THE ROLE OF THE DIMENSION IN THE CONTROL ON SOME NETWORKED MULTIAGENT SYSTEMS

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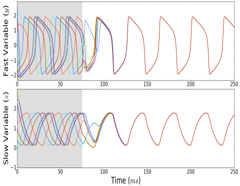
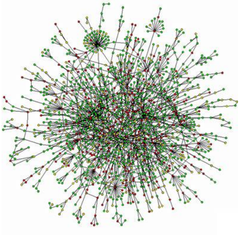
INTRODUCTION

Collective behavior models

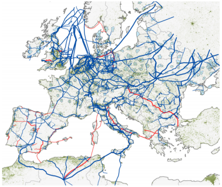
- They describe the dynamics of a system of interacting individuals.
- They are applied in a large spectrum of subjects such as synchronization of coupled oscillators, random networks, multi-area power grid, opinion propagation,...



Fitz-Hugh-Nagumo oscillators [Davison et al., Allerton 2016]



Yeast's protein interactions [Jeong et al., Nature, 2001]



European natural gas pipeline network [www.offiziere.ch]

Collective behavior

- We have a collection of individuals that interact by means of simple rules
- The whole system, however, might show very complex patterns
- One of the main interests is consensus, if the agents will arrive to a configuration in which all of them will share the same state.

Collective behavior

General first order model, let $a : \mathbb{R}^2 \to \mathbb{R}$ to be a globally Lipschitz function.

$$\frac{d}{dt}\theta_i = \frac{1}{N}\sum_{j=1}^{N} a(\theta_i, \theta_j) \tag{1}$$

Control to consensus

Can we build a function $\boldsymbol{u} \in L^2([0, T]; \mathbb{R}^M)$ such that we can drive the whole system to consensus by means of acting on some agents?

- Can we control for any N?
- How much time do we need?

Other works

These equations can have a mean field representation

$$\mu_t(\theta, t) + div_\theta(V[\mu]\mu) = 0, \quad \text{where} \quad V[\mu] := \int_{\mathbb{R}} a(\omega, \theta)\mu(\omega, t) d\omega.$$

Control to consensus has been already tackled also for the mean field case: Caponigro, Fornasier, Piccoli, Rossi, Trélat...

Example: The Kuramoto Model

From ODE models for coupled neurons (Hodgkin-Huxley) it can be justified the so called Kuramoto model.

$$\frac{d}{dt}\theta_i = \frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) \tag{2}$$

Models in networks

But in many applications it is more normal to consider a model with a network interaction. Let $G(V,E)$ be a connected graph with $|V| = N$ and let A be its adjacency matrix. Consider the dynamics

$$\frac{d}{dt}\theta_i = \sum_{j=1}^{N} \frac{1}{\kappa_i} A_{i,j} \sin(\theta_j - \theta_i) \tag{3}$$

where κ_i is the degree of node i.

Goal

Can we control the system with a control that ‘‘does not depend a lot on the dimension N‘‘?

We study the cost of the control for a simplified system.

Results

- Biccari, Ko, Zuazua, ''Dynamics and control for multi-agent networked systems: a finite difference approach'' , Math. Models Methods Appl. Sci
- Ruiz-Balet, Zuazua, ''A parabolic approach to the control of opinion spreading'', Applied Wave Mathematics II

Motivation: Kuramoto in a chain network

Consider the Kuramoto model in a chain network

$$\frac{d}{dt}\theta_i = \sin(\theta_{i+1} - \theta_i) + \sin(\theta_{i-1} - \theta_i) \tag{4}$$

If we linearize around consensus $\theta_i = \theta_j$ for all i, j we obtain

$$\frac{d}{dt}\theta_i = (\theta_{i+1} - \theta_i) + (\theta_{i-1} - \theta_i) \tag{5}$$

The model: chain network

$$\frac{d}{dt}\theta_i = (\theta_{i-1} - \theta_i) + (\theta_{i+1} - \theta_i) \quad t \in [0, N^2 T] \tag{6}$$

$$\frac{d}{dt}\boldsymbol{\theta} = A\boldsymbol{\theta} + Bu \quad t \in [0, N^2 T] \tag{7}$$

$$A = \begin{pmatrix} -1 & 1 & 0 & \dots & \dots & 0 \\ 1 & -2 & 1 & \dots & \dots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \dots & \dots & 1 & -2 & 1 \\ 0 & \dots & \dots & \dots & 1 & -1 \end{pmatrix}$$

Change of variables and heat equation

By performing a change of variables $N^2\tau = t$ we end up with

$$\frac{d}{dt}\boldsymbol{y} = N^2 A\boldsymbol{y} + Bv \qquad \tau \in [0, T] \tag{8}$$

with $\boldsymbol{y}(\tau) = \boldsymbol{\theta}(N^2\tau)$, $v(\tau) = N^2 u(N^2\tau)$

Result

Applying known results on Finite differences for the heat equation we can obtain a control with a control cost with bounded L^2 by $K\exp\{C/T\}$.

But the price to pay is that that the time horizon grows.

Adding a nonlinearity

Limitation

If we add a nonlinearity to the system we can lose the uniform boundedness of the control cost.

Let $g \in L^\infty$

$$\frac{d}{dt}\theta_i = (\theta_{i-1} - \theta_i) + (\theta_{i+1} - \theta_i) + g(\theta_i)\theta_i \quad t \in [0, N^2 T] \tag{9}$$

After a change of variables it can be represented as:

$$\frac{d}{d\tau}\boldsymbol{y} = N^2 A\boldsymbol{y} + N^2 \boldsymbol{G}(\boldsymbol{y}) + Bv \quad \tau \in [0, T] \tag{10}$$

Known results of numerical semidiscretizations

Let $\sigma > 0$ be a constant diffusivity, and the nonlinearity G to be Lipschitz.

$$\partial_\tau \boldsymbol{y} - \sigma N^2 A\boldsymbol{y} - \boldsymbol{G}(\boldsymbol{y}) = B_N \boldsymbol{u}, \tag{11a}$$

$$\boldsymbol{y}(0) = \boldsymbol{y}^0. \tag{11b}$$

where $B_N \in \mathbb{R}^N \times \mathbb{R}^M$ and $M/N > 0$ is kept constant,

Known results of numerical semidiscretizations

Theorem (Boyer, F. and Le Rosseau, J.)

The system (11) *is uniformly controllable as* $N \to \infty$ *for any* $T > 0$, *in the sense that for all initial data there are controls assuring that*

$$|\boldsymbol{y}(T)|_2 \leq C(T)e^{-C_0 N}|\boldsymbol{y}^0|_2 \tag{12}$$

$$\|\boldsymbol{u}\|_{L^2((0,T);\mathbb{R}^M)} \leq C(T)|\boldsymbol{y}^0|_2 \tag{13}$$

$$C(T) = e^{C_1(1+1/T+\|g\|_\infty T+\|g\|_\infty^{2/3})} \tag{14}$$

with $C_0, C_1 > 0$ *depending on the location of the controlled components and* σ, *but independent of* g *and* T.

A bit further and limitations of the framework

The control cost

The control cost is not in general independent of N after the change of variables, $K_N = \exp\{C_1(1 + 1/T + TN^2\|g_N\|_\infty + N^{4/3}\|g_N\|_\infty^{2/3})\}$.

We can just obtain a control cost that it is independent of N if our nonlinearity depends on N so that it can be understood as a discretization of a semilinear heat equation:

$$\frac{d}{dt}\theta = N^2 A\theta + \frac{1}{N^2}\boldsymbol{G}(\theta) \quad t \in [0, N^2 T] \tag{15}$$

Remark

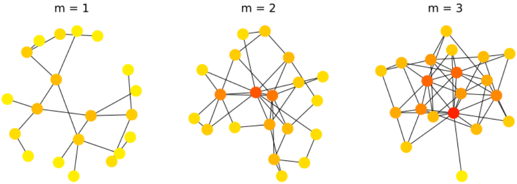
The exponential tail depends on the cost of the control.

General perspectives and Open problem

Networks expample: Barabassi-Albert

Take $m \in \mathbb{N}$ and $m \leq m_0 \in \mathbb{N}$

$$p_i = \frac{\kappa_i}{\sum_j \kappa_j} \tag{16}$$



Networks

Other networks with different properties also exist

- Watts-Stogatz model: small world network $L \sim \log N$
- Random graphs
- Lattice networks

Important issues in applications

- In most real world problems where there several agents interact they are not fully connected
- Furthermore, the generation of the network can come from a natural proces that can lead to the construction of a sequence of graphs.

Perspective and Open question

Let G_N be a sequence of graphs, let $a : \mathbb{R}^2 \to \mathbb{R}$ be a globally Lipschitz and consider:

$$\frac{d}{dt}\theta_i = \sum_{j=1}^{N} A_{i,j}^{(N)} a(\theta_j, \theta_i) \tag{17}$$

where $A^{(N)}$ is the adjacency matrix of G_N.

- How can we control those systems?
- How are the control properties being affected when N grows?

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