilon_1\epsilon_2) \\
&\qquad - (k_2\epsilon_1)(\epsilon_2\epsilon_3) + (k_2\epsilon_3)(\epsilon_2\epsilon_1) \\
&\qquad - (k_3\epsilon_2)(\epsilon_3\epsilon_1) + (k_3\epsilon_1)(\epsilon_3\epsilon_2)
\end{aligned}
$$

$$
\begin{aligned}
&- (k_1\epsilon_3)(\epsilon_1\epsilon_2) + (k_1\epsilon_2)(\epsilon_1\epsilon_3) \\
&+ (k_2\epsilon_3)(\epsilon_2\epsilon_1) - (k_2\epsilon_1)(\epsilon_2\epsilon_3) \\
&+ (k_3\epsilon_1)(\epsilon_3\epsilon_2) - (k_3\epsilon_2)(\epsilon_3\epsilon_1)] \\
= g f_{abc}\ &\{[(k_1\epsilon_3) - (k_2\epsilon_3)](\epsilon_1\epsilon_2) \\
&+ (k_2\epsilon_1) - (k_3\epsilon_1)](\epsilon_2\epsilon_3) \\
&+ (k_3\epsilon_2) - (k_1\epsilon_2)](\epsilon_3\epsilon_1)\}
\end{aligned}
$$

$$
\Rightarrow \Gamma_{\mu\nu\lambda}[V^a(k_1), V^b(k_2), V^c(k_3)] = g f_{abc}\ [g_{\mu\nu}(k_1-k_2)_\lambda + g_{\nu\lambda}(k_2-k_3)_\mu + g_{\lambda\mu}(k_3-k_1)_\nu]
$$



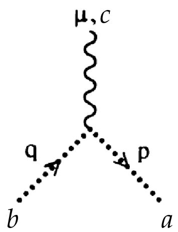
$$
\mathcal{L}_{\text{quartic}} = -\frac{1}{4}g^2 \sum_{abcde} f_{abe}f_{cde}\ W^a_\mu W^b_\nu W^{c,\mu} W^{d,\nu}
$$

$$
\begin{aligned}
\Rightarrow \quad -\mathrm{i}g^2\ [\ & f_{abe}f_{cde}\ (g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \\
+\ & f_{ace}f_{dbe}\ (g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho}) \\
+\ & f_{ade}f_{bce}\ (g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho})]
\end{aligned}
$$

Proof:
Fixing $abcd$ the summation over repeated indices leads to the following set of permutations:

$$
\begin{aligned}
&\Gamma_{\mu\nu\lambda\rho}[V^a(k_1), V^b(k_2), V^c(k_3), V^d(k_4)]\epsilon_1^\mu\epsilon_2^\nu\epsilon_3^\lambda\epsilon_4^\rho \\
&= -\frac{\mathrm{i}}{4}g^2\{f_{abe}f_{cde}\ [(\epsilon_a\epsilon_c)(\epsilon_b\epsilon_d) - (ab) - (cd) + (ab)(cd)] \\
&\qquad + f_{cbe}f_{ade}\ [(\epsilon_c\epsilon_a)(\epsilon_b\epsilon_d) - (bc) - (ad) + (bc)(ad)] \qquad \Leftarrow a \leftrightarrow c \\
&\qquad + f_{dbe}f_{cae}\ [(\epsilon_d\epsilon_c)(\epsilon_b\epsilon_a) - (bd) - (ac) + (bd)(ac)] \qquad \Leftarrow a \leftrightarrow d \\
&\qquad + f_{ace}f_{bde}\ [(\epsilon_a\epsilon_b)(\epsilon_c\epsilon_d) - (ac) - (bd) + (ac)(bd)] \qquad \Leftarrow b \leftrightarrow c \\
&\qquad + f_{ade}f_{cbe}\ [(\epsilon_a\epsilon_c)(\epsilon_d\epsilon_b) - (ad) - (bc) + (ad)(bc)] \qquad \Leftarrow b \leftrightarrow d \\
&\qquad + f_{dce}f_{bae}\ [(\epsilon_d\epsilon_b)(\epsilon_c\epsilon_a) - (cd) - (ab) + (cd)(ab)]\} \qquad \Leftarrow a \leftrightarrow d\ \&\ b \leftrightarrow c \\
&= -\frac{\mathrm{i}}{4}g^2\{f_{abe}f_{cde}\ [(\epsilon_1\epsilon_3)(\epsilon_2\epsilon_4) - (\epsilon_2\epsilon_3)(\epsilon_1\epsilon_4) - (\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3) + (\epsilon_2\epsilon_4)(\epsilon_1\epsilon_3)] \\
&\qquad + f_{cbe}f_{ade}\ [(\epsilon_3\epsilon_1)(\epsilon_2\epsilon_4) - (\epsilon_2\epsilon_1)(\epsilon_3\epsilon_4) - (\epsilon_3\epsilon_4)(\epsilon_2\epsilon_1) + (\epsilon_2\epsilon_4)(\epsilon_3\epsilon_1)] \\
&\qquad + f_{dbe}f_{cae}\ [(\epsilon_4\epsilon_3)(\epsilon_2\epsilon_1) - (\epsilon_2\epsilon_3)(\epsilon_4\epsilon_1) - (\epsilon_4\epsilon_1)(\epsilon_2\epsilon_3) + (\epsilon_2\epsilon_1)(\epsilon_4\epsilon_3)] \\
&\qquad + f_{ace}f_{bde}\ [(\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4) - (\epsilon_3\epsilon_2)(\epsilon_1\epsilon_4) - (\epsilon_1\epsilon_4)(\epsilon_3\epsilon_2) + (\epsilon_3\epsilon_4)(\epsilon_1\epsilon_2)] \\
&\qquad + f_{ade}f_{cbe}\ [(\epsilon_1\epsilon_3)(\epsilon_4\epsilon_2) - (\epsilon_4\epsilon_3)(\epsilon_1\epsilon_2) - (\epsilon_1\epsilon_2)(\epsilon_4\epsilon_3) + (\epsilon_4\epsilon_2)(\epsilon_1\epsilon_3)] \\
&\qquad + f_{dce}f_{bae}\ [(\epsilon_4\epsilon_2)(\epsilon_3\epsilon_1) - (\epsilon_3\epsilon_2)(\epsilon_4\epsilon_1) - (\epsilon_4\epsilon_1)(\epsilon_3\epsilon_2) + (\epsilon_3\epsilon_1)(\epsilon_4\epsilon_2)]\} \\
&= -\mathrm{i}g^2\{f_{abe}f_{cde}\ [(\epsilon_1\epsilon_3)(\epsilon_2\epsilon_4) - (\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3)] \\
&\qquad + f_{ace}f_{dbe}\ [(\epsilon_1\epsilon_4)(\epsilon_2\epsilon_3) - (\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4)] \\
&\qquad + f_{ade}f_{bce}\ [(\epsilon_1\epsilon_2)(\epsilon_3\epsilon_4) - (\epsilon_1\epsilon_3)(\epsilon_2\epsilon_4)]\}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \Gamma_{\mu\nu\lambda\rho} = -\mathrm{i}g^2\ [\ & f_{abe}f_{cde}\ (g_{\mu\lambda}g_{\nu\rho} - g_{\mu\rho}g_{\nu\lambda}) \\
+\ & f_{ace}f_{dbe}\ (g_{\mu\rho}g_{\nu\lambda} - g_{\mu\nu}g_{\lambda\rho}) \\
+\ & f_{ade}f_{bce}\ (g_{\mu\nu}g_{\lambda\rho} - g_{\mu\lambda}g_{\nu\rho})]
\end{aligned}
$$



$$\mathcal{L}_{\rm FP} \supset -\, g f_{abc} (\partial^\mu \overline{c}_a) c_b W_\mu^c$$

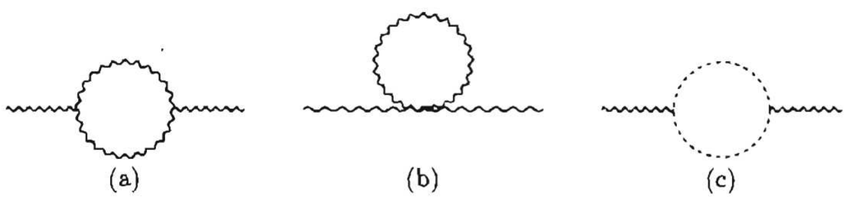
$$\Rightarrow \quad g f_{abc} p_\mu$$

Proof:
Momentum p is outgoing. Then:

$$\Gamma_\mu = -\mathrm{i} g f_{abc} \mathrm{i} p_\mu = g f_{abc} p_\mu$$

Exercise 4: Faddeev-Popov ghosts and gauge invariance

Consider the 1-loop self-energy diagrams for non-Abelian gauge theories in the figure. Calculate the diagrams in the 't Hooft-Feynman gauge and show that the sum does not have the tensor structure $g_{\mu\nu}k^2 - k_\mu k_\nu$ required by the gauge invariance of the theory unless diagram (c) involving ghost fields is included.



Hint: Take Feynman rules from previous excercise and use dimensional regularization. It is convenient to use the Passarino-Veltman tensor decomposition of loop integrals:

$$\frac{\mathrm{i}}{16\pi^2}\{B_0, B_\mu, B_{\mu\nu}\} = \mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{\{1, q_\mu, q_\mu q_\nu\}}{q^2(q+k)^2}$$

$$\text{where} \quad B_0 = \Delta_\epsilon + \text{finite}$$

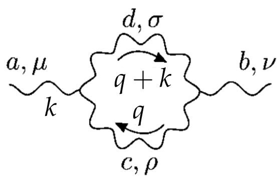
$$B_\mu = k_\mu B_1 \,, \quad B_1 = -\frac{\Delta_\epsilon}{2} + \text{finite}$$

$$B_{\mu\nu} = g_{\mu\nu} B_{00} + k_\mu k_\nu B_{11} \,, \quad B_{00} = -\frac{k^2}{12}\Delta_\epsilon + \text{finite} \,, \quad B_{11} = \frac{\Delta_\epsilon}{3} + \text{finite}$$

with $\Delta_\epsilon = 2/\epsilon - \gamma + \ln 4\pi$ and $D = 4 - \epsilon$. You may check that the ultraviolet divergent part has the expected structure or find the final result in terms of scalar integrals, that for massless fields read:

$$B_1 = -\frac{1}{2} B_0 \,, \quad B_{00} = -\frac{k^2}{4(D-1)} B_0 \,, \quad B_{11} = \frac{D}{4(D-1)} B_0 \,.$$

Do not forget a symmetry factor (1/2) in front of (a) and (b), and a factor (-1) in (c).



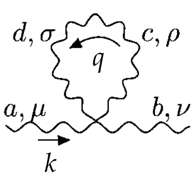
$$
\begin{aligned}
&= \frac{1}{2}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{-\mathrm{i}}{q^2}\frac{-\mathrm{i}}{(q+k)^2} g^2 f_{acd} f_{bcd} N_{\mu\nu} \\
&= -\frac{g^2 C_2(G)\delta_{ab}}{2}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{N_{\mu\nu}}{q^2(q+k)^2}
\end{aligned}
$$

where $f_{acd}f_{bcd} \equiv C_2(G)\delta_{ab}$ (Einstein summation convention) $[C_2(G) = N$ for $G = SU(N)]$

and $N_{\mu\nu} = \Gamma^{\mu\sigma\rho}(k, -q-k, q)\Gamma^{\nu}_{\sigma\rho}(-k, q+k, -q)$

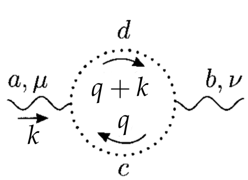
$$
= -g_{\mu\nu}(2q^2 + 5k^2 + 2qk) - 10 q_\mu q_\nu - 5(q_\mu k_\nu + q_\nu k_\mu) + 2k_\mu k_\nu
$$

$$
\begin{aligned}
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\Big\{-g_{\mu\nu}(2g_{\rho\sigma}B^{\rho\sigma} + 5k^2 B_0 + 2k_\rho B^\rho) - 10B_{\mu\nu} - 5(B_\mu k_\nu + B_\nu k_\mu) + 2k_\mu k_\nu B_0\Big\} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\Big\{-g_{\mu\nu}[(2D+10)B_{00} + 2k^2 B_{11} + 2k^2 B_1 + 5k^2 B_0] - k_\mu k_\nu(10B_{11} + 10B_1 - 2B_0)\Big\} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\frac{B_0}{2(D-1)}\Big\{-(8D-13)g_{\mu\nu}k^2 + (9D-14)k_\mu k_\nu\Big\} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2}\Delta_\epsilon\left\{-\frac{19}{6}g_{\mu\nu}k^2 + \frac{11}{3}k_\mu k_\nu\right\} + \text{finite}
\end{aligned}
$$



$$
\begin{aligned}
&= \frac{1}{2}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{-\mathrm{i}g^{\rho\sigma}}{q^2}\delta_{cd}(-\mathrm{i}g^2)[\ f_{abe}f_{cde}\ (g_{\mu\sigma}g_{\nu\rho} - g_{\mu\rho}g_{\nu\sigma}) \\
&\qquad + f_{ace}f_{dbe}\ (g_{\mu\rho}g_{\nu\sigma} - g_{\mu\nu}g_{\sigma\rho}) \\
&\qquad + f_{ade}f_{bce}\ (g_{\mu\nu}g_{\sigma\rho} - g_{\mu\sigma}g_{\nu\rho})] \\
&= -g^2 C_2(G)\delta_{ab}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{(D-1)g_{\mu\nu}}{q^2} \\
&= -g^2 C_2(G)\delta_{ab}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{(D-1)g_{\mu\nu}}{q^2}\frac{q^2+2qk+k^2}{(q+k)^2} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(D-1)g_{\mu\nu}(g_{\rho\sigma}B^{\rho\sigma} + 2k_\rho B^\rho + k^2 B_0) \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(D-1)g_{\mu\nu}(DB_{00} + k^2 B_{11} + 2k^2 B_1 + k^2 B_0) \\
&= 0
\end{aligned}
$$

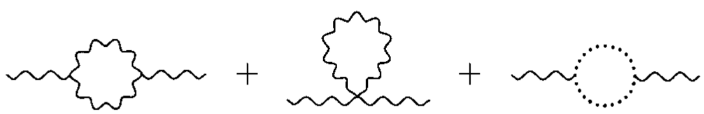
(as expected, since *scaleless* integrals always vanish in dimensional regularization.)



$$
= (-1)\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D}\frac{\mathrm{i}}{q^2}\frac{\mathrm{i}}{(q+k)^2} g^2 f_{dca}\ (q_\mu + k_\mu)\ f_{cdb}\ q_\nu
$$

$$
\begin{aligned}
&= -\mathrm{i}g^2 C_2(G)\delta_{ab}\mu^\epsilon \int \frac{\mathrm{d}^D q}{(2\pi)^D} \frac{(q_\mu + k_\mu)q_\nu}{q^2(q+k)^2} \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(B_{\mu\nu} + k_\mu B_\nu) \\
&= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2}(g_{\mu\nu}B_{00} + k_\mu k_\nu B_{11} + k_\mu k_\nu B_1) \\
&= \frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2} \frac{B_0}{2(D-1)} \left\{ g_{\mu\nu}k^2 + (D-2)k_\mu k_\nu \right\} \\
&= \frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{32\pi^2} \Delta_\epsilon \left\{ \frac{1}{6} g_{\mu\nu}k^2 + \frac{1}{3}k_\mu k_\nu \right\} + \text{finite}
\end{aligned}
$$

Summing all three diagrams (actually diagram (b) does not contribute):



$$
= -\frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2} \frac{B_0}{D-1}(2D-3) \left\{ g_{\mu\nu}k^2 - k_\mu k_\nu \right\} = \frac{\mathrm{i}g^2 C_2(G)\delta_{ab}}{16\pi^2} \Delta_\epsilon \frac{5}{3} \left\{ g_{\mu\nu}k^2 - k_\mu k_\nu \right\} + \text{finite}
$$

Exercise 5: Propagator of a massive vector boson field

Consider the Proca Lagrangian of a massive vector boson field

$$
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu , \quad \text{with} \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .
$$

Show that the propagator of A_μ is

$$
\widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}0} \left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right]
$$

$$
\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M^2 A_\mu A^\mu = -\frac{1}{2}(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu) + \frac{1}{2}M^2 A_\mu A^\mu
$$

Euler-Lagrange: $\displaystyle \frac{\partial \mathcal{L}}{\partial A_\nu} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad \Rightarrow \quad M^2 A^\nu + \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0$

$$
\Rightarrow \quad [g^{\mu\nu}(\Box + M^2) - \partial^\mu \partial^\nu]A_\mu = 0
$$

Note that if $M \neq 0$ from the first equation one has $\partial_\mu A^\mu = 0$ (and this is not due to a gauge symmetry). The propagator is i times the inverse of the operator in square brackets. In momentum space:

$$
\widetilde{D}_{\mu\nu}(k) = \mathrm{i}[-g^{\mu\nu}(k^2 - M^2) + k^\mu k^\nu]^{-1} = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}0} \left[-g_{\mu\nu} + \frac{k_\mu k_\nu}{M^2} \right]
$$

where the Feynman's prescription has been included. To show that this is actually the inverse, check:

$$
\widetilde{D}_{\mu\nu}(k)[-g^{\nu\rho}(k^2 - M^2) + k^\nu k^\rho] = \mathrm{i}\delta_\mu^\rho
$$

Exercise 6: Propagator of a massive gauge field

Consider the U(1) gauge invariant Lagrangian $\mathcal{L}$ with gauge fixing $\mathcal{L}_{\mathrm{GF}}$:

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + (D_\mu\phi)^\dagger(D^\mu\phi) - \mu^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

$$\mathcal{L}_{\mathrm{GF}} = -\frac{1}{2\xi}(\partial_\mu A^\mu - \xi M_A\chi)^2\,, \quad \text{with} \quad D_\mu = \partial_\mu + \mathrm{i}eA_\mu\,, \quad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where $M_A = ev$ after spontaneous symmetry breaking ($\mu^2 < 0$, $\lambda > 0$) when the complex scalar field ϕ acquires a VEV and is parameterized by

$$\phi(x) = \frac{1}{\sqrt{2}}[v + \varphi(x) + \mathrm{i}\chi(x)]\,, \quad \mu^2 = -\lambda v^2.$$

Show that the propagators of φ, χ and the gauge field A_μ are respectively

$$\widetilde{D}^\varphi(k) = \frac{\mathrm{i}}{k^2 - M_\varphi^2 + \mathrm{i}0} \quad \text{with } M_\varphi^2 = -2\mu^2 = 2\lambda v^2$$

$$\widetilde{D}^\chi(k) = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}0}\,, \quad \widetilde{D}_{\mu\nu}(k) = \frac{\mathrm{i}}{k^2 - M_A^2 + \mathrm{i}0}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]$$

Writing ϕ in terms of φ and χ:

$$\begin{aligned}\mathcal{L} + \mathcal{L}_{\mathrm{GF}} = &-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}M_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\\ &+\frac{1}{2}(\partial_\mu\chi)(\partial^\mu\chi) - \frac{1}{2}\xi M_A^2\chi^2\\ &+\frac{1}{2}(\partial_\mu\varphi)(\partial^\mu\varphi) - \lambda v^2\varphi^2 + \dots\end{aligned}$$

- Propagator of A_μ:

  Euler-Lagrange: $$\frac{\partial\mathcal{L}}{\partial A_\nu} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = 0 \quad\Rightarrow\quad \partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) + \frac{1}{\xi}\partial^\nu\partial^\mu A_\mu + M_A^2 A^\nu = 0$$
  $$\Rightarrow\quad \left[g^{\mu\nu}(\Box + M_A^2) - \left(1 - \frac{1}{\xi}\right)\partial^\mu\partial^\nu\right]A_\mu = 0$$

  The propagator is i times the inverse of the operator in square brackets. In momentum space:

  $$\widetilde{D}_{\mu\nu}(k) = \mathrm{i}\left[-g^{\mu\nu}(k^2 - M_A^2) + \left(1 - \frac{1}{\xi}\right)k^\mu k^\nu\right]^{-1} = \frac{\mathrm{i}}{k^2 - M^2 + \mathrm{i}0}\left[-g_{\mu\nu} + (1-\xi)\frac{k_\mu k_\nu}{k^2 - \xi M_A^2}\right]$$

  where the Feynman's prescription has been included. In fact,

  $$\widetilde{D}_{\mu\nu}(k)\left[-g^{\nu\rho}(k^2 - M_A^2) + \left(1 - \frac{1}{\xi}\right)k^\nu k^\rho\right] = \mathrm{i}\delta_\mu^\rho$$

- Propagator of χ:

  Euler-Lagrange: $$\frac{\partial\mathcal{L}}{\partial\chi} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\chi)} = 0 \quad\Rightarrow\quad [\Box\chi + \xi M_A^2]\chi = 0$$

  The propagator is $-\mathrm{i}$ times the inverse of the operator in square brackets. In momentum space:

  $$\widetilde{D}(k) = -\mathrm{i}[-k^2 + \xi M_A^2]^{-1} = \frac{\mathrm{i}}{k^2 - \xi M_A^2 + \mathrm{i}0}$$

– Propagator of φ. Similarly to previous case:

$$\widetilde{D}(k) = \frac{\mathrm{i}}{k^2 - M_\varphi^2 + \mathrm{i}0}\,, \quad M_\varphi^2 = 2\lambda v^2$$

Exercise 7: The conjugate Higgs doublet

Show that $\Phi^c \equiv i\sigma_2\Phi^* = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix}$ transforms under SU(2) like $\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix}$, with $\phi^- = (\phi^+)^*$. What are the weak isospins, hypercharges and electric charges of ϕ^0, ϕ^{0*}, ϕ^+, ϕ^-?
Hint: Use the property of Pauli matrices: $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$.

Consider an infinitesimal SU(2) transformation:

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \mapsto \left(1 - \mathrm{i}\frac{\sigma_i}{2}\delta\theta^i\right)\Phi$$

$$\Rightarrow \Phi^c = \begin{pmatrix} \phi^{0*} \\ -\phi^- \end{pmatrix} = \mathrm{i}\sigma_2\Phi^* \mapsto \mathrm{i}\sigma_2\left(1 - \mathrm{i}\frac{\sigma_i}{2}\delta\theta^i\right)^*\Phi^* = \left(1 + \mathrm{i}\sigma_2\frac{\sigma_i^*}{2}\sigma_2\delta\theta^i\right)\mathrm{i}\sigma_2\Phi^*$$
$$= \left(1 - \mathrm{i}\frac{\sigma_i}{2}\delta\theta^i\right)\Phi^c$$

Under a U(1) transformation:

$$\Phi \mapsto \mathrm{e}^{-\mathrm{i}\alpha Y}\Phi \quad \Rightarrow \quad \Phi^c \mapsto \mathrm{e}^{\mathrm{i}\alpha Y}\Phi^c$$

Then, using $Q = T_3 + Y$ and taking Φ with hypercharge $y = \frac{1}{2}$ we have

	ϕ^0	ϕ^+	ϕ^{0*}	ϕ^-
T_3	$-\frac{1}{2}$	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$
Y	$\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$
Q	0	1	0	-1

Exercise 8: Lagrangian and Feynman rules of the Standard Model

Try to reproduce the Lagrangian and the corresponding Feynman rules of as many Standard Model interactions as you can. Of particular interest/difficulty are [VVV] and [VVVV].

Check your results in http://www.ugr.es/local/jillana/SM/FeynmanRulesSM.pdf (taken from FeynArts)

Exercise 9: Z pole observables at tree level

Show that

(a) $\Gamma(f\bar{f}) \equiv \Gamma(Z \to f\bar{f}) = N_c^f \dfrac{\alpha M_Z}{3}(v_f^2 + a_f^2)\,, \quad N_c^f = 1\ (3)$ for $f =$ lepton (quark)

(b) $$\sigma_{\text{had}} = 12\pi \frac{\Gamma(e^+e^-)\Gamma(\text{had})}{M_Z^2\Gamma_Z^2}$$

(c) $$A_{FB} = \frac{3}{4}A_f\,, \quad \text{with } A_f = \frac{2v_f a_f}{v_f^2 + a_f^2}$$

(a) Amplitude for $Z \to f(p_1)\overline{f}(p_2)$ in the SM at tree level:

$$\mathrm{i}\mathcal{M} = \overline{u}(p_1)\mathrm{i}e\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\epsilon_\mu(\lambda)$$

Averaging over the 3 initial polarizations and summing over final polarizations:

$$\begin{aligned}\widetilde{\sum}|\mathcal{M}|^2 &= \frac{e^2}{3}\sum_{\lambda=\pm,0}\epsilon_\mu(\lambda)\epsilon_\nu^*(\lambda)\ \overline{u}(p_1)\gamma^\mu(v_f - a_f\gamma_5)v(p_2)\overline{v}(p_2)(v_f + a_f\gamma_5)\gamma^\nu u(p_1) \\ &= -\frac{e^2}{3}\mathrm{Tr}\{\not{p}_1\gamma^\mu(v_f - a_f\gamma_5)\not{p}_2(v_f + a_f\gamma_5)\gamma_\mu\} \\ &= \frac{e^2}{3}8p_1p_2(v_f^2 + a_f^2) = \frac{e^2}{3}4M_Z^2(v_f^2 + a_f^2)\end{aligned}$$

where fermion masses have been neglected and we have inserted:

$$\sum_{\lambda=\pm,0}\epsilon_\mu(\lambda)\epsilon_\nu^*(\lambda) = -g_{\mu\nu} + \frac{k_\mu k_\nu}{M_Z^2}$$

(the second term does not contribute in the limit of massless fermions) and substituted:

$$2p_1p_2 = (p_1 + p_2)^2 = M_Z^2$$

The differential width is then:

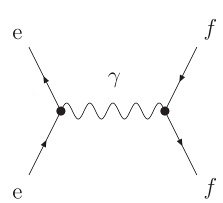
$$\frac{\mathrm{d}\Gamma_Z}{\mathrm{d}\Omega} = \frac{1}{32\pi^2}\frac{|\vec{p}|}{M_Z^2}|\mathcal{M}|^2 = \frac{1}{64\pi^2 M_Z}|\mathcal{M}|^2 = N_c^f\frac{\alpha}{12\pi}M_Z(v_f^2 + a_f^2)$$

(isotropic) where a factor has been included to account for a sum over 3 colors in the case of the fermion being a quark, and the total width is:

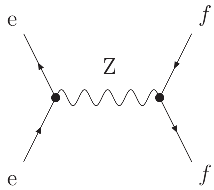
$$\Gamma_Z = 4\pi\frac{\mathrm{d}\Gamma_Z}{\mathrm{d}\Omega} = N_c^f\frac{\alpha M_Z}{3}(v_f^2 + a_f^2)$$

(b) Amplitude for $e^+(p_1)e^-(p_2) \to f(p_3)\overline{f}(p_4)$ with $f \neq e$ in the SM at tree level (unitary gauge):

$$\mathcal{M} = \mathcal{M}_\gamma + \mathcal{M}_Z$$



$$\mathrm{i}\mathcal{M}_\gamma = \overline{u}(p_3)\ (-\mathrm{i}eQ_f)\gamma^\mu\ v(p_4)\frac{-\mathrm{i}g_{\mu\nu}}{s}\ \overline{v}(p_1)\ (-\mathrm{i}eQ_e)\gamma^\nu\ u(p_2)$$



$$i\mathcal{M}_Z = \overline{u}(p_3)\ ie\gamma^\mu (v_e - a_e\gamma_5)\ v(p_4) \frac{i(-g_{\mu\nu} + k_\mu k_\nu / M_Z^2)}{s - M_Z^2 + iM_Z\Gamma_Z}$$
$$\times\ \overline{v}(p_1)\ ie\gamma^\nu (v_f - a_f\gamma_5)\ u(p_2)$$

where the term proportional to $k_\mu k_\nu$ is irrelevant in the limit of $m_e = 0$. The cross-section in the CM (unpolarized case and $m_e = 0$) is, after some Diracology:

$$\frac{d\sigma}{d\Omega} = \frac{\beta_f}{64\pi^2}\frac{N_c^f}{4}\widetilde{\sum}|\mathcal{M}|^2$$
$$= N_c^f \frac{\alpha^2}{4s}\beta_f \left\{ \left[1 + \cos^2\theta + (1-\beta_f^2)\sin^2\theta\right] G_1(s) + 2(\beta_f^2 - 1)G_2(s) + 2\beta_f \cos\theta G_3(s) \right\}$$

with $G_1(s) = Q_e^2 Q_f^2 + 2Q_e Q_f v_e v_f \mathrm{Re}\chi_Z(s) + (v_e^2 + a_e^2)(v_f^2 + a_f^2)|\chi_Z(s)|^2$

$$G_2(s) = (v_e^2 + a_e^2)a_f^2 |\chi_Z(s)|^2$$
$$G_3(s) = 2Q_e Q_f a_e a_f \mathrm{Re}\chi_Z(s) + 4v_e v_f a_e a_f |\chi_Z(s)|^2$$

$$\Rightarrow \quad \sigma(s) = N_c^f \frac{2\pi\alpha^2}{3s}\beta_f \left[(3-\beta_f^2)G_1(s) - 3(1-\beta_f^2)G_2(s)\right]$$

where $\chi_Z(s) \equiv \dfrac{s}{s - M_Z^2 + iM_Z\Gamma_Z}$ and $N_c^f = 1$ (3) for $f =$ lepton (quark). It is easy to guess which terms come from the exchange of γ, Z and the interference.

At the Z pole ($s = M_Z^2$) the interference terms vanish. Neglecting all fermion masses and the γ exchange, the total cross-section is

$$\sigma(M_Z^2) = N_c^f \frac{4\pi\alpha^2}{3M_Z^2} G_1(M_Z^2) = N_c^f \frac{4\pi\alpha^2}{3\Gamma_Z^2}(v_e^2 + a_e^2)(v_f^2 + a_f^2)$$

We see that in fact

$$\sigma_{\text{had}} = 12\pi \frac{\Gamma(e^+e^-)\Gamma(\text{had})}{M_Z^2\Gamma_Z^2} = 3\frac{4\pi\alpha^2}{3\Gamma_Z^2}(v_e^2 + a_e^2)(v_f^2 + a_f^2)$$

(c) The forward-backward asymmetry at the Z pole, neglecting fermion masses and γ exchange, is:

$$A_{FB} = \frac{\sigma_F - \sigma_B}{\sigma_F + \sigma_B} = \frac{3}{4}\frac{G_3(M_Z^2)}{G_1(M_Z^2)} = \frac{3}{4}A_f A_e \quad \text{with } A_f = \frac{2v_f a_f}{v_f^2 + a_f^2}$$

Exercise 10: Higgs partial decay widths at tree level

Show that

(a) $$\Gamma(H \to f\bar{f}) = N_c^f \frac{G_F M_H}{4\pi\sqrt{2}} m_f^2 \left(1 - \frac{4m_f^2}{M_H^2}\right)^{3/2}, \quad N_c^f = 1\ (3) \text{ for } f = \text{lepton (quark)}$$

(b) $$\Gamma(H \to W^+W^-) = \frac{G_F M_H^3}{8\pi\sqrt{2}}\sqrt{1 - \frac{4M_W^2}{M_H^2}}\left(1 - \frac{4M_W^2}{M_H^2} + \frac{12M_W^4}{M_H^4}\right)$$

$$\Gamma(H\to ZZ)=\frac{G_F M_H^3}{16\pi\sqrt{2}}\sqrt{1-\frac{4M_Z^2}{M_H^2}}\left(1-\frac{4M_Z^2}{M_H^2}+\frac{12M_Z^4}{M_H^4}\right)$$

(a) Amplitude for $H\to f(p_1)\overline{f}(p_2)$ (unpolarized) in the SM at tree level:

$$\begin{aligned}
\mathrm{i}\mathcal{M}&=-\mathrm{i}\frac{em_f}{2s_W M_W}\overline{u}(p_1)v(p_2)\\
\Rightarrow\quad \widetilde{\sum}|\mathcal{M}|^2&=\frac{e^2m_f^2}{4s_W^2M_W^2}\overline{u}(p_1)v(p_2)\overline{v}(p_2)u(p_1)\\
&=\frac{e^2m_f^2}{4s_W^2M_W^2}\mathrm{Tr}\{(\not p_1+m)(\not p_2-m)\}\\
&=\frac{e^2m_f^2}{4s_W^2M_W^2}(4p_1p_2-4m_f^2)\\
&=\frac{e^2m_f^2}{4s_W^2M_W^2}2M_H^2\left(1-\frac{4m_f^2}{M_H^2}\right)
\end{aligned}$$

where we have used that $(p_1+p_2)^2=2m_f^2+2p_1p_2=M_H^2$.

The unpolarized decay is isotropic, so the total width is 4π times the differential decay width:

$$\Gamma=4\pi\frac{1}{32\pi^2}\frac{|\vec{p}|}{M_H^2}|\mathcal{M}|^2=\frac{1}{16\pi M_H}\sqrt{1-\frac{4m_f^2}{M_H^2}}|\mathcal{M}|^2$$

where we have used that the CM trimomentum of the final state femions is $|\vec{p}|=\frac{M_H}{2}\sqrt{1-\frac{4m_f^2}{M_H^2}}$.

Therefore, summing over $N_c^f=3$ colors in the case of quarks:

$$\Gamma=N_c^f\frac{\alpha m_f^2M_H}{8M_W^2s_W^2}\left(1-\frac{4m_f^2}{M_H^2}\right)^{3/2}=N_c^f\frac{G_FM_H}{4\pi\sqrt{2}}m_f^2\left(1-\frac{4m_f^2}{M_H^2}\right)^{3/2}$$

(b) Amplitude for $H\to V(p_1)V(p_2)$ (unpolarized) in the SM at tree level:

$$\begin{aligned}
\mathrm{i}\mathcal{M}&=\mathrm{i}eK\epsilon_\mu^*(p_1,\lambda_1)\epsilon^\mu(p_2,\lambda_2)\\
\Rightarrow\quad \widetilde{\sum}|\mathcal{M}|^2&=e^2K^2\left(-g_{\mu\nu}+\frac{p_{1\mu}p_{1\nu}}{M_V^2}\right)\left(-g^{\mu\nu}+\frac{p_2^\mu p_2^\nu}{M_V^2}\right)\\
&=\frac{e^2K^2}{4M_V^4}(M_H^4-4M_V^2M_H^2+12M_V^4)
\end{aligned}$$

where $p_1p_2=(M_H^2-2M_V^2)/2$ and

$$K=\frac{M_W}{s_W}\quad\text{for } HW^+W^-\ ,\quad K=\frac{M_W}{s_Wc_W^2}\quad\text{for } HZZ\ .$$

The unpolarized decay is isotropic, so the total width is 4π times the differential decay width for HW^+W^-:

$$\Gamma(H\to W^+W^-)=\frac{1}{16\pi M_H}\sqrt{1-\frac{4M_W^2}{M_H^2}}\ \widetilde{\sum}|\mathcal{M}|^2$$

$$= \frac{G_F M_H^3}{8\pi\sqrt{2}} \sqrt{1 - \frac{4M_W^2}{M_H^2}} \left(1 - \frac{4M_W^2}{M_H^2} + \frac{12M_W^4}{M_H^4}\right)$$

and 2π times the differential decay width for HZZ, since there are two identical particles in the final state:

$$\Gamma(H \to ZZ) = \frac{G_F M_H^3}{16\pi\sqrt{2}} \sqrt{1 - \frac{4M_Z^2}{M_H^2}} \left(1 - \frac{4M_Z^2}{M_H^2} + \frac{12M_Z^4}{M_H^4}\right)$$