Games for the evolution problem associated to the eigenvalues of the Hessian

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Benasque, 2019

Eigenvalues of D^2u

For a C^2 function $u : \Omega \subset \mathbb{R}^n \mapsto \mathbb{R}$ we denote its Hessian as

$$D^2u = \left(\frac{\partial^2 u}{\partial x_i \partial x_j} \right)_{i,j}$$

and

$$\lambda_1(D^2u) \leq \lambda_2(D^2u) \leq .... \leq \lambda_j(D^2u) \leq ....\lambda_n(D^2u)$$

the ordered eigenvalues of the Hessian D^2u.

Notice that

$$\Delta u = \lambda_1(D^2u) + ... + \lambda_n(D^2u).$$

Main goals

For the problem

$$\begin{cases} \lambda_j(D^2 u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

• Relate solutions to convex/concave envelopes of the boundary datum g.

• Show a connection with probability (game theory).

• Find sufficient condition on the domain Ω in such a way that this problem has a viscosity solution that is continuous in $\overline{\Omega}$ for every $g \in C(\partial\Omega)$.

• Study a parabolic version of this problem.

$$\begin{cases} u_t - \lambda_j(D^2 u) = 0, & \text{in } \Omega \times]0, \infty[, \\ u = g, & \text{on } \partial\Omega \times]0, \infty[, \\ u = u_0, & \text{in } \Omega, \end{cases}$$

Let Ω be a bounded domain in $\mathbb{R}^n$.

Convex envelope

Given $g : \partial\Omega \mapsto \mathbb{R}$ the convex envelope of g in Ω is

$$u^*(x) = \sup_{u \text{ convex},\ u|_{\partial\Omega} \leq g} u(x).$$

That is, u^* is the largest convex function that is below g on $\partial\Omega$.

Concave envelope

Given $g : \partial\Omega \mapsto \mathbb{R}$ the concave envelope of g in Ω is

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Convex envelopes

If $u \in C^2$ is convex then $D^2u(x)$ must be positive semidefinite,

$$\langle D^2u(x)v, v\rangle \geq 0, \qquad \text{for any } v \in \mathbb{R}^n.$$

In terms of the eigenvalues of D^2u this can be written as

$$\lambda_1(D^2u(x)) = \inf_{|v|=1} \langle D^2u(x)v, v\rangle \geq 0.$$

A. Oberman – L. Silvestre (2011)

Moreover, the convex envelope of g in Ω, u^*, is the largest viscosity solution to

$$\begin{cases} \lambda_1(D^2u) = 0, & \text{in } \Omega, \\ u \leq g, & \text{on } \partial\Omega. \end{cases}$$

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Convex envelopes

Notice that in an interval $(a, b) \subset \mathbb{R}$, it holds that

$$u^*(x) = u_*(x) = \frac{(u(b) - u(a))}{b - a}(x - a) + u(a).$$

Therefore, a convex function, u, has the following property: for every segment (a, b) inside Ω we have

$$u(s) \leq v(s) \qquad s \in (a, b)$$

being v the concave envelope of the boundary values $u(a)$, $u(b)$ in (a, b).

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Concave / convex envelopes

Let H_j be the set of functions v such that

$$v \leq g \qquad \text{on } \partial\Omega,$$

and have the following property: for every j-dimensional ball B_j inside Ω it holds that

$$v \leq z \qquad \text{in } B_j$$

where z is the concave envelope of $v|_{\partial B_j}$ in B_j.

Theorem [P. Blanc, J.D. Rossi (2018)]
The function

$$u(x) = \sup_{v \in H_j} v(x)$$

is the largest viscosity solution to

$$\lambda_j(D^2 u) = 0 \qquad in \ \Omega,$$

with $u \leq g$ on $\partial\Omega$.

The equation for the concave envelope of $g|_{\partial\Omega}$ in Ω is just $\lambda_n = 0$; while the equation for the convex envelope is $\lambda_1 = 0$.

Remark If u is the largest viscosity solution of

$$\lambda_j(D^2 u) = 0 \text{ in } \Omega, \quad \text{with } u \leq g \text{ on } \partial\Omega,$$

then $v = -u$ is the smallest viscosity solution of

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F.R. Harvey, H.B. Jr. Lawson, (2009)

A comparison principle (hence uniqueness) was proved for the equation $\lambda_j(D^2u) = 0$.

For the existence, it is assumed the following condition: the domain is smooth (at least C^2) and such that $\kappa_1 \le \kappa_2 \le ... \le \kappa_{n-1}$, the main curvatures of $\partial\Omega$, verify

$$\kappa_j(x) > 0 \qquad \text{and} \qquad \kappa_{n-j+1}(x) > 0, \qquad \forall x \in \partial\Omega. \tag{H}$$

This condition is sufficient but not necessary. There is a weaker sufficient condition which applies also for nonsmooth domains.

Theorem *Let $\Omega \subset \mathbb{R}^n$ be a smoothly bounded domain (at least C^2) satisfying (H). Then, for any $g \in C(\partial\Omega)$, there exists $u \in C(\overline{\Omega})$ which is the unique viscosity solution of*

$$\begin{cases} \lambda_j(D^2u) = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Let us consider a final payoff function

$$g : \mathbb{R}^n \setminus \Omega \mapsto \mathbb{R}.$$

In a random walk with steps of size ε from x the position of the particle can move to

$$x \pm \varepsilon \boldsymbol{e}_j,$$

each movement being chosen at random with the same probability $1/2n$.

Let x_τ be the position of the particle the first time it leaves the domain Ω. The payoff is then given by $g(x_\tau)$.

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Let

$$u_\varepsilon(x) = \mathbb{E}^x(g(x_N))$$

be the expected final payoff when we move with steps of size ε. Applying conditional expectations we get

$$u_\varepsilon(x) = \sum_{j=1}^{n} \left(\frac{1}{2n} u_\varepsilon(x + \varepsilon e_j) + \frac{1}{2n} u_\varepsilon(x - \varepsilon e_j) \right).$$

That is,

$$0 = \sum_{j=1}^{n} \Big\{ u_\varepsilon(x + \varepsilon e_j) - 2u_\varepsilon(x) + u_\varepsilon(x - \varepsilon e_j) \Big\}.$$

Now, one shows that u_ε converge as $\varepsilon \to 0$ to a continuous function u uniformly in $\overline{\Omega}$.

Then, we get that u is a **viscosity solution** to the Laplace equation

$$\begin{cases} -\Delta u = 0 & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

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A game

Tug-of-war game

A game

Rules

- Two-person, zero-sum game: two players are in contest and the total earnings of one are the losses of the other. Player I, plays trying to minimize his expected outcome, while Player II is trying to maximize.
- $\Omega \subset \mathbb{R}^n$, bounded domain and $g : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ a final payoff function.
- Starting point $x_0 \in \Omega$. At each turn, Player I chooses a subspace S of dimension j and then Player II chooses $v \in S$ with $|v| = 1$.
- The new position of the game is $x + \varepsilon v$ or $x - \varepsilon v$, each with probability $1/2$.
- The game ends when $x_N \notin \Omega$, and then Player I pays $g(x_N)$ to Player II.

Remark

The sequence of positions $\{x_0, x_1, \cdots, x_N\}$ has some probability, which depends on

- The starting point x_0.
- The strategies of players, S_I and S_{II}.

Expected result: For a given initial position x_0 and two strategies S_I, S_{II}, we can compute:

$$\mathbb{E}^{x_0}_{S_I, S_{II}}(g(x_N))$$

We assume the Players I and II are "Smart":

- Player I tries to choose at each step a strategy which minimizes the result.
- Player II tries to choose at each step a strategy which maximizes the result.

- Value of the game for Player I:

$$u_I(x) = \inf_{S_I} \sup_{S_{II}} \mathbb{E}^x_{S_I, S_{II}}(g(x_N))$$

- Value of the game for Player II:

$$u_{II}(x) = \sup_{S_{II}} \inf_{S_I} \mathbb{E}^x_{S_I, S_{II}}(g(x_N))$$

The game has a value $\Leftrightarrow$ $u_I = u_{II}$.

Theorem *This game has a value $u_\varepsilon(x)$, which satisfies the following* **Dynamic Programming Principle**

$$u_\varepsilon(x) = \inf_{dim(S)=j} \sup_{v \in S, |v|=1} \left\{ \frac{1}{2} u_\varepsilon(x + \varepsilon v) + \frac{1}{2} u_\varepsilon(x - \varepsilon v) \right\}$$

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Idea

If $\lambda_1 \leq ... \leq \lambda_n$ are the eigenvalues of $D^2 u(x)$, the $j-$st eigenvalue verifies

$$\inf_{dim(S)=j} \sup_{v \in S, |v|=1} \langle D^2 u(x) v, v \rangle = \lambda_j.$$

Theorem *Assume $\Omega \subset \mathbb{R}^n$ is a bounded domain satisfying (G). Then*

$$u_\varepsilon \to u, \qquad \text{as } \varepsilon \to 0,$$

uniformly in $\overline{\Omega}$.

The limit u is the unique viscosity solution to

$$\begin{cases} \lambda_j(D^2 u) = 0, & \text{in } \Omega, \\ u = g, & \text{on } \partial\Omega. \end{cases}$$

Condition (G): Given $y \in \partial\Omega$ we assume that for every $r > 0$ there exists $\delta > 0$ such that for every $x \in B_\delta(y) \cap \Omega$ and $S \subset \mathbb{R}^n$ a subspace of dimension j, there exists $v \in S$ of norm 1 such that

$$(G_j) \qquad\qquad \{x + tv\}_{t \in \mathbb{R}} \cap B_r(y) \cap \partial\Omega \neq \emptyset.$$

We say that Ω satisfies condition (G) if it satisfies both (G_j) and (G_{n-j+1}).

Remark: For the case $j = 1$ and $j = n$, condition (G) is equivalent to say that Ω is strictly convex.

A parabolic version

Consider the evolution problem associated to the previous elliptic problem:

$$\begin{cases} u_t(x,t) - \lambda_j(D^2u(x,t)) = 0, & \text{in } \Omega \times [0,+\infty), \\ u(x,t) = g(x), & \text{on } \partial\Omega \times [0,+\infty), \\ u(x,0) = u_0(0), & \text{in } \Omega. \end{cases}$$

Parabolic game

Consider $\Omega \subset \mathbb{R}^n$ and two continuous payoff functions

$$g : \mathbb{R}^N \setminus \Omega \longrightarrow \mathbb{R} \quad \text{and} \quad u_0 : \Omega \longrightarrow \mathbb{R}.$$

At each step, the new position of the game is

$$(x_n, t_n) = (x_{n-1} + \varepsilon v_n, t_{n-1} - \frac{\varepsilon^2}{2}) \quad \text{or} \quad (x_n, t_n) = (x_{n-1} - \varepsilon v_n, t_{n-1} - \frac{\varepsilon^2}{2})$$

with same probability, being v_n a vector chosen by Players I and II as before. The game stops when $(x_\tau, t_\tau) \notin \Omega \times]0,\infty[$, and then, Player I pays $R(x_\tau, t_\tau)$ to Player II, where $R(x_\tau, t_\tau) := \begin{cases} g(x_\tau) & \text{if } x_\tau \notin \Omega, \\ u_0(x_\tau) & \text{if } t_\tau \leq 0. \end{cases}$

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Value of the game for Players I and II:

$$u_I^\varepsilon(x_0,t_0)=\inf_{S_I}\sup_{S_{II}}\mathbb{E}_{S_I,S_{II}}^{x_0,t_0}(R(x_\tau,t_\tau)),\qquad u_{II}^\varepsilon(x_0,t_0)=\sup_{S_{II}}\inf_{S_I}\mathbb{E}_{S_I,S_{II}}^{x_0,t_0}(R(x_\tau,t_\tau)).$$

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uniformly in $\overline{\Omega}$.

The limit u is the unique viscosity solution to

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Theorem (asymptotic behaviour) *There exist positive constants C (depending on the initial condition u_0) and $\mu > 0$ (depending only on Ω), such that*

$$\|u(\cdot, t) - z(\cdot)\|_\infty \leq Ce^{-\mu t},$$

where z is the unique stationary solution.

Asymptotic behaviour

In addition, we also describe an interesting behaviour of the solution to

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with u_0 any continuous function satisfying $u_0|_{\partial\Omega} = 0$.

Theorem *There exists $T > 0$ depending only on Ω, such that the viscous solution u satisfies*

- *if $j = 1$, then $u(x,t) \leq z(x)$, for any $t > T$.*
- *if $j = N$, then $u(x,t) \geq z(x)$, for any $t > T$.*
- *if $1 < j < N$, then $u(x,t) = z(x)$, for any $t > T$.*

The previous result is a consequence of the fact that equation

$$-\lambda_1(D^2\varphi) = \mu\varphi, \quad \text{in } \Omega,$$

admits a positive solution for any $\mu > 0$,

$$\varphi(r) = Ce^{-\mu\frac{r^2}{2}},$$

and the equation $-\lambda_n(D^2\psi) = \mu\psi, \quad \text{in } \Omega$, admits a negative solution for any $\mu > 0$, $\psi(r) = -\varphi(r)$, where $r = |x|$.

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THANKS !!!   GRACIAS !!!.