Control issues and linear constraints on the control and the controlled trajectory

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August 29, 2019

Outline

Framework

Linear controlled system

$$y' = Ay + Bu, \quad \text{for } t \in (0, T), \qquad y(0) = y_0.$$

- $y = y(t)$ is the state.
- $u \in L^2(0, T; U)$ is the control.
- y_0 is the initial datum.

(H1) A generates a C_0 semigroup on an Hilbert space H,

(H2) B is the control operator, $\in \mathscr{L}(U; H)$, for an Hilbert space U.

$$\left\{ \begin{array}{l} y_0 \in H, \\ u \in L^2(0, T; U) \end{array} \right. \quad \Rightarrow \quad y \in C^0([0, T]; H).$$

$$y' = Ay + Bu, \qquad \text{for } t \in (0, T), \qquad y(0) = y_0.$$

Some control questions: What states can be reached at time T?

Approximate controllability

For any $y_0, y_1 \in H$ and $\varepsilon > 0$, find u such that the solution y satisfies $\|y(T) - y_1\|_H \leq \varepsilon$.

Exact controllability

For any $y_0, y_1 \in H$, find u such that the solution y satisfies $y(T) = y_1$.

Null controllability / controllability to trajectories

For any $y_0 \in H$, find u such that the solution y satisfies $y(T) = 0$.

Classical approach

To solve these problems, one usually relies on duality theory. Introducing

$$F_T : u \in L^2(0, T; U) \mapsto \int_0^T e^{(T-t)A} Bu(t)\, dt,$$

we have $y(T) = e^{TA} y_0 + F_T u$. Therefore,

$$\text{Approximate controllability } \Leftrightarrow \overline{\text{Ran } F_T} = H \Leftrightarrow \text{Ker } F_T^* = \{0\}.$$

$$\text{Exact controllability} \Leftrightarrow \text{Ran } F_T = H$$

$$\Leftrightarrow \exists C > 0,\ \forall z_T \in H,\ \|z_T\| \leq C \|F_T^* z_T\|_{L^2(0,T;U)}.$$

$$\text{Null controllability} \Leftrightarrow \text{Ran } F_T = \text{Ran}\,(e^{TA})$$

$$\Leftrightarrow \exists C > 0,\ \forall z_T \in H,\ \|e^{TA^*} z_T\| \leq C \|F_T^* z_T\|_{L^2(0,T;U)}.$$

What is $F_T^* z_T$?

$$F_T^* z_T = B^* z(t),$$

where z is the solution of

$$z' + A^* z = 0, \qquad \text{for } t \in (0,T), \qquad z(T) = z_T.$$

Consequently,

Approximate controllability $\Leftrightarrow$ Unique continuation property

$$\begin{cases} z' + A^* z = 0, & \text{for } t \in (0,T), \\ z(T) = z_T \in H, & \\ B^* z = 0, & \text{for } t \in (0,T), \end{cases} \qquad \text{then} \qquad z_T = 0.$$

A constructive approach

Given $y_0, y_1 \in H$, and $\varepsilon > 0$, to find an approximate control, one can minimize

$$J(z_T, f) = \frac{1}{2}\int_0^T \|B^* z(t)\|_U^2 \, dt + \frac{1}{2}\int_0^T \|f(t)\|_H^2 \, dt \\ + \langle y_0, z(0)\rangle_H - \langle y_1, z_T\rangle_H + \varepsilon \|z_T\|_H ,$$

for $(z_T, f) \in H \times L^2(0, T; H)$, where z satisfies

$$z' + A^* z = f, \quad \text{for } t \in (0, T), \quad z(T) = z_T.$$

Lemmata

- J is strictly convex and coercive on $H \times L^2(0, T; H)$
  $\rightsquigarrow$ Consequence of unique continuation.
- If (Z_T, F) is the minimizer, $y = F$ and $u = B^* Z$ solves the approximate control problem.

Our goal today

Control the linear system $y' = Ay + Bu$ at time T and impose linear constraints on the control and the controlled trajectory.

(H3) $\mathscr{G}$ is a closed vector space of $L^2(0,T;U)$, and $\mathbb{P}_{\mathscr{G}}$ is the orthogonal projection on $\mathscr{G}$ in $L^2(0,T;U)$.

(H4) $\mathscr{W}$ is a closed vector space of $L^2(0,T;H)$, and $\mathbb{P}_{\mathscr{W}}$ is the orthogonal projection on $\mathscr{W}$ in $L^2(0,T;H)$.

Approximate controllability with constraints

For any $y_0, y_1 \in H$, $\varepsilon > 0$, $g_* \in \mathscr{G}$, $w_* \in \mathscr{W}$, find a control function $u \in L^2(0,T;U)$ such that the control u and the controlled trajectory y satisfy

$$\|y(T) - y_1\|_H \leq \varepsilon,$$
$$\mathbb{P}_{\mathscr{G}} u = g_*,$$
$$\mathbb{P}_{\mathscr{W}} y = w_*.$$

Relevant unique continuation property is

$$\begin{cases} z' + A^* z = w, & \text{for } t \in (0,T), \\ z(T) = z_T, & \\ B^* z = g, & \text{for } t \in (0,T), \\ \text{with } (z_T, g, w) \in H \times \mathscr{G} \times \mathscr{W} & \end{cases} \Rightarrow \begin{cases} z_T = 0, \\ g = 0, \\ w = 0. \end{cases} \qquad \text{(UC)}$$

Theorem

Assume that $\mathscr{W}$ is of finite dimension.

Unique continuation property (UC)

$\Leftrightarrow$ Approximate controllability with constraints

Proof of $\Leftarrow$: If $\exists (z_T, g, w) \in H \times \mathscr{G} \times \mathscr{W} \setminus \{(0,0,0)\}$ such that

$$\begin{aligned} & z' + A^* z = w, \qquad \text{for } t \in (0, T), \\ & z(T) = z_T, \\ & B^* z = g \end{aligned}$$

then, for $y' = Ay + Bu$, with $y(0) = 0$,

$$0 = \langle y(T), z_T \rangle_H - \int_0^T \langle y(t), w(t) \rangle_H \, dt - \int_0^T \langle u(t), g(t) \rangle_U \, dt.$$

In particular, if one imposes $\mathbb{P}_{\mathscr{W}} y = w$, $\mathbb{P}_{\mathscr{G}} u = g$, we should have

$$\|y(T) + z_T\|_H \|z_T\|_H \geq \|z_T\|_H^2 + \|w\|_{L^2(0,T;H)}^2 + \|g\|_{L^2(0,T;U)}^2,$$

hence $y(T)$ cannot approximate $-z_T$.

Proof of $\Rightarrow$: If we suppose that (UC) holds, let $y_0, y_1 \in H$, $\varepsilon > 0$, $g \in \mathscr{G}_*$, and $w_* \in \mathscr{W}$. Minimize

$$J(z_T, g, w, f) = \frac{1}{2}\int_0^T \|B^* z(t) + g(t)\|_U^2 \, dt + \frac{1}{2}\int_0^T \|f(t) + w(t)\|_H^2 \, dt$$
$$+ \langle y_0, z(0)\rangle_H - \langle y_1, z_T\rangle_H + \varepsilon \|z_T\|_H$$
$$+ \int_0^T \langle B^* z(t), g_*(t)\rangle_U \, dt + \int_0^T \langle f(t), w_*(t)\rangle_H \, dt,$$

for $(z_T, g, w, f) \in H \times \mathscr{G} \times \mathscr{W} \times L^2(0,T;H)$, and $z' + A^* z = f$, $z(T) = z_T$.

Lemmata

- Unique continuation (UC) $\Rightarrow$ J is coercive.
- If (Z_T, G, W, F) is the minimizer of J, $y = F + W + w_*$, and $u = B^* Z + G + g_*$ solves the control problem.

Exact controllability with constraints

For any $y_0, y_1 \in H$, $g_* \in \mathscr{G}$, $w_* \in \mathscr{W}$, find a control function $u \in L^2(0, T; U)$ such that the control u and the controlled trajectory y satisfy

$$y(T) = y_1, \quad \mathbb{P}_{\mathscr{G}} u = g_*, \quad \mathbb{P}_{\mathscr{W}} y = w_*.$$

Theorem

Assume the observability inequality: $\exists C > 0$, such that for all z satisfying $z' + A^* z = f$, $z(T) = z_T$,

$$\|(z_T, g, w, f)\|_{H \times \mathscr{G} \times \mathscr{W} \times L^2(0,T;H)} \leq C \left(\|B^* z + g\|_{L^2(0,T;U)} + \|f + w\|_{L^2(0,T;H)} \right). \quad \text{(ExObs)}$$

Then Exact controllability with constraints holds.

Recall that classical exact controllability of $y' = Ay + Bu$ is equivalent to

$$\|z_T\| \leq C\|F_T^* z_T\|_{L^2(0,T;U)} = C\|B^* z\|_{L^2(0,T;U)}, \qquad \text{(ClassExObs)}$$

for z solving $z' + A^* z = 0$, $z(T) = z_T$.

Lemma

If $\mathscr{G}$ and $\mathscr{W}$ are of finite dimension,
Unique continuation (UC) + Classical Observability (ClassExObs)
$\Rightarrow$ Observability inequality (ExObs).

Null controllability with constraints

For any $y_0 \in H$, $g_* \in \mathscr{G}$, $w_* \in \mathscr{W}$, find a control function $u \in L^2(0, T; U)$ such that the control u and the controlled trajectory y satisfy

$$y(T) = 0, \quad \mathbb{P}_{\mathscr{G}} u = g_*, \quad \mathbb{P}_{\mathscr{W}} y = w_*.$$

Theorem

Assume the observability inequality: $\exists C > 0$, such that for all z satisfying $z' + A^* z = f$, $z(T) = z_T$,

$$\begin{aligned} &\|(z(0), g, w, f)\|_{H \times \mathscr{G} \times \mathscr{W} \times L^2(0,T;H)} \\ &\qquad \leq C \left(\|B^* z + g\|_{L^2(0,T;U)} + \|f + w\|_{L^2(0,T;H)} \right). \quad \text{(NullObs)} \end{aligned}$$

Then Null controllability with constraints holds.

Lemma

If $\mathscr{G}$ and $\mathscr{W}$ are of finite dimension, and $\exists \tilde{T} \in (0, T)$ s.t.

$$\left\{ \begin{array}{ll} z' + A^* z = w, & \text{for } t \in (0, \tilde{T}), \\ z(\tilde{T}) = z_{\tilde{T}}, & \\ B^* z = g, & \text{for } t \in (0, \tilde{T}), \\ \text{ with } (z_{\tilde{T}}, g, w) \in H \times \mathscr{G} \times \mathscr{W}, & \end{array} \right. \Rightarrow \left\{ \begin{array}{l} z_{\tilde{T}} = 0, \\ g = 0, \\ w = 0, \end{array} \right.$$

and $\exists C > 0$ such that for z solving $z' + A^* z = 0$ in $(0, T)$,

$$\left\| z(\tilde{T}) \right\|_H \leq C \left\| B^* z \right\|_{L^2(0,T;U)} . \qquad \text{(Ineq)}$$

Then the observability inequality (NullObs) holds.

Rk: The observability estimate (Ineq) implies null-controllability of $y' = Ay + Bu$ at time T, and null-controllability of $y' = Ay + Bu$ at time T' implies (Ineq).

Summary

Relevant unique continuation property is

$$\begin{cases} z' + A^* z = w, & \text{for } t \in (0,T), \\ z(T) = z_T, & \\ B^* z = g, & \text{for } t \in (0,T), \\ \text{with } (z_T, g, w) \in H \times \mathscr{G} \times \mathscr{W} & \end{cases} \Rightarrow \begin{cases} z_T = 0, \\ g = 0, \\ w = 0. \end{cases} \qquad \text{(UC)}$$

Main remaining difficulty

How to check this unique continuation property in practice?

Example 1

We consider a linear control system

$$y' = Ay + Bu$$

which is

- approximately controllable in time T_{AC}
- exactly controllable in time T_{EC}
- $T_{EC} > T_{AC}$.

Typical example

The wave equation. In the unit square observed from a neighborhood of two consecutive sides, $T_{AC} = 2$, $T_{EC} = 2\sqrt{2}$.

We choose $T = T_{EC}$, and

$$\mathscr{G} \subset \{u \in L^2(0, T; U),\ u = 0 \text{ on } (T - T_{AC}, T)\}, \quad \mathscr{W} = 0,$$

Then (UC) holds.

$\Omega = (0,1)^2$, $\omega =$ neighborhood of two consecutive sides:

$$\begin{cases} \partial_{tt} y - \Delta y = u\chi_\omega, & \text{for } (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & \text{for } (t,x) \in (0,T) \times \partial\Omega, \\ (y(0,\cdot), \partial_t y(0,\cdot)) = (y_0, y_1) & \in H_0^1(\Omega) \times L^2(\Omega). \end{cases}$$

Let $T \geq 2\sqrt{2}$, and $\mathscr{G}$ be a finite dimensional subspace of

$$\{u \in L^2(0,T; L^2(\omega)),\ u = 0 \text{ on } (T-2,T) \times \omega\},$$

Theorem

Given any $(y_0, y_1), (y_0^T, y_1^T) \in H_0^1(\Omega) \times L^2(\Omega)$, any $g \in \mathscr{G}$, there exists $u \in L^2(0,T; L^2(\omega))$ such that the solution of the wave equation satisfies

$$(y(T,\cdot), \partial_t y(T,\cdot)) = (y_0^T, y_1^T) \text{ and } \mathbb{P}_{\mathscr{G}} u = g.$$

Rk: This theorem cannot be true when

$$\mathscr{G} = \{u \in L^2(0,T; L^2(\omega)),\ u = 0 \text{ on } (T-2,T) \times \omega\}$$

Ex. 2: $\mathscr{G} = \{0\}$, and $B^* z = 0$ implies that $w = 0$.

Controlled heat equation in Ω bounded domain of $\mathbb{R}^d$, ω open subset of Ω.

$$\begin{cases} \partial_t y - \Delta y = u\chi_\omega, & \text{for } (t,x) \in (0,T) \times \Omega, \\ y(t,x) = 0, & \text{for } (t,x) \in (0,T) \times \partial\Omega, \\ y(0,\cdot) = y_0, & \text{in } \Omega. \end{cases}$$

$\mathscr{G} = \{0\}$ and $\mathscr{W}$ a subspace of $L^2(0,T; L^2(\Omega))$ such that

$$\Pi_\omega : f \mapsto f|_\omega \text{ satisfies } \operatorname{Ker}\left(\Pi_\omega|_{\mathscr{W}}\right) = \{0\}.$$

Then (UC) holds:

$$\begin{cases} \partial_t z + \Delta z = w, & \text{for } (t,x) \in (0,T) \times \Omega, \\ z(t,x) = 0, & \text{for } (t,x) \in (0,T) \times \partial\Omega, \\ z(T,\cdot) = z_T, & \text{in } \Omega, \\ z(t,x) = 0 & \text{in } (0,T) \times \omega. \end{cases} \Rightarrow w = 0 \text{ and } z = 0.$$

In particular, if we further assume that $\mathscr{W}$ is of finite dimension, Null controllability with constraint holds in time T.

- Inspired by works on sentinels: [Lions '92, Nakoulima '04, Mophou-Nakoulima '08, '09, Gao '15].
- Can be done when $\mathscr{W}$ and $\mathscr{G}$ are non zero under the condition

$\exists$ two linear operators K and L s.t.

$$\begin{cases} K : L^2(0, T; H) \mapsto \mathcal{H} \text{ for some Hilbert space } \mathcal{H}, \\ L : L^2(0, T; U) \mapsto \mathcal{H}, \\ K(\partial_t + A^*) = LB^*, \\ \mathrm{Ker}\left((g, w) \in \mathscr{G} \times \mathscr{W} \mapsto Lg + Kw\right) = \{0\}. \end{cases}$$

Example 3

Theorem

Let A be the generator of an analytic semigroup on H.
Let $K \in \mathbb{N}$, $(\mu_k)_{k\in\{1,\cdot,K\}}$ two by two distinct real numbers, $\mathcal{W}_k$ be a family of closed vector spaces included in H such that

$$\forall z \in \mathcal{D}(A^*), \quad (\mu_k + A^*)z \in \mathcal{W}_k, \text{ and } B^*z = 0 \Rightarrow z = 0,$$

and $\mathscr{W} = \text{Span}\,\{e^{\mu_k t} w_k, k \in \{1, \cdots, K\}, \text{ and } w_k \in \mathcal{W}_k\}$.
Let $J \in \mathbb{N}$, $(\rho_j)_{j\in\{1,\cdot,J\}}$ two by two distinct real numbers, $\mathcal{G}_j$ be a family of closed vector spaces included in U such that

$$\forall z \in \mathcal{D}(A^*) \text{ satisfying } (\rho_j + A^*)z = 0, \ B^*z \in \mathcal{G}_j \Rightarrow z = 0,$$

and $\mathscr{G} = \text{Span}\,\{e^{\rho_j t} g_j, j \in \{1, \cdots, J\}, \text{ and } g_j \in \mathcal{G}_j\}$.
We also assume $\mu_k \neq \rho_j$ for all j, k and the classical unique continuation property $(z' + A^*z = 0 \ \& \ B^*z = 0) \Rightarrow z \equiv 0$.
Then the unique continuation property (UC) is satisfied.

Proof. If z satisfies

$$z' + A^* z = w, \quad B^* z = g,$$

with $w \in \mathscr{W}$ and $g \in \mathscr{G}$, applying

$$P = \prod_{k=1}^{K} (\partial_t - \mu_k) \prod_{j=1}^{J} (\partial_t - \rho_j),$$

we obtain $(Pz)' + A^*(Pz) = 0$ and $B^*(Pz) = 0$. By the classical unique continuation property, $Pz = 0$, hence

$$z(t) = \sum_{k=1}^{K} z_k e^{\mu_k t} + \sum_{j=1}^{J} z_j e^{\mu_j t},$$

$$\text{with } (\mu_k + A^*) z_k \in \mathscr{W}_k, \quad \text{and} \quad B^* z_k = 0,$$
$$\text{and } (\rho_j + A^*) z_j = 0, \quad \text{and} \quad B^* z_k \in \mathscr{G}_j.$$

Therefore, $z = 0$, and thus $w = 0$ and $g = 0$.

This idea was used in the context of Navier-Stokes equations:

$$\Omega = \mathbb{T} \times (0,1), \text{ where } \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$

$$\begin{cases} \partial_t y + (y \cdot \nabla) y - \Delta y + \nabla p = 0, & \text{in } (0,\infty) \times \Omega, \\ \text{div } y = 0, & \text{in } (0,\infty) \times \Omega, \\ y(t, x_1, 0) = (0,0), & \text{on } (0,\infty) \times \mathbb{T}, \\ y(t, x_1, 1) = (0, u(t, x_1)), & \text{on } (0,\infty) \times \mathbb{T}, \\ y(0, x_1, x_2) = y^0(x_1, x_2), & \text{in } \Omega. \end{cases}$$

- $y = y(t, x_1, x_2) \in \mathbb{R}^2$ is the velocity.
- $p = p(t, x_1, x_2)$ is the pressure.
- $u = u(t, x_1)$ is the control function, acting on the normal component only.

Theorem: Stabilization at any exponential rate [Chowdhury SE 19].

For any $\omega > 0$, $\exists \varepsilon > 0$, $\forall y_0 \in V_0^1(\Omega)$ satisfying $\|y_0\|_{H^1} \leq \varepsilon$, $\exists u \in L^2(0,\infty; L^2(0,1))$, such that $\|y(t)\|_{H^1(\Omega)} \leq Ce^{-\omega t}$.

Here, the control system is of the form

$$y' = Ay + F(y) + Bu,$$

where

- F is a quadratic term.
- The space $H = L^2_\sigma(\Omega)$ can be decomposed into $H = H_0 \oplus H_1$, and

$$y' = Ay + Bu \Leftrightarrow \left\{ \begin{array}{l} y_0' = A_0 y_0, \\ y_1' = A_1 y_1 + B_1 u, \end{array} \right. \quad \text{where} \quad \left\{ \begin{array}{l} y_0 = P_{H_0} y, \\ y_1 = P_{H_1} y. \end{array} \right.$$

In fact, $H_0 = \{y \in L^2_\sigma(\Omega),\, y = y(x_2)\}$, and $H_1 = H_0^\perp$.
Consequently, the projection y_0 cannot be controlled on the linearized equations.

Strategy: Expand $y = \varepsilon\alpha + \varepsilon^2\beta$ and use the non-linear term to control the projection in H_0.
Inspired by [Coron Crépeau '04, Cerpa '07, Cerpa Crépeau '09, Coron Rivas '15].

- $\varepsilon > 0$ small.
- α, β of order 1.
- $\alpha \in H_1$.

Up to lower order terms,

$$\begin{cases} \alpha' = A_1\alpha + B_1 u, \\ \alpha(0) = \alpha_0 \in H_1 \end{cases} \qquad \begin{cases} \beta' = A\beta + Bu + F(\alpha), \\ \beta(0) = u_0 - \alpha_0. \end{cases}$$

Difficulty: Controlling $P_0\beta = \beta_0$.

$$\begin{cases} \beta_0' = A_0\beta_0 + F_0(\alpha), \\ \beta_0(0) = \beta_{00} \in H_0. \end{cases}$$

In particular, our arguments rely on the following construction: Let

$$H_{0,\omega} = \text{Span}\,\{\Psi \text{ eigenvector of } A_0 \text{ corresponding to eigenvalue } \lambda > \omega\}.$$

This space is of finite dimensional in our case.

Lemma

For any $h_0 \in H_{0,\omega}$, there exists a control function u such that the solution (α, β_0) of

$$\left\{ \begin{array}{l} \alpha' = A_1 \alpha + B_1 u, \\ \alpha(0) = 0, \end{array} \right. \qquad \left\{ \begin{array}{l} \beta_0' = A_0 \beta_0 + F_0(\alpha), \\ \beta_0(0) = 0, \end{array} \right.$$

satisfies

$$\alpha(T) = 0 \quad \text{and } P_{H_{0,\omega}} \beta_0(T) = h_0.$$

Indirect controllability result through a non-linear coupling.

Intermediate result

If Ψ is an eigenvector of A_0 corresponding to an eigenvalue λ, there exist two controls $u_\pm$ such that the solutions $(\alpha_\pm, \beta_{0,\pm})$ of

$$\begin{cases} \alpha_\pm' = A_1 \alpha_\pm + B_1 u_\pm, \\ \alpha_\pm(0) = 0, \end{cases} \qquad \begin{cases} \beta_{0,\pm}' = A_0 \beta_{0,\pm} + F_0(\alpha_\pm), \\ \beta_{0,\pm}(0) = 0, \end{cases}$$

satisfies

$$\alpha_\pm(T) = 0 \quad \text{and } \langle \beta_{0,\pm}(T), \Psi \rangle = \pm 1.$$

Proof

When taking two controls u_a and u_b such that

$$\begin{cases} \alpha_a' = A_1 \alpha_a + B_1 u_a, \\ \alpha_a(0) = 0, \end{cases} \qquad \begin{cases} \alpha_b' = A_1 \alpha_b + B_1 u_b, \\ \alpha_b(0) = 0, \end{cases}$$

satisfy $\alpha_a(T) = \alpha_b(T) = 0$, for all $a, b \in \mathbb{R}$, if β_0 satisfies

$$\begin{cases} \beta_0' = A_0 \beta_0 + F_0(a\alpha_a + b\alpha_b), \\ \beta_0(0) = 0, \end{cases}$$

satisfy

$$\langle \beta_0(T), \Psi \rangle = a^2 Q(u_a, u_a) + 2ab Q(u_a, u_b) + b^2 Q(u_b, u_b),$$

where Q is a bilinear form.

A structure result: If u is odd or even in space, $Q(u, u) = 0$.

$\rightsquigarrow$ We look for $u_a = u_a(t)\sin(x_1)$ odd, $u_b = u_b(t)\cos(x_1)$ even and $Q(u_a, u_b) \neq 0$.

$$Q(u^a, u^b) = \pi^{5/2} \int_0^T u_a(t) q_b(t, 1)\, dt,$$

where q_b is obtained by solving

$$\begin{cases} -\partial_t Z_b + Z_b - \partial_{22} Z_b + \begin{pmatrix} q_b \\ \partial_2 q_b \end{pmatrix} = F_b(t, x_2), & \text{in } (0,T)\times(0,1), \\ -Z_{1,b} + \partial_2 Z_{2,b} = 0, & \text{in } (0,T)\times(0,1), \\ Z_b(t,0) = Z_b(t,1) = (0,0), & \text{in } (0,T), \\ Z_b(T, x_2) = 0, & \text{in } (0,1). \end{cases}$$

with $F_b(t, x_2) = \cos(\pi x_2) e^{\pi^2 t} \begin{pmatrix} \alpha_{2,b}(t, x_2) \\ \alpha_{1,b}(t, x_2) \end{pmatrix}$, and

$$\alpha_b = \begin{pmatrix} \alpha_{1,b}(t, x_2)\sin(x_1) \\ \alpha_{2,b}(t, x_2)\cos(x_1) \end{pmatrix}$$

solves $\alpha_b' = A\alpha_b + B u_b$.

Two steps:

- Find a control function u_b such that $\alpha_b(T) = 0$ and $q_b(t,1)$ non-zero.
- Find a control function u_a such that $\alpha_a(T) = 0$ and $\int_0^T u_a(t) q_b(t,1)\, dt \neq 0$.

This further requires $q_b(t,1)$ to be such that we can impose the projection on $q_b(t,1)$.

We do that by imposing $\alpha_b(t) = e^{\mu t}\overline{\alpha}$ and $u_b(t) = e^{\mu t}$ on some subinterval (T_1, T_2) of $(0, T)$ for a suitable choice of μ guaranteeing that

$$q_b(t,1) = c_0 e^{(\mu+\pi^2)t} + \sum a_j e^{(\lambda_j+\pi^2)(t-T_1)} + \sum b_j e^{\lambda_j(T_2-t)}$$

for λ_j in the spectrum of A, and $c_0 \neq 0$.
Then our argument applies if $\mu \notin \{\lambda_j, -\lambda_j - \pi^2\}$ and a null control u_a exists for which we have $\int_0^T u_a(t) q_b(t,1)\, dt = 1$.

Comments

- Time-dependent coefficients: Provided well-posedness is ensured, the control system $y' = A(t)y + Bu$ can be controlled with linear constraints on u in $\mathscr{G}$ and y in $\mathscr{W}$ provided the following unique continuation property holds

$$\begin{cases} z' + A(t)^* z = w, & \text{for } t \in (0, T), \\ z(T) = z_T, & \\ B^* z = g, & \text{for } t \in (0, T), \\ \text{with } (z_T, g, w) \in H \times \mathscr{G} \times \mathscr{W} & \end{cases} \Rightarrow \begin{cases} z_T = 0, \\ g = 0, \\ w = 0. \end{cases} \tag{UC}$$

- Unbounded control operators: The same results and proofs apply when B is an admissible control operator.

Open problem

- Quantifying the cost of controlling in $\mathscr{G}$ and $\mathscr{W}$.
  Similarly as the discussion in the work [Fernandez-Cara, Zuazua '00] for controlling the state exactly on some finite-dimensional space $E \subset L^2(\Omega)$ in the context of approximate controllability of the heat equation:

$$P_E y(T) = P_E y_1 \quad \text{and} \quad \|y(T) - y_1\|_{L^2(\Omega)} \leq \varepsilon.$$

  Estimates of the cost:

$$\exp\left(\frac{C(1+\exp(T\mu(E))}{\varepsilon}\right),$$

$$\text{where } \mu(E) = \max_{\varphi \in E\setminus\{0\}} \frac{\|\nabla\varphi\|^2_{L^2(\Omega)}}{\|\varphi\|^2_{L^2(\Omega)}}.$$

Thank you for your attention!

Comments Welcome

Reference:
Control issues and linear constraints on the control and on the controlled trajectory.
Sylvain Ervedoza, 2019.