Freie Universität Berlin

Majorana dimer models: Tensor networks and holographic quantum error correction

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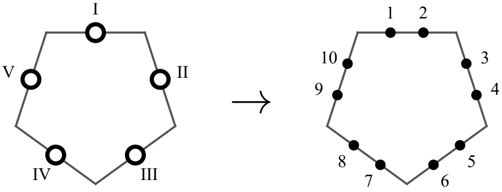
Benasque, February 26, 2019

Introduction to Majorana dimers

From spins to Majorana modes

We can map a closed chain of N spins to $2N$ Majorana modes:



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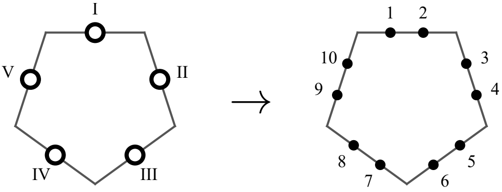

This is performed by a *Jordan-Wigner transformation* from spin operators to fermionic Majorana modes:

$$\gamma_{2k-1} = (\sigma^z)^{\otimes(k-1)} \otimes \sigma^x \otimes \mathbb{1}^{\otimes(N-k)},$$
$$\gamma_{2k} = (\sigma^z)^{\otimes(k-1)} \otimes \sigma^y \otimes \mathbb{1}^{\otimes(N-k)},$$

In terms of standard fermionic operators f_k and $\mathrm{f}_k^\dagger$:

$$\mathrm{f}_k = (\gamma_{2k-1} + \mathrm{i}\,\gamma_{2k})/2\ , \quad \mathrm{f}_k^\dagger = (\gamma_{2k-1} - \mathrm{i}\,\gamma_{2k})/2\ .$$

Majorana dimer states

A special class of fermionic Gaussian states:
Majorana dimers

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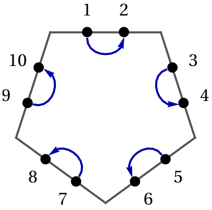

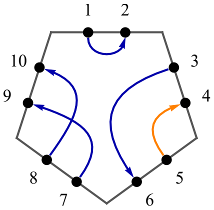

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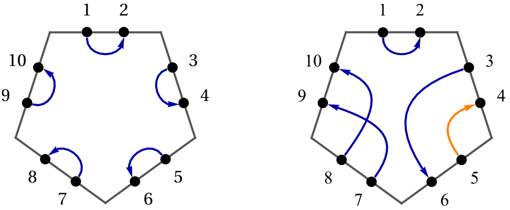
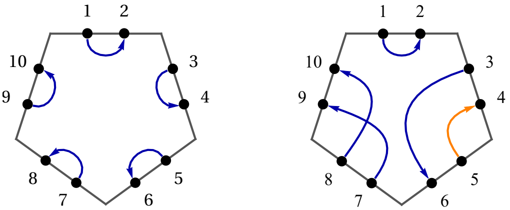


An arrow $j \to k$ indicates that the operator $\gamma_j + \mathrm{i}\,\gamma_k$ annihilates the state vector. Ordering of j and k defines *local parity* $p_{j,k}$:

$$p_{j,k} = \begin{cases} +1 & \text{for } j < k \text{ (blue)} \\ -1 & \text{for } j > k \text{ (orange)} \end{cases}$$

Majorana dimer states



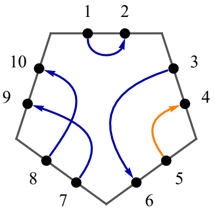
A Majorana dimer state is also the ground state of a Hamiltonian (Ware et al., 2016)

$$H = \frac{\mathrm{i}}{2} \sum_{k=1}^{N} \gamma_{\sigma(2k-1)} \gamma_{\sigma(2k)} \quad ,$$

where $\sigma(k)$ is a permutation of the $2N$ Majorana modes.

Majorana dimer states

The physical properties are determined by the *covariance matrix* $\Gamma^{\psi}_{j,k} = \langle\psi|\, \mathrm{i}\,[\gamma_j, \gamma_k]\, |\psi\rangle / 2$ which can be read from the diagram:



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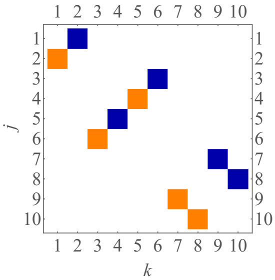


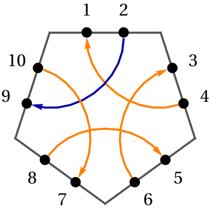
(orange$= +1$, blue$= -1$)

$$\Gamma^{\psi}_{j,k} = \begin{cases} -1 & \text{for an arrow } j \to k \\ +1 & \text{for an arrow } k \to j \\ 0 & \text{if no arrow connects } j \text{ and } k \end{cases} .$$

Majorana dimer states: Examples

N-fermion GHZ states $|\pm\rangle_N^{\mathsf{GHZ}} = \frac{1}{\sqrt{2}}\big(|+\mathrm{i}\rangle^{\otimes N} \pm |-\mathrm{i}\rangle^{\otimes N}\big)$:

$|+\rangle_5^{\mathsf{GHZ}} =$



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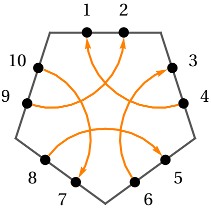

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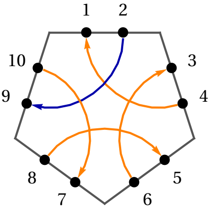

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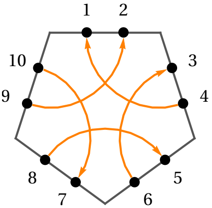
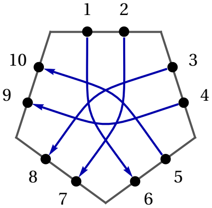
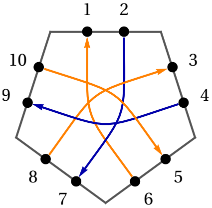


For every N, the GHZ state exhibits a $2k+2 \to 2k-1$ pairing.

Majorana dimer states: Examples

$[[5,1,3]]$ quantum error-correcting code, with two logical states:

$|\bar{0}\rangle_5 =$ 

$|\bar{1}\rangle_5 =$ 

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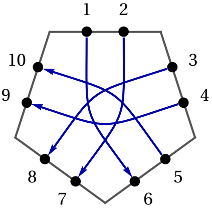
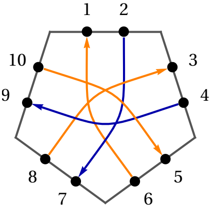
$\left|\bar{0}\right\rangle_5 =$ 

$\left|\bar{1}\right\rangle_5 =$ 

The corresponding stabilizer Hamiltonian $H = \sum_i S_i$ consists of

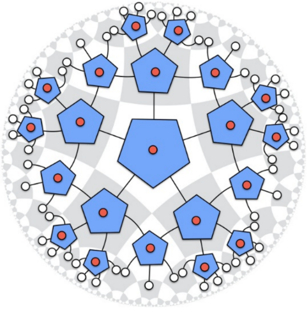
$$S = \{\, \mathrm{i}\,\mathcal{P}_{\text{tot}}\,\gamma_1\,\gamma_6,\ \mathrm{i}\,\gamma_2\,\gamma_7,\ \mathrm{i}\,\mathcal{P}_{\text{tot}}\,\gamma_3\,\gamma_8,\ \mathrm{i}\,\gamma_4\,\gamma_9,\ \mathrm{i}\,\mathcal{P}_{\text{tot}}\,\gamma_5\,\gamma_{10} \}\ ,$$

with the total parity operator $\mathcal{P}_{\text{tot}}$ distinguishing $\left|\bar{0}\right\rangle_5$ from $\left|\bar{1}\right\rangle_5$.

The HaPPY code

The HaPPY code: Definition

The $[[5,1,3]]$ code is the building block of the holographic toy model by Harlow, Pastawski, Preskill and Yoshida (HaPPY, 2015):



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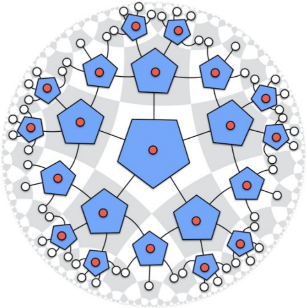
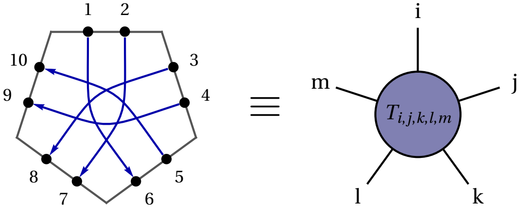


This hyperbolic $\{5, 4\}$ tiling is filled with the tensors encoding the [[5, 1, 3]] code, giving a mapping between **bulk** logical qubits and physical **boundary** spins.

HaPPY code → Majorana dimers?

For each fixed logical input $\bar{0}$ or $\bar{1}$, the HaPPY code is a tiling of Majorana dimer states. Each "tile" corresponds to a tensor:



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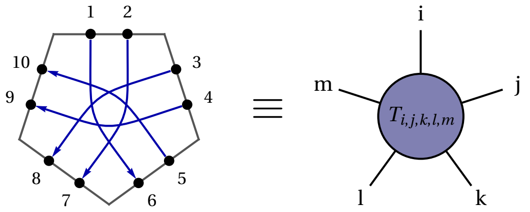

But what happens with Majorana dimers under contraction?

Intermezzo: Contraction rules for Majorana dimers

...after a lot of Grassmann algebra, we find that Majorana dimer states are closed under contraction. Examples:

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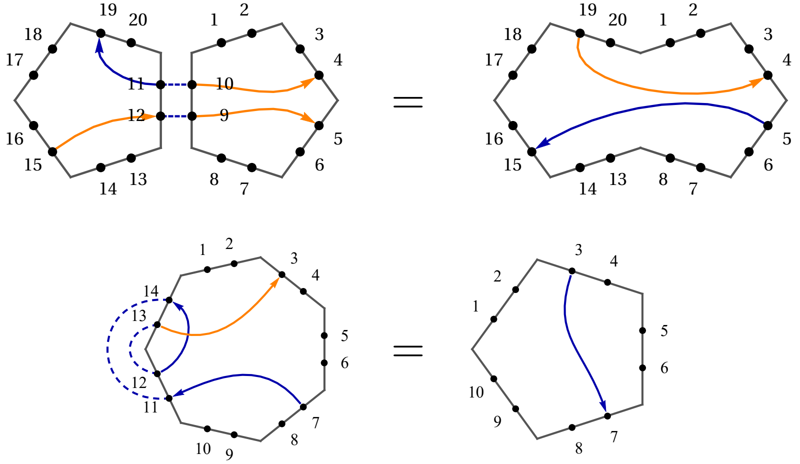

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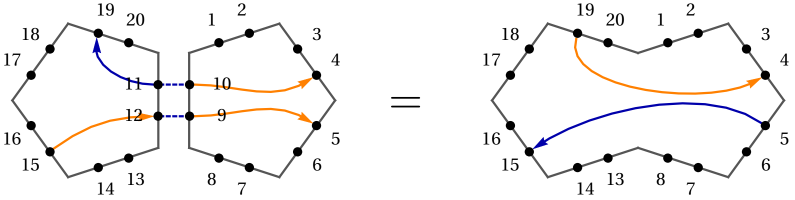

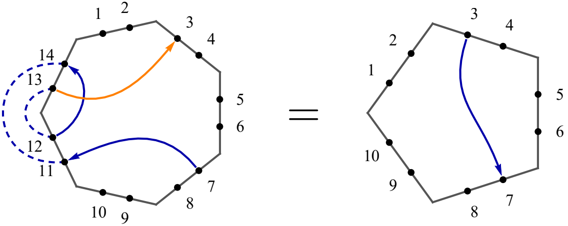

Dimers "fuse" under contraction!

Contracting Majorana dimers: The rules

- Contracting neighboring edges k and $k+1$ removes the Majorana modes $\{2k-1, 2k, 2k+1, 2k+2\}$. The dimers ending on $2k-1$ and $2k+2$ as well as $2k$ and $2k+1$ are fused into larger dimers.

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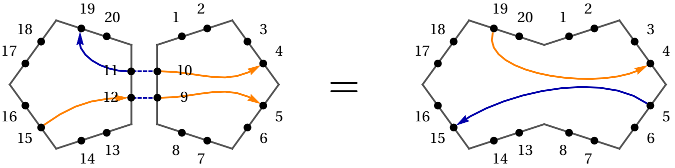

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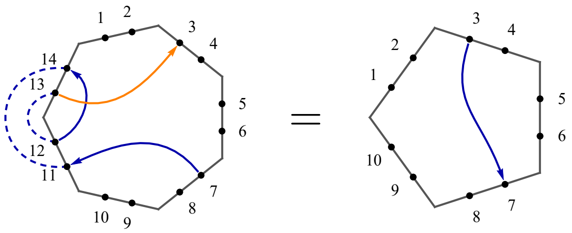

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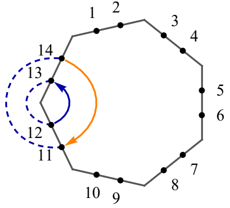
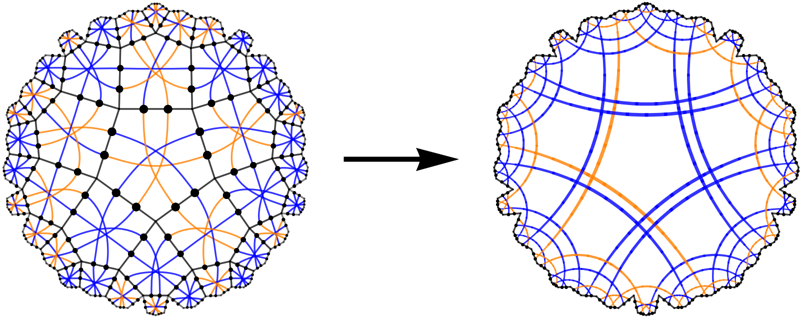


$= \; 0$

Results

HaPPY code $\rightarrow$ Majorana dimers

Given local inputs $\bar{0}$ and $\bar{1}$ in the bulk, we can now perform contractions:



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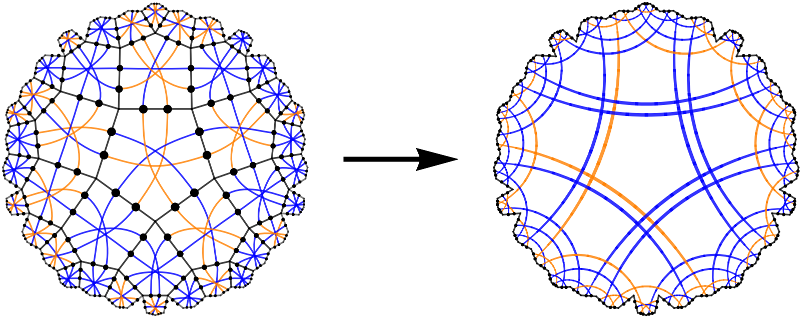
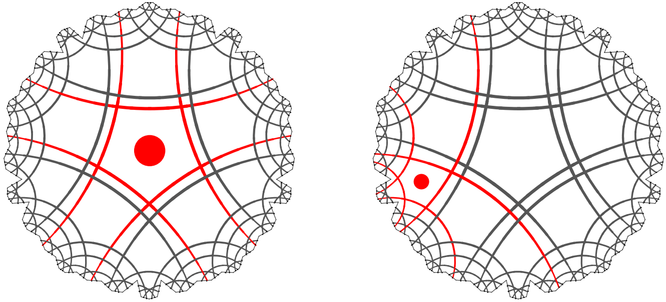


The dimer structure follows discrete *bulk geodesics*!

Bulk/boundary mapping

The action of any bulk operator can be mapped directly on a set of Majorana sites on the boundary:



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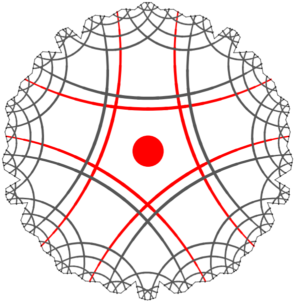

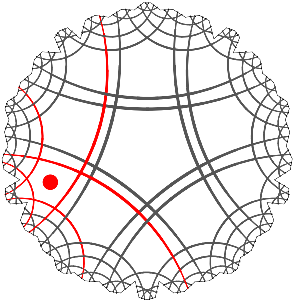

Conversely, the bulk state can be reconstructed from a subset of these boundary sites. (Almheiri et al., 2015)

HaPPY code $\to$ Majorana dimers

Full HaPPY code: Local superpositions of $\bar{0}$ and $\bar{1}$ inputs.

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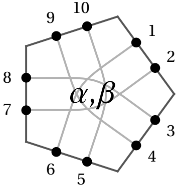
$\Rightarrow$ We need to consider superpositions of dimers:



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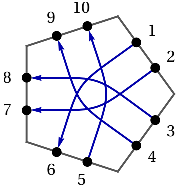
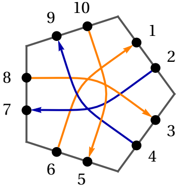


$+ \beta$



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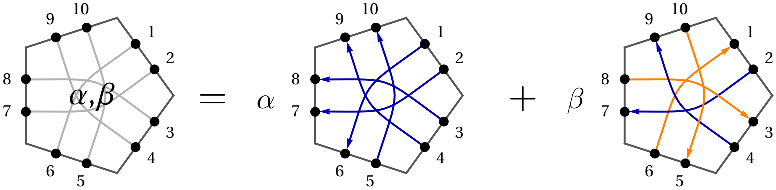

Now let's replace each tile with such a superposition...

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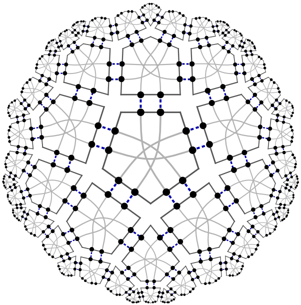
$|\psi\rangle \sim$ 

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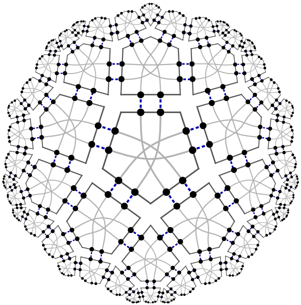
$|\psi\rangle \sim$ 

Contraction of n tiles $\equiv$ sum of 2^n Majorana dimer states $|\psi_k\rangle$.
But: $\langle\psi_j|\,\gamma_a\,\gamma_b\,|\psi_k\rangle = 0$ for any $j \neq k$.

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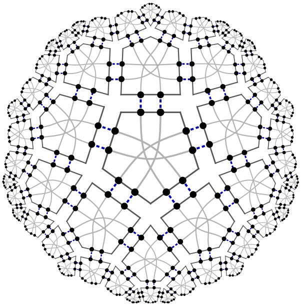
$|\psi\rangle \sim$ 

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Thus, the HaPPY code's two-point functions are just convex sums of Gaussian covariance matrices!

Entanglement entropy

The entanglement entropy $S_A = -\mathrm{tr}[\rho_A \log \rho_A]$ of a subsystem A of Majorana sites can be computed graphically: Each dimer connecting A with $\bar{A}$ contributes $\frac{1}{2}\log 2$ to S_A.

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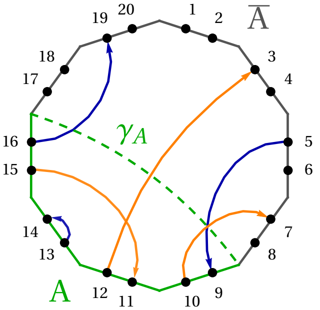
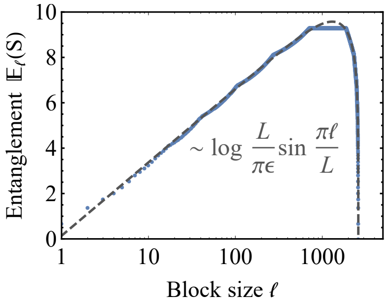


For example, this diagrams implies $S_A = 2\log 2$. For a contraction, S_A is bounded above by $|\gamma_A|\log 2$ along a minimal cut γ_A, resembling the holographic entanglement entropy formula (Ryu/Takayanagi, 2006).

Entanglement entropy

EE scaling of the HaPPY code can now be computed (for fixed input $\bar{0}$ or $\bar{1}$):



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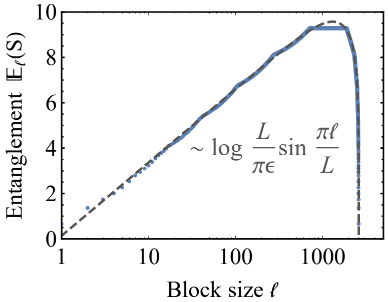

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We find a logarithmic scaling expected for CFTs
(Calabrese/Cardy, 2004) with central charge $c \approx 4.2$.

Further entanglement properties

Full HaPPY code: Local superpositions of $\bar{0}$ and $\bar{1}$ inputs.

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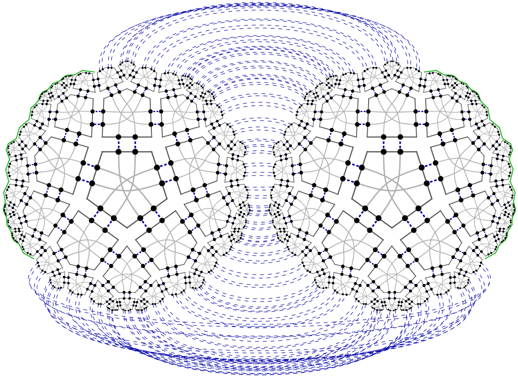

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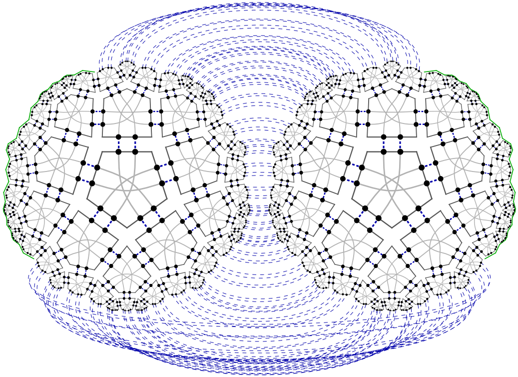

Reduced density matrices ρ_A can become complicated...

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- $\rho_A^2 = \rho_A \Rightarrow$ Flat spectrum of Rényi entropies $S_A^{(n)}$

(for suitable regions A, B, C)

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Future questions:

- Can we generalize to other holographic models?
- What is the connection to translation-invariant CFTs (MERA)?
- Is this a realization of the bit thread proposal? (Freedman/Headrick, 2017)
- Can Majorana dimers be understood in terms of spin structures?