Control issues and linear constraints on the control and the controlled trajectory

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Outline

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Framework

Linear controlled system

\[ y' = Ay + Bu, \quad \text{for } t \in (0, T), \quad y(0) = y_0. \]

- \( y = y(t) \) is the state.
- \( u \in L^2(0, T; U) \) is the control.
- \( y_0 \) is the initial datum.

(H1) \( A \) generates a \( C_0 \) semigroup on an Hilbert space \( H \),
(H2) \( B \) is the control operator, \( \in \mathcal{L}(U; H) \), for an Hilbert space \( U \).

\[
\begin{cases}
  y_0 \in H, \\
  u \in L^2(0, T; U)
\end{cases}
\Rightarrow \quad y \in C^0([0, T]; H).
\]
\[ y' = Ay + Bu, \quad \text{for } t \in (0, T), \quad y(0) = y_0. \]

Some control questions: What states can be reached at time \( T \)?

**Approximate controllability**

For any \( y_0, y_1 \in H \) and \( \varepsilon > 0 \), find \( u \) such that the solution \( y \) satisfies \( \| y(T) - y_1 \|_H \leq \varepsilon \).

**Exact controllability**

For any \( y_0, y_1 \in H \), find \( u \) such that the solution \( y \) satisfies \( y(T) = y_1 \).

**Null controllability / controllability to trajectories**

For any \( y_0 \in H \), find \( u \) such that the solution \( y \) satisfies \( y(T) = 0 \).
Classical approach

To solve these problems, one usually relies on duality theory. Introducing

$$F_T : u \in L^2(0, T; U) \mapsto \int_0^T e^{(T-t)A} Bu(t) \, dt,$$

we have $y(T) = e^{TA}y_0 + F_T u$. Therefore,

Approximate controllability $\iff$ Ran $F_T = H \iff$ Ker $F_T^* = \{0\}$.

Exact controllability $\iff$ Ran $F_T = H$

$\iff \exists C > 0, \forall z_T \in H, \|z_T\| \leq C \|F_T^* z_T\|_{L^2(0, T; U)}$.

Null controllability $\iff$ Ran $F_T = \text{Ran}(e^{TA})$

$\iff \exists C > 0, \forall z_T \in H, \|e^{TA} z_T\| \leq C \|F_T^* z_T\|_{L^2(0, T; U)}$. 
What is $F_T^*z_T$?

$$F_T^*z_T = B^*z(t),$$

where $z$ is the solution of

$$z' + A^*z = 0, \quad \text{for } t \in (0, T), \quad z(T) = z_T.$$

Consequently,

**Approximate controllability** $\iff$ **Unique continuation property**

\[
\begin{cases}
  z' + A^*z = 0, & \text{for } t \in (0, T), \\
  z(T) = z_T \in H, & \text{then } \quad z_T = 0, \\
  B^*z = 0, & \text{for } t \in (0, T),
\end{cases}
\]
A constructive approach

Given $y_0, y_1 \in H$, and $\varepsilon > 0$, to find an approximate control, one can minimize

$$J(z_T, f) = \frac{1}{2} \int_0^T \|B^*z(t)\|_U^2 \, dt + \frac{1}{2} \int_0^T \|f(t)\|_H^2 \, dt$$

$$+ \langle y_0, z(0) \rangle_H - \langle y_1, z_T \rangle_H + \varepsilon \|z_T\|_H,$$

for $(z_T, f) \in H \times L^2(0, T; H)$, where $z$ satisfies

$$z' + A^*z = f, \quad \text{for } t \in (0, T), \quad z(T) = z_T.$$

Lemmata

- $J$ is strictly convex and coercive on $H \times L^2(0, T; H)$
  $\sim \rightarrow$ Consequence of unique continuation.

- If $(Z_T, F)$ is the minimizer, $y = F$ and $u = B^*Z$ solves the approximate control problem.
Our goal today

Control the linear system \( y' = Ay + Bu \) at time \( T \) and impose linear constraints on the control and the controlled trajectory.

\((H3)\) \( \mathcal{G} \) is a closed vector space of \( L^2(0, T; U) \), and \( \mathbb{P}_\mathcal{G} \) is the orthogonal projection on \( \mathcal{G} \) in \( L^2(0, T; U) \).

\((H4)\) \( \mathcal{W} \) is a closed vector space of \( L^2(0, T; H) \), and \( \mathbb{P}_\mathcal{W} \) is the orthogonal projection on \( \mathcal{W} \) in \( L^2(0, T; H) \).

Approximate controllability with constraints

For any \( y_0, y_1 \in H \), \( \varepsilon > 0 \), \( g_* \in \mathcal{G} \), \( w_* \in \mathcal{W} \), find a control function \( u \in L^2(0, T; U) \) such that the control \( u \) and the controlled trajectory \( y \) satisfy

\[
\| y(T) - y_1 \|_H \leq \varepsilon,
\]

\[
\mathbb{P}_\mathcal{G} u = g_* ,
\]

\[
\mathbb{P}_\mathcal{W} y = w_* .
\]
Relevant unique continuation property is

\[
\begin{aligned}
  z' + A^* z &= w, \quad \text{for } t \in (0, T), \\
  z(T) &= z_T, \\
  B^* z &= g, \quad \text{for } t \in (0, T), \\
  \text{with } (z_T, g, w) &\in H \times G \times W
\end{aligned}
\]

\[\Rightarrow \begin{cases}
  z_T = 0, \\
  g = 0, \\
  w = 0.
\end{cases} \quad (UC)
\]

**Theorem**

Assume that \( W \) is of finite dimension.

Unique continuation property (UC)

\[\Leftrightarrow \quad \text{Approximate controllability with constraints} \]
Proof of $\iff$: If $\exists (z_T, g, w) \in H \times G \times W \setminus \{(0,0,0)\}$ such that

$$z' + A^* z = w, \text{ for } t \in (0, T),$$

$$z(T) = z_T,$$

$$B^* z = g$$

then, for $y' = Ay + Bu$, with $y(0) = 0$, $0 = \langle y(T), z_T \rangle_H - \int_0^T \langle y(t), w(t) \rangle_H dt - \int_0^T \langle u(t), g(t) \rangle_U dt$.

In particular, if one imposes $P_W y = w$, $P_G u = g$, we should have

$$\|y(T) + z_T\|_H \|z_T\|_H \geq \|z_T\|^2_H + \|w\|_{L^2(0,T;H)}^2 + \|g\|_{L^2(0,T;U)}^2,$$

hence $y(T)$ cannot approximate $-z_T$. 

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Proof of $\Rightarrow$: If we suppose that (UC) holds, let $y_0, y_1 \in H$, $\varepsilon > 0$, $g \in G_*$, and $w_* \in W$. Minimize

$$J(z_T, g, w, f) = \frac{1}{2} \int_0^T \|B^*z(t) + g(t)\|_U^2 \ dt + \frac{1}{2} \int_0^T \|f(t) + w(t)\|_H^2 \ dt$$

$$+ \langle y_0, z(0) \rangle_H - \langle y_1, z_T \rangle_H + \varepsilon \|z_T\|_H$$

$$+ \int_0^T \langle B^*z(t), g_*(t) \rangle_U \ dt + \int_0^T \langle f(t), w_*(t) \rangle_H \ dt,$$

for $(z_T, g, w, f) \in H \times G \times W \times L^2(0, T; H)$, and $z' + A^*z = f$, $z(T) = z_T$.

**Lemmata**

- **Unique continuation (UC) $\Rightarrow$** $J$ is coercive.
- If $(Z_T, G, W, F)$ is the minimizer of $J$, $y = F + W + w_*$, and $u = B^*Z + G + g_*$ solves the control problem.
Exact controllability with constraints

For any $y_0, y_1 \in H$, $g_* \in G$, $w_* \in W$, find a control function $u \in L^2(0, T; U)$ such that the control $u$ and the controlled trajectory $y$ satisfy

$$y(T) = y_1, \quad \mathbb{P}_G u = g_*, \quad \mathbb{P}_W y = w_*.$$

**Theorem**

Assume the observability inequality: $\exists C > 0$, such that for all $z$ satisfying $z' + A^* z = f$, $z(T) = z_T,$

$$\|(z_T, g, w, f)\|_{H \times G \times W \times L^2(0, T; H)} \leq C \left(\|B^* z + g\|_{L^2(0, T; U)} + \|f + w\|_{L^2(0, T; H)}\right).$$  \((\text{ExObs})\)

Then Exact controllability with constraints holds.
Recall that classical exact controllability of $y' = Ay + Bu$ is equivalent to

$$\|z_T\| \leq C\|F_T^*z_T\|_{L^2(0,T;U)} = C\|B^*z\|_{L^2(0,T;U)}, \quad \text{(ClassExObs)}$$

for $z$ solving $z' + A^*z = 0$, $z(T) = z_T$.

**Lemma**

If $G$ and $W$ are of finite dimension, Unique continuation (UC) + Classical Observability (ClassExObs) $\Rightarrow$ Observability inequality (ExObs).
Null controllability with constraints

For any \( y_0 \in H, \ g_\ast \in G, \ w_\ast \in W \), find a control function \( u \in L^2(0, T; U) \) such that the control \( u \) and the controlled trajectory \( y \) satisfy

\[
y(T) = 0, \quad P_G u = g_\ast, \quad P_W y = w_\ast.
\]

Theorem

Assume the observability inequality: \( \exists C > 0 \), such that for all \( z \) satisfying \( z' + A^*z = f \), \( z(T) = z_T \),

\[
\|(z(0), g, w, f)\|_{H \times G \times W \times L^2(0, T; H)} \leq C \left( \|B^* z + g\|_{L^2(0, T; U)} + \|f + w\|_{L^2(0, T; H)} \right). \quad \text{(NullObs)}
\]

Then Null controllability with constraints holds.
Lemma

If $\mathcal{G}$ and $\mathcal{W}$ are of finite dimension, and $\exists \tilde{T} \in (0, T)$ s.t.

\[
\begin{cases}
z' + A^*z = w, & \text{for } t \in (0, \tilde{T}), \\
z(\tilde{T}) = z_{\tilde{T}}, \\
B^*z = g, & \text{for } t \in (0, \tilde{T}),
\end{cases}
\]

with $(z_{\tilde{T}}, g, w) \in H \times \mathcal{G} \times \mathcal{W}$,

and $\exists C > 0$ such that for $z$ solving $z' + A^*z = 0$ in $(0, T)$,

\[
\left\|z(\tilde{T})\right\|_H \leq C \left\|B^*z\right\|_{L^2(0, T; U)}.
\]  

(Ineq)

Then the observability inequality (NullObs) holds.

Rk: The observability estimate (Ineq) implies null-controllability of $y' = Ay + Bu$ at time $T$, and null-controllability of $y' = Ay + Bu$ at time $T'$ implies (Ineq).
### Summary

**Relevant unique continuation property** is

\[
\begin{cases}
z' + A^*z = w, & \text{for } t \in (0, T), \\
z(T) = z_T, & \text{for } t \in (0, T), \\
B^*z = g, & \text{for } t \in (0, T),
\end{cases}
\]

with \((z_T, g, w) \in H \times G \times W\)  \Rightarrow

\[
\begin{cases}
z_T = 0, \\
g = 0, \\
w = 0.
\end{cases}
\]  \hspace{1cm} (UC)

### Main remaining difficulty

How to check this unique continuation property in practice?
Example 1

We consider a linear control system

\[ y' = Ay + Bu \]

which is

- approximately controllabe in time \( T_{AC} \)
- exactly controllable in time \( T_{EC} \)
- \( T_{EC} > T_{AC} \).

Typical example

The wave equation. In the unit square observed from a neighborhood of two consecutive sides, \( T_{AC} = 2, T_{EC} = 2\sqrt{2} \).

We choose \( T = T_{EC} \), and

\[ \mathcal{G} \subset \{ u \in L^2(0, T; U), u = 0 \text{ on } (T - T_{AC}, T) \}, \quad \mathcal{W} = 0, \]

Then (UC) holds.
\[ \Omega = (0, 1)^2, \omega = \text{neighborhood of two consecutive sides}: \]

\[
\begin{cases}
\partial_{tt} y - \Delta y = u \chi_\omega, & \text{for } (t, x) \in (0, T) \times \Omega, \\
y(t, x) = 0, & \text{for } (t, x) \in (0, T) \times \partial \Omega, \\
(y(0, \cdot), \partial_t y(0, \cdot)) = (y_0, y_1) & \in H^1_0(\Omega) \times L^2(\Omega).
\end{cases}
\]

Let \( T \geq 2\sqrt{2} \), and \( \mathcal{G} \) be a finite dimensional subspace of

\[
\{ u \in L^2(0, T; L^2(\omega)), u = 0 \text{ on } (T - 2, T) \times \omega \},
\]

**Theorem**

*Given any* \((y_0, y_1), (y_0^T, y_1^T) \in H^1_0(\Omega) \times L^2(\Omega), \text{any } g \in \mathcal{G}, \text{there exists } u \in L^2(0, T; L^2(\omega)) \text{ such that the solution of the wave equation satisfies}*

\[
(y(T, \cdot), \partial_t y(T, \cdot)) = (y_0^T, y_1^T) \text{ and } \mathbb{P}_\mathcal{G} u = g.
\]

**Rk:** This theorem cannot be true when

\[
\mathcal{G} = \{ u \in L^2(0, T; L^2(\omega)), u = 0 \text{ on } (T - 2, T) \times \omega \}.
\]
Ex. 2: $\mathcal{G} = \{0\}$, and $B^* z = 0$ implies that $w = 0$.

Controlled heat equation in $\Omega$ bounded domain of $\mathbb{R}^d$, $\omega$ open subset of $\Omega$.

$$
\begin{aligned}
\left\{
\begin{array}{ll}
\partial_t y - \Delta y = u \chi_\omega, & \text{for } (t, x) \in (0, T) \times \Omega, \\
y(t, x) = 0, & \text{for } (t, x) \in (0, T) \times \partial \Omega, \\
y(0, \cdot) = y_0, & \text{in } \Omega.
\end{array}
\right.
\end{aligned}
$$

$\mathcal{G} = \{0\}$ and $\mathcal{W}$ a subspace of $L^2(0, T; L^2(\Omega))$ such that

$$
\Pi_\omega : f \mapsto f|_\omega \text{ satisfies } \text{Ker}(\Pi_\omega|_\mathcal{W}) = \{0\}.
$$

Then (UC) holds:

$$
\begin{aligned}
\left\{
\begin{array}{ll}
\partial_t z + \Delta z = w, & \text{for } (t, x) \in (0, T) \times \Omega, \\
z(t, x) = 0, & \text{for } (t, x) \in (0, T) \times \partial \Omega, \\
z(T, \cdot) = z_T, & \text{in } \Omega, \\
z(t, x) = 0 & \text{in } (0, T) \times \omega.
\end{array}
\right. 
\Rightarrow w = 0 \text{ and } z = 0.
\end{aligned}
$$
In particular, if we further assume that $\mathcal{W}$ is of finite dimension, Null controllability with constraint holds in time $T$.

- Inspired by works on sentinels: [Lions ’92, Nakoulima ’04, Mophou-Nakoulima ’08, ’09, Gao ’15].
- Can be done when $\mathcal{W}$ and $\mathcal{G}$ are non zero under the condition

$$\exists \text{ two linear operators } K \text{ and } L \text{ s.t.}$$

$$\begin{cases} 
K : L^2(0, T; H) \mapsto \mathcal{H} \text{ for some Hilbert space } \mathcal{H}, \\
L : L^2(0, T; U) \mapsto \mathcal{H}, \\
K(\partial_t + A^*) = LB^*, \\
\text{Ker } ((g, w) \in \mathcal{G} \times \mathcal{W} \mapsto Lg + Kw) = \{0\}.
\end{cases}$$
Example 3

**Theorem**

Let $A$ be the generator of an **analytic semigroup** on $H$.

Let $K \in \mathbb{N}$, $(\mu_k)_{k \in \{1, \ldots, K\}}$ two by two distinct real numbers, $\mathcal{W}_k$ be a family of closed vector spaces included in $H$ such that

$$\forall z \in \mathcal{D}(A^*), \quad (\mu_k + A^*)z \in \mathcal{W}_k, \text{ and } B^*z = 0 \Rightarrow z = 0,$$

and $\mathcal{W} = \text{Span}\{e^{\mu_k t}w_k, k \in \{1, \ldots, K\}, \text{ and } w_k \in \mathcal{W}_k\}$.

Let $J \in \mathbb{N}$, $(\rho_j)_{j \in \{1, \ldots, J\}}$ two by two distinct real numbers, $\mathcal{G}_j$ be a family of closed vector spaces included in $U$ such that

$$\forall z \in \mathcal{D}(A^*) \text{ satisfying } (\rho_j + A^*)z = 0, \quad B^*z \in \mathcal{G}_j \Rightarrow z = 0,$$

and $\mathcal{G} = \text{Span}\{e^{\rho_j t}g_j, j \in \{1, \ldots, J\}, \text{ and } g_j \in \mathcal{G}_j\}$.

We also assume $\mu_k \neq \rho_j$ for all $j, k$ and the classical unique continuation property $(z' + A^*z = 0 \& B^*z = 0) \Rightarrow z \equiv 0$.

Then the **unique continuation property (UC)** is satisfied.
Proof. If $z$ satisfies

$$z' + A^*z = w, \quad B^*z = g,$$

with $w \in \mathcal{W}$ and $g \in \mathcal{G}$, applying

$$P = \prod_{k=1}^{K} (\partial_t - \mu_k) \prod_{j=1}^{J} (\partial_t - \rho_j),$$

we obtain $(Pz)' + A^*(Pz) = 0$ and $B^*(Pz) = 0$. By the classical unique continuation property, $Pz = 0$, hence

$$z(t) = \sum_{k=1}^{K} z_k e^{\mu_k t} + \sum_{j=1}^{J} z_j e^{\mu_j t},$$

with $(\mu_k + A^*)z_k \in \mathcal{W}_k$, and $B^*z_k = 0$, and $(\rho_j + A^*)z_j = 0$, and $B^*z_k \in \mathcal{G}_j$.

Therefore, $z = 0$, and thus $w = 0$ and $g = 0$. 
This idea was used in the context of Navier-Stokes equations:

$$\Omega = \mathbb{T} \times (0, 1), \text{ where } \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}.$$  

$$\begin{cases} 
\partial_t y + (y \cdot \nabla)y - \Delta y + \nabla p = 0, & \text{in } (0, \infty) \times \Omega, \\
\text{div } y = 0, & \text{in } (0, \infty) \times \Omega, \\
y(t, x_1, 0) = (0, 0), & \text{on } (0, \infty) \times \mathbb{T}, \\
y(t, x_1, 1) = (0, u(t, x_1)), & \text{on } (0, \infty) \times \mathbb{T}, \\
y(0, x_1, x_2) = y^0(x_1, x_2), & \text{in } \Omega. 
\end{cases}$$

- $y = y(t, x_1, x_2) \in \mathbb{R}^2$ is the velocity.
- $p = p(t, x_1, x_2)$ is the pressure.
- $u = u(t, x_1)$ is the control function, acting on the normal component only.

**Theorem: Stabilization at any exponential rate [Chowdhury SE 19].**

For any $\omega > 0$, $\exists \varepsilon > 0$, $\forall y_0 \in V_0^1(\Omega)$ satisfying $\|y_0\|_{H^1} \leq \varepsilon$, $\exists u \in L^2(0, \infty; L^2(0, 1))$, such that $\|y(t)\|_{H^1(\Omega)} \leq Ce^{-\omega t}$. 
Here, the control system is of the form

\[ y' = Ay + F(y) + Bu, \]

where

- \( F \) is a quadratic term.
- The space \( H = L^2_0(\Omega) \) can be decomposed into \( H = H_0 \oplus H_1 \), and

\[
\begin{align*}
y' = Ay + Bu &\iff \begin{cases}
y'_0 = A_0y_0, \\
y'_1 = A_1y_1 + B_1u,
\end{cases} \quad \text{where} \quad \begin{cases}
y_0 = P_{H_0}y, \\
y_1 = P_{H_1}y.
\end{cases}
\end{align*}
\]

In fact, \( H_0 = \{ y \in L^2_\sigma(\Omega), \ y = y(x_2) \} \), and \( H_1 = H_0^\perp \). Consequently, the projection \( y_0 \) cannot be controlled on the linearized equations.
Strategy: Expand \( y = \varepsilon \alpha + \varepsilon^2 \beta \) and use the non-linear term to control the projection in \( H_0 \).

Inspired by [Coron Crépeau '04, Cerpa '07, Cerpa Crépeau '09, Coron Rivas '15].

- \( \varepsilon > 0 \) small.
- \( \alpha, \beta \) of order 1.
- \( \alpha \in H_1 \).

Up to lower order terms,

\[
\begin{align*}
\alpha' &= A_1 \alpha + B_1 u, \\
\alpha(0) &= \alpha_0 \in H_1
\end{align*}
\]

\[
\begin{align*}
\beta' &= A \beta + Bu + F(\alpha), \\
\beta(0) &= u_0 - \alpha_0.
\end{align*}
\]

Difficulty: Controlling \( P_0 \beta = \beta_0 \).

\[
\begin{align*}
\beta'_0 &= A_0 \beta_0 + F_0(\alpha), \\
\beta_0(0) &= \beta_{00} \in H_0.
\end{align*}
\]
In particular, our arguments rely on the following construction: Let

$$H_{0, \omega} = \text{Span} \{ \Psi \text{ eigenvector of } A_0 \text{ corresponding to eigenvalue } \lambda > \omega \}.$$ 

This space is of finite dimensional in our case.

**Lemma**

For any $h_0 \in H_{0, \omega}$, there exists a control function $u$ such that the solution $(\alpha, \beta_0)$ of

$$\begin{cases}
\alpha' = A_1 \alpha + B_1 u, \\
\alpha(0) = 0,
\end{cases} \quad \begin{cases}
\beta_0' = A_0 \beta_0 + F_0(\alpha), \\
\beta_0(0) = 0,
\end{cases}$$

satisfies

$$\alpha(T) = 0 \quad \text{and} \quad P_{H_{0, \omega}} \beta_0(T) = h_0.$$ 

Indirect controllability result through a non-linear coupling.
Intermediate result

If $\Psi$ is an eigenvector of $A_0$ corresponding to an eigenvalue $\lambda$, there exist two controls $u_\pm$ such that the solutions $(\alpha_\pm, \beta_{0,\pm})$ of

$$\begin{cases} 
\alpha_\prime_\pm = A_1 \alpha_\pm + B_1 u_\pm, \\
\alpha_\pm(0) = 0,
\end{cases}$$

and

$$\begin{cases} 
\beta_{0,\prime} = A_0 \beta_{0,\pm} + F_0(\alpha_\pm), \\
\beta_{0,\pm}(0) = 0,
\end{cases}$$

satisfies

$$\alpha_\pm(T) = 0 \quad \text{and} \quad \langle \beta_{0,\pm}(T), \Psi \rangle = \pm 1.$$
When taking two controls $u_a$ and $u_b$ such that

$$\begin{cases}
\alpha_a' = A_1 \alpha_a + B_1 u_a, \\
\alpha_a(0) = 0,
\end{cases} \quad \begin{cases}
\alpha_b' = A_1 \alpha_b + B_1 u_b, \\
\alpha_b(0) = 0,
\end{cases}$$

satisfy $\alpha_a(T) = \alpha_b(T) = 0$, for all $a, b \in \mathbb{R}$, if $\beta_0$ satisfies

$$\begin{cases}
\beta_0' = A_0 \beta_0 + F_0(a \alpha_a + b \alpha_b), \\
\beta_0(0) = 0,
\end{cases}$$

satisfy

$$\langle \beta_0(T), \psi \rangle = a^2 Q(u_a, u_a) + 2abQ(u_a, u_b) + b^2 Q(u_b, u_b),$$

where $Q$ is a bilinear form.

**A structure result:** If $u$ is odd or even in space, $Q(u, u) = 0$. 
We look for \( u_a = u_a(t) \sin(x_1) \) odd, \( u_b = u_b(t) \cos(x_1) \) even and \( Q(u_a, u_b) \neq 0 \).

\[
Q(u^a, u^b) = \pi^{5/2} \int_0^T u_a(t) q_b(t, 1) \, dt,
\]

where \( q_b \) is obtained by solving

\[
\begin{cases}
-\partial_t Z_b + Z_b - \partial_{22} Z_b + \begin{pmatrix} q_b \\ \partial_2 q_b \end{pmatrix} = F_b(t, x_2), & \text{in } (0, T) \times (0, 1), \\
-Z_{1,b} + \partial_2 Z_{2,b} = 0, & \text{in } (0, T) \times (0, 1), \\
Z_b(t, 0) = Z_b(t, 1) = (0, 0), & \text{in } (0, T), \\
Z_b(T, x_2) = 0, & \text{in } (0, 1),
\end{cases}
\]

with \( F_b(t, x_2) = \cos(\pi x_2) e^{\pi^2 t} \begin{pmatrix} \alpha_{2,b}(t, x_2) \\ \alpha_{1,b}(t, x_2) \end{pmatrix} \), and

\[
\alpha_b = \begin{pmatrix} \alpha_{1,b}(t, x_2) \sin(x_1) \\ \alpha_{2,b}(t, x_2) \cos(x_1) \end{pmatrix}
\]

solves \( \alpha'_b = A\alpha_b + Bu_b \).
Two steps:

- Find a control function $u_b$ such that $\alpha_b(T) = 0$ and $q_b(t, 1)$ non-zero.
- Find a control function $u_a$ such that $\alpha_a(T) = 0$ and $\int_0^T u_a(t)q_b(t, 1)\,dt \neq 0$.

This further requires $q_b(t, 1)$ to be such that we can impose the projection on $q_b(t, 1)$.

We do that by imposing $\alpha_b(t) = e^{\mu t \alpha}$ and $u_b(t) = e^{\mu t}$ on some subinterval $(T_1, T_2)$ of $(0, T)$ for a suitable choice of $\mu$ guaranteeing that

\[ q_b(t, 1) = c_0 e^{(\mu + \pi^2)t} + \sum a_j e^{(\lambda_j + \pi^2)(t-T_1)} + \sum b_j e^{\lambda_j(T_2-t)} \]

for $\lambda_j$ in the spectrum of $A$, and $c_0 \neq 0$.

Then our argument applies if $\mu \notin \{\lambda_j, -\lambda_j - \pi^2\}$ and a null control $u_a$ exists for which we have $\int_0^T u_a(t)q_b(t, 1)\,dt = 1$. 
**Time-dependent coefficients:** Provided well-posedness is ensured, the control system $y' = A(t)y + Bu$ can be controlled with linear constraints on $u$ in $G$ and $y$ in $W$ provided the following unique continuation property holds

\[
\begin{align*}
&z' + A(t)^*z = w, \quad \text{for } t \in (0, T), \\
&z(T) = z_T, \\
&B^*z = g, \quad \text{for } t \in (0, T), \\
\end{align*}
\]

with $(z_T, g, w) \in H \times G \times W$

\[
\Rightarrow \begin{cases} 
z_T = 0, \\
g = 0, \\
w = 0. 
\end{cases}
\]

(UC)

**Unbounded control operators:** The same results and proofs apply when $B$ is an admissible control operator.
Quantifying the cost of controlling in $\mathcal{G}$ and $\mathcal{W}$.
Similarly as the discussion in the work [Fernandez-Cara, Zuazua '00] for controlling the state exactly on some finite-dimensional space $E \subset L^2(\Omega)$ in the context of approximate controllability of the heat equation:

$$P_E y(T) = P_E y_1 \quad \text{and} \quad \|y(T) - y_1\|_{L^2(\Omega)} \leq \varepsilon.$$ 

Estimates of the cost:

$$\exp \left( \frac{C(1 + \exp(T \mu(E)))}{\varepsilon} \right),$$

where $\mu(E) = \max_{\varphi \in E \setminus \{0\}} \frac{\|\nabla \varphi\|_{L^2(\Omega)}^2}{\|\varphi\|_{L^2(\Omega)}^2}$. 

Sylvain Ervedoza 28/08/19
Thank you for your attention!

Comments Welcome

Reference:
Control issues and linear constraints on the control and on the controlled trajectory.
Sylvain Ervedoza, 2019.