Exact nodal profile controllability for the Saint Venant system on networks with cycles

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Joint work with

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Benasque VIII

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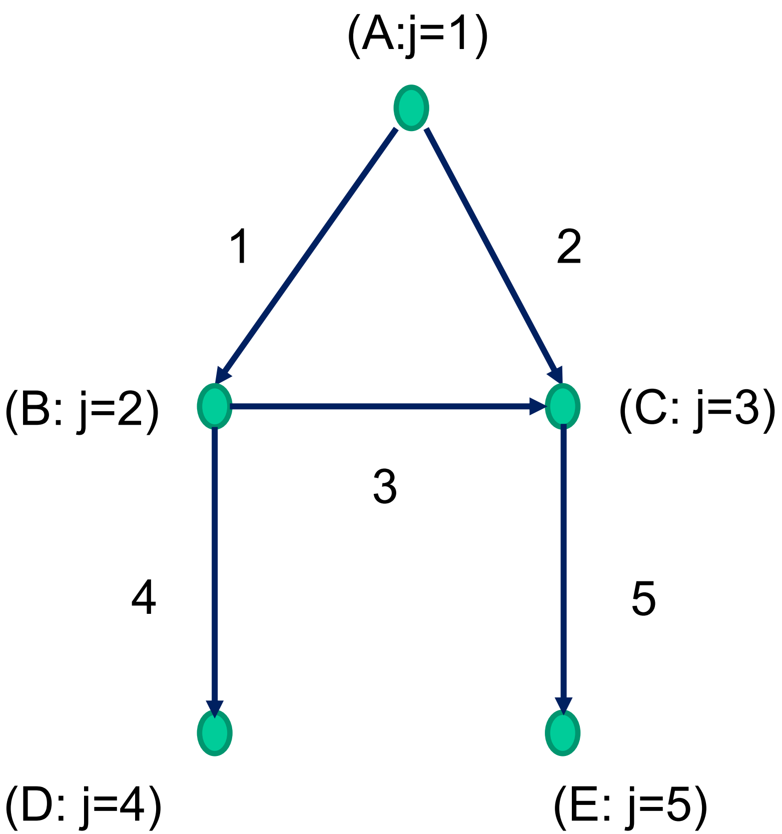
Department MATHEMATIK

CONTINUOUS OPTIMIZATION

The special network for this presentation



Flux controls at D and E

Edges

$$\mathcal{I} := \{i = 1, \dots, 5\}$$

Nodes

$$\begin{aligned}\mathcal{J} &:= \{j = 1, \dots, 5\} = \mathcal{J}^M \cup \mathcal{J}^S \\ &= \{1, 2, 3\} \cup \{4, 5\}\end{aligned}$$

$$d_{ij} := \begin{cases} -1 & \text{if the edge } i \text{ starts at node } j \\ 1 & \text{if the edge } i \text{ ends at node } j \\ 0 & \text{else} \end{cases}$$

Incidence matrix and incident edges

$$\mathcal{I}_j := \{i \in \mathcal{I} : d_{ij} \neq 0\}$$

The Saint Venant system

We consider the Saint Venant system on a network with a cycle:

$$\partial_t A_i + \partial_x (A_i V_i) = 0$$

$$\partial_t V_i + \partial_x S_i = 0,$$

where

$$\begin{aligned} A_i(t,x) &\hat{=} \text{ wetted cross section} \\ V_i(t,x) &\hat{=} \text{ average velocity over the cross section} \\ S_i(t,x) &=: \frac{1}{2} V_i 1^2 + g h_i(A_i) + g Y_{bi} \\ h_i(t,x) &\hat{=} \text{ water height} \\ Y_{bi} &\hat{=} \text{ bottom profile} \end{aligned}$$

Initial, boundary and transmission conditions

We have initial conditions

$$A_i(0,x) = A_{i0}(x),\ V_i(0,x) = V_{i0}(x),\ x \in [0, \ell_i],$$

transmission conditions at the multiple nodes: here A $\#1$, B $\#2$, C $\#$ 3;
$\mathcal{J}^M := \{1,2,3\}$

$$\sum_{i \in \mathcal{I}_j} d_{ij}(A_i V_i(t, v_j) = q_j(t),\ j \in \mathcal{I}_j,\ t \in [0,\ T]$$

$$S_i(t, v_j) = S_k(t, v_j), i, k \in \mathcal{I}_j$$

and flux boundary conditions at the simple nodes: here D $\#4$, E$\#5$;
$\mathcal{J}^S := \{4,5\}$

$$(A_i V_i)(t, v_j) = q_j(t),\ j \in \mathcal{J}^S,\ i\mathcal{I}_j,\ t \in [0,\ T].$$

Equilibrium and characteristics

We will always refer to an equilibrium state by (A_i^0, V_i^0) such that

$$|V_i^0| < \sqrt{gA_i^0 h_i'(A_i^0)}$$

and (A_i, V_i) satisfy all homogeneous transmission and boundary conditions as well as C^1-compatibility conditions at the nodes. We consider subcritical states $A_i, V_i)$ in the neighborhood of A_i^0, V_i^0 such that the eigenvalues of the system matrix are:

$$\lambda_i^1 = V_i - \sqrt{gA_i h_i'(A_i)} < 0 < \lambda_i^2 + \sqrt{gA_i h_i'(A_i)}$$

The corresponding characteristic equations are denoted as

$$t = f_i^1(x), \; t = f_i^2(x),$$

where f_i^1 is the incoming characteristic at $x = 0$ and f_i^2 the outgoing one.

Theorem: Existence and uniqueness of semi-global classical solutions

For any $T > 0$ and any given initial data $(A_{i0}, V_{i0}), i \in \mathcal{I},\ q_j, j \in \mathcal{J}$ with small norms:

$$\sum_{i\in\mathcal{I}} \|A_{i0} - A_i^0, V_{i0} - V_i^0\|_{C^1(0,\ell_i)},\ \|q_j - \sum_{i\in\mathcal{I}_j} d_{ij} A_i^0 V_i^0\|_{C^1(0,T)}, j \in \mathcal{J}^M$$

$$\|q_j - A_i^0 V_i^0\|_{C^1(0,T)}, j \in \mathcal{J}^S$$

such that C^1-compatibility conditions hold at all nodes. Then the network IBVP above admits a unique semi-global classical solution (A_i, V_i) with small piecewise $C1$-norm on the domain $R(T) = \cup_{i\in\mathcal{I}}\{(t,x)|0 \le t \le T,\ 0 \le x \le \ell_i\}$

Theorem: Nodal profile exact controllability

Lat T be larger than

$$T_0 := \max\left(\frac{\ell_1}{|\lambda_1^1(A_1^0, V_1^0)|}, \frac{\ell_2}{|\lambda_2^1(A_2^0, V_2^0)|}\right) + \max\left(\frac{\ell_4}{|\lambda_4^1(A_4^0, V_4^0)|}, \frac{\ell_5}{|\lambda_5^1(A_5^0, V_5^0)|}\right)$$

let $\bar{T} > T$ be given. Moreover, let initial data $(A_{i0}, V_{i0}), i \in \mathcal{I},\ q_j, j \in \mathcal{J}$ be given as in the last theorem. Further more, we prescribe nodal data $(\tilde{A}_{iA}(t), \tilde{V}_{iA}(t)), i = 1, 2$ at the multiple node A $(j = 1)$ for $T \leq t \leq \bar{T}$, satisfying the transmission conditions at A and having small norm $\|(\tilde{A}_{iA} - A_i^0, \tilde{V}_{iA} - V_i^0)\|_{C^1}$. Then there exist boundary controls q_4, q_5 with small norm $\|q_i - A_i^0 V_i^0\|_{C^1}$ such that the corresponding unique semi-global classical solution (A_i, V_i) with small piecewise $C1$-norm on the domain $R(T) = \cup_{i \in \mathcal{I}}\{(t, x) | 0 \leq t \leq \bar{T},\ 0 \leq x \leq \ell_i\}$ satisfies the prfile condition at A:

$$(A_i(t, 0), V_i(t, 0) = (\tilde{A}_i(t), \tilde{V}_i(t)),\ \ T \leq t \leq \bar{T},\ \ i = 1, 2.$$

Further travel times and outline of the proof

Proof: we define the following times:

$$T_1 := \max \sup_{\sum_{i=1,2} \|A_i - A_i^0, V_i - V_i^0\| \leq \epsilon} \left(\frac{\ell_1}{|\lambda_1^1(A_1, V_1)|}, \frac{\ell_2}{|\lambda_2^1(A_2, V_2)|} \right)$$

$$T_2 := \max \sup_{\sum_{i=4,5} \|A_i - A_i^0, V_i - V_i^0\| \leq \epsilon} \left(\frac{\ell_4}{|\lambda_4^1(A_4, V_4)|}, \frac{\ell_5}{|\lambda_5^1(A_5, V_5)|} \right)$$

We set $\hat{T} := T_1 + T_2$ Let $\epsilon > 0$ be small enough such that $T > \hat{T}$. We proceed in 5 steps:
(1) Forward solve of the whole system on $[0, \hat{T}]$, take traces at the nodes
(2) Extend nodal data at node A and perform a rightway solve for edges #1 and #2, interchanging x and t
(3) Include edge #3 using the nodal values at B,C complete the Kirchhoff conditions there
(4) Perform reightway solves fro the edges #4. #5 (5) Read off f_4, f_5.

Proof: Step 1

We perform a forward solve on the entire system with boundary conditions at D and E:

$$(A_4 V_4)(t, \ell_4) = f_4(t), \; (A_5 V_5)(t, \ell_5) = f_5(t)$$

, where f_i are arbitrary $C^1(0, \hat(T))$ functions with small norm. We denote the corresponding unique solution of this problem by

$$(A_i^f(t,x), V_i^f(t,x)), \; i = 1, \ldots, 5, (t,x) \in R(\hat(T))$$

Step 1

We can now uniquely determine the nodal values:

$$
\begin{aligned}
A :& (A_i^f, V_i^f)(t,0) =: (A_{iA}^f(t), V_{iA}^f(t)), t \in [0, \hat{T}], i = 1,2 \\
B :& (A_1^f, V_1^f)(t,\ell_1) =: (A_{1B}^f(t), V_{1B}^f(t)), \\
& (A_i^f, V_i^f)(t,0) =: (A_{iB}^f(t), V_{iB}^f(t)), t \in [0, \hat{T}], i = 3,4 \\
& S_1(t,\ell_1) = S_3(t,0) = S_4(t,0) =: S_B^f(t), t \in [0, \hat{T}] \\
C :& (A_i^f, V_i^f)(t,\ell_i) =: (A_{iC}^f(t), V_{iC}^f(t)), t \in [0, \hat{T}], i = 2,3 \\
& (A_5^f, V_5^f)(t,0) =: (A_{5C}^f(t), V_{5C}^f(t)), \\
& S_2(t,\ell_2) = S_3(t,\ell_3) = S_5(t,0) =: S_C^f(t), t \in [0, \hat{T}]
\end{aligned}
$$

Step 2

We now extend the nodal data at node A: Since $T > \hat{T}$ there exist $C^1(0, \bar{T})$ functions $(a_{iA}(t), v_{iA}(t)), i = 1, 2$ with small norm such that

$$(a_{iA}(t), v_{iA}(t) = \begin{cases} (A^f_{iA}(t), V^f_{iA}(t)) & 0 \leq t \leq \hat{T} \\ (\tilde{A}_{iA}(t), \tilde{V}_{iA}(t)) & T \leq t \leq \bar{T} \end{cases}$$

We then change the role of x and t and use the extended data as initial conditions at $x = 0$ for $i = 1, 2$:

$$(A_i, V_i)(t, 0) = (a_{iA}(t), v_{iA}(t)),\ 0 \leq t \leq \bar{T}$$

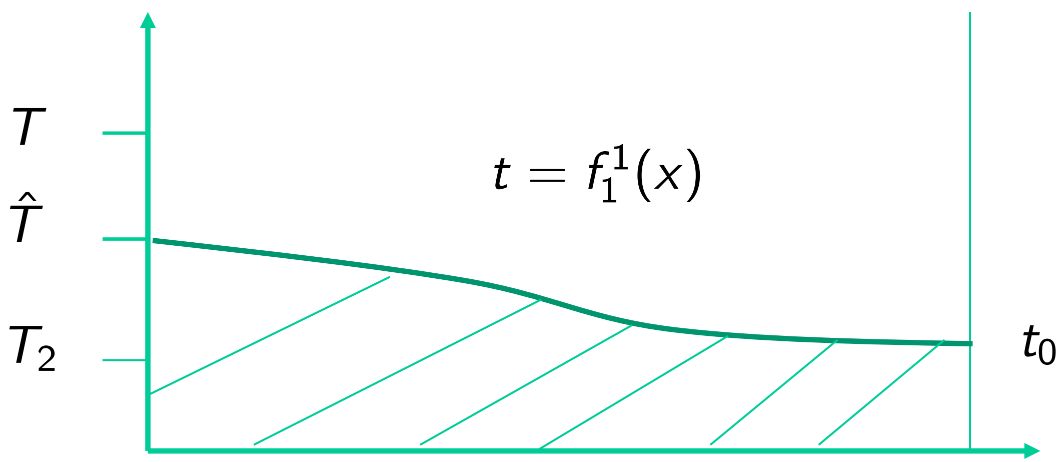
and take as boundary conditions

$$\begin{aligned} t = 0 :& A_i V_i(0, x) = A_{i0}(x) V_{i0}(x), & 0 \leq x \leq \ell_i \\ t = \bar{T} :& A_i V_i(\bar{T}, x) = g_i(x), & 0 \leq x \leq \ell_i \end{aligned}$$

Step 2

We now do a rightway solve for the edges #1, #2 and obtain a unique C^1 solution $(A_i, V_i) = (A_i(t,x), V_i(t,x)), i = 1, 2$ on $R_i(\bar{T})$.
In fact, $(A_i(t,x), V_i(t,x))$ and $(A_i^f(t,x), V_i^f(t,x))$ satify simulaneously the conditions

$$A_i(t,0)V_i(t,0) = A_{i0}V_{i0},\ 0 \le x \le \hat{T},$$
$$(A_i(0,x), V_i(0,x)) = (A_{iA}^f(t), V_{iA}^f(t)),\ 0 \le t \le \ell_i$$



$(A_1, V_1) = (A_1^f, V_1^f)$ on $R_1(T_2)$

$$R_1(T_2) = \{(t,x) | 0 \le t \le f_1^1(x), 0 \le x \le \ell_1\}$$

Step 2

We can in particular uniquely determine the values $(A_1(t,x), V_1(t,x))$ and $S_1(t,x)$ at the node B ($j=2$):

$$\begin{aligned} &(A_1(t,\ell_1), V_1(t,\ell_1)) = (A_{1B}(t), V_{1B}(t)), && 0 \le t \le \bar{T} \\ &S_1(t,\ell_1) = S_B(t), && 0 \le t \le \bar{T} \end{aligned}$$

We notice that we also have

$$\begin{aligned} &(A_{1B}(t), V_{1B}(t)) = (A^f_{1B}(t), V^f_{1B}(t)), && 0 \le t \le T_2 \\ &S_B(t) = S^f_B(t), && 0 \le t \le T_2 \end{aligned}$$

Step 2

We can follow the same procedure for channel $\#2$ $(A_2(t,x), V_2(t,x))$ and $S_2(t,x)$ at the node C ($j = 3$):

$$
\begin{aligned}
&(A_2(t,\ell_2), V_2(t,\ell_2)) = (A_{2C}(t), V_{2C}(t)), && 0 \le t \le \bar{T} \\
&S_2(t,\ell_1) = S_C(t), && 0 \le t \le \bar{T}
\end{aligned}
$$

We notice that we also have

$$
\begin{aligned}
&(A_{2C}(t), V_{2C}(t)) = (A^f_{2C}(t), V^f_{2C}(t)), && 0 \le t \le T_2 \\
&S_C(t) = S^f_C(t), && 0 \le t \le T_2
\end{aligned}
$$

Step 3

We are now in the position to handle channel #3:
Indeed, we have two boundary condtions

$$S_3(t,0) = S_B(t),\ S_3(t,\ell_3) = S_C(t),\ 0 \leq t \leq \bar{T}$$

with the original initial conditions. There exists a unique solution $(A_3(t,x), V_3(t,x))$ with small norm on

$$R_3(\hat{T}) := \{(t,x) | 0 \leq t \leq \bar{T}, 0 \leq x \leq \ell_3\}$$

Thus, we can evaluate at nodes B and C:

$$\begin{aligned}(A_3(t,0), V_3(t,0)) &= (A_{3B}(t), V_{3B}(t)) \\ (A_3(t,\ell_3), V_3(t,\ell_3)) &= (A_{3C}(t), V_{3C}(t)),\end{aligned}$$

for $0 \leq t \leq \bar{T}$. We have the same data for (A^f_{3B}, V^f_{3B}) and (A^f_{3C}, V^f_{3C}), respectively.

Step 4

Finally, we look at the channels #4 and #5:

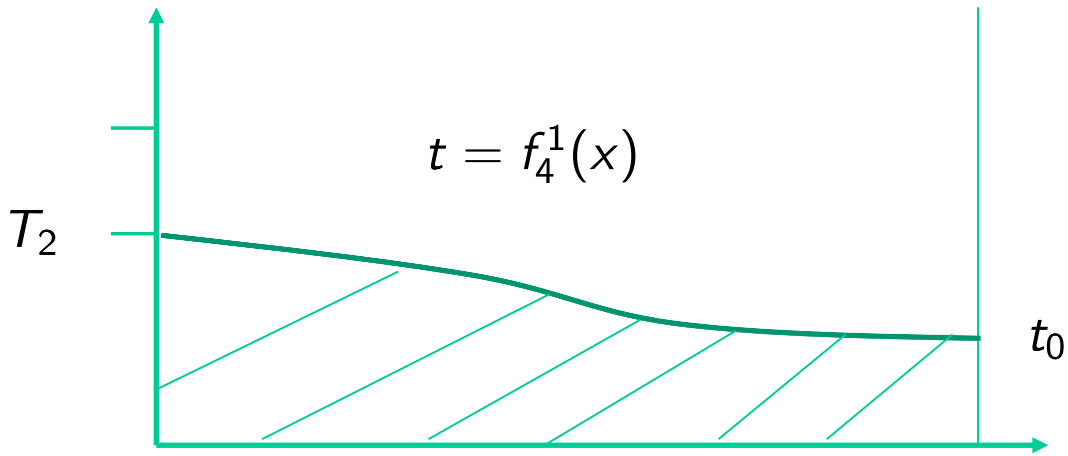
$$(A_4(t,0), V_4(t,0)) = (A_{4B}(t), V_{4B}(t)),\ 0 \le t \le \bar{T}$$
$$(A_{4B}(t), V_{4B}(t)) = (A^f_{4B}(t), V^f_{4B}(t)),\ 0 \le t \le T_2.$$

We then solve the rightward IBVP for channel #4 witht initial data above and boundary conditions

$$t = 0 : A_4 V_4(0,x) = A_{40} V_{40}(x),\ 0 \le x \le \ell_4$$
$$t = \bar{T} : A_4 V_4(\bar{T},x) = g_4(x),\ 0 \le x \le \ell_4$$

where, again, $g_4(x)$ is an arbitrary C^1 function with small norm. We obtain a uniquel solution (A_4, V_4) on $R_4(\bar{T})$. We can prove that the solution satisfies the original initial data. Indeed, the solutions (A_4, V_4) and (A^f_f, V^f_4) satisfy simultaneously the same rightway problem with the nitial condition $A_4 V_4(0,x) = g_4(x)$

Step 4 and 5



$$(A_4(0,x), V_4(0,x)) = (A_4^f(0,x), V_4^f(0,x)) = (A_{40}(x), V_{40}(x)), 0 \leq x \leq \ell_4$$

The same procedure applies to channel #5!

Finally, we collect all solutions $A_i(t,x), V_i(t,x)$ $i = 1, \dots, 5$ and then take the traces at

$$A_4 V_4(t, \ell_4) =: f_4(t),\ A_5 V_5(t, \ell_5) =: f_5(t)$$

Thank you for your attention!