Approximation of the controls for the 2-D wave equation

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Controlled wave equation

Let $\Omega = (0,1) \times (0,1) \subset \mathbb{R}^2$ and divide its boundary in two parts $\partial\Omega = \Gamma_0 \cup \Gamma_1$ such that $\Gamma_0 = \{(1,y) \,|\, 0 \le y \le 1\} \cup \{(x,1) \,|\, 0 \le x \le 1\}$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$.

Given any $T > 0$ and initial data $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} \in \mathcal{H} := L^2(\Omega) \times H^{-1}(\Omega)$ the exact controllability in time T of the two–dimensional controlled linear wave equation

$$\begin{cases} u_{tt}(t,x,y) - \Delta u(t,x,y) = 0 & t > 0,\ (x,y) \in \Omega \\ u(t,x,y) = v(t,x,y) & t > 0,\ (x,y) \in \Gamma_0 \\ u(t,x,y) = 0 & t > 0,\ (x,y) \in \Gamma_1 \\ u(0,x,y) = u^0(x,y), \quad u_t(0,x,y) = u^1(x,y) & (x,y) \in \Omega \end{cases} \tag{1}$$

consists of finding a function

$$v(t,x,y) = \begin{cases} v_1(t,x,y) & x = 1,\ 0 \le y \le 1 \\ v_2(t,x,y) & y = 1,\ 0 \le x \le 1, \end{cases} \quad \text{in } L^2((0,T) \times \Gamma_0),$$

called control, such that the corresponding solution (u, u_t) of (1) verifies

$$u(T,x,y) = u_t(T,x,y) = 0 \qquad ((x,y) \in \Omega). \tag{2}$$

Variational result

The function $v \in L^2((0,T) \times \Gamma_0)$ is a control which drives to zero the solution of (1) in time T if and only if, the following relation holds

$$\int_0^T \int_{\Gamma_0} v(t,x,y) \frac{\partial \overline{\varphi}}{\partial \nu}(t,x,y)\,dtdxdy =$$

$$= \langle u^1, \varphi(0) \rangle_{H^{-1},H_0^1}, - \int_\Omega u^0(x,y) \overline{\varphi_t}(0,x,y)\,dxdy, \tag{3}$$

for every $\begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$, where $\begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix} \in H_0^1(\Omega) \times L^2(\Omega)$ is the solution of the following adjoint backward problem

$$\begin{cases} \varphi_{tt}(t,x,y) - \Delta\varphi(t,x,y) = 0 & t > 0,\ (x,y) \in \Omega \\ \varphi(t,x,y) = 0 & t > 0,\ (x,y) \in \partial\Omega \\ \varphi(T,x,y) = \varphi^0(x,y) & (x,y) \in \Omega \\ \varphi_t(T,x,y) = \varphi^1(x,y) & (x,y) \in \Omega. \end{cases} \tag{4}$$

Spectral analysis

By denoting $W = \begin{pmatrix} \varphi \\ \varphi_t \end{pmatrix}$, equation (4) is equivalent with

$$\begin{cases} W_t + AW = 0 \\ W(T) = W^0 = \begin{pmatrix} \varphi^0 \\ \varphi^1 \end{pmatrix}, \end{cases} \tag{5}$$

where $$A = \begin{pmatrix} 0 & -I \\ \Delta & 0 \end{pmatrix}.$$

Eigenvalues of A: $(i\lambda_{mn}^{\pm})_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$, where

$$\lambda_{mn}^{\pm} = \pm\pi\sqrt{m^2+n^2}.$$

Eigenfunctions of A form an orthogonal basis in $H_0^1(\Omega)\times L^2(\Omega)$:

$$\phi_{mn}^{\pm} = \sqrt{2}\begin{pmatrix} \frac{1}{i\lambda_{mn}^{\pm}} \\ -1 \end{pmatrix}\sin(m\pi x)\sin(n\pi y). \tag{6}$$

Moment problem for the wave equation

The null-controllability of the wave equation is equivalent to solve the following moment problem:

For any $\begin{pmatrix} u^0 \\ u^1 \end{pmatrix} = \sum_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*} a_{mn}\phi_{mn}^{\pm}$, find $v \in L^2((0,T)\times\Gamma_0)$ such that

$$\int_0^T \int_{\Gamma_0} e^{-i\lambda_{mn}^{\pm}t} v(t,s)\,ds\,dt = a_{mn}, \quad \Leftrightarrow \tag{7}$$

$$\int_0^T \int_0^1 e^{-i\lambda_{mn}^{\pm}t} v^1(t,y)\,dy\,dt + \int_0^T \int_0^1 e^{-i\lambda_{mn}^{\pm}t} v^2(t,x)\,dx\,dt = a_{mn}. \tag{8}$$

A solution (v^1, v^2) of the moment problem may be constructed by means of two biorthogonal sequences to the families $\left(e^{i\lambda_{mn}^{\pm}t}\right)_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$.

Biorthogonal sequence

Definition

Let $m \in \mathbb{N}^*$ be fixed. The sequence $\left(\theta_{mn}^{1,\pm}\right)_{n\in\mathbb{N}^*} \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is $(1, m)$-biorthogonal to the family $\left(e^{i\lambda_{mn}^{\pm} t}\right)_{n\in\mathbb{N}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{1,\pm}(t) e^{-i\lambda_{mq}^{\pm} t} dt = \delta_{nq}, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{1,\mp}(t) e^{-i\lambda_{mq}^{\pm} t} dt = 0 \quad (n, q \in \mathbb{N}^*).$$

Let $n \in \mathbb{N}^*$ be fixed. The sequence $\left(\theta_{mn}^{2,\pm}\right)_{m\in\mathbb{N}^*} \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ is $(2, n)$-biorthogonal to the family $\left(e^{i\lambda_{mn}^{\pm} t}\right)_{m\in\mathbb{N}^*}$ in $L^2\left(-\frac{T}{2}, \frac{T}{2}\right)$ if

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{2,\pm}(t) e^{-i\lambda_{pn}^{\pm} t} dt = \delta_{mp}, \quad \int_{-\frac{T}{2}}^{\frac{T}{2}} \theta_{mn}^{2,\mp}(t) e^{-i\lambda_{pn}^{\pm} t} dt = 0 \quad (m, p \in \mathbb{N}^*).$$

If $\left(\theta^{1,\pm}_{mn}\right)_{n\in\mathbb{N}^*}$ is $(1,m)$-biorthogonal to the family $\left(e^{i\lambda^{\pm}_{mn}t}\right)_{n\in\mathbb{N}^*}$ and $\left(\theta^{2,\pm}_{mn}\right)_{m\in\mathbb{N}^*}$ is $(2,n)$-biorthogonal to the family $\left(e^{i\lambda^{\pm}_{mn}t}\right)_{m\in\mathbb{N}^*}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ then a "formal" solution of the moment problem is given by $\begin{pmatrix} v^1(t,y) \\ v^2(t,x) \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} v^1_n(t)\sin(n\pi y) \\ \sum_{m=1}^{\infty} v^2_m(t)\sin(m\pi x) \end{pmatrix}$, where

$$\begin{cases} v^1_n(t) = \displaystyle\sum_{m\in\mathbb{N}^*;\, m\geq n}^{\infty} a^1_{mn}\theta^{2,\pm}_{mn}(t) & (n\in\mathbb{N}^*) \\ v^2_m(t) = \displaystyle\sum_{n\in\mathbb{N}^*;\, n>m}^{\infty} a^2_{mn}\theta^{1,\pm}_{mn}(t) & (m\in\mathbb{N}^*). \end{cases} \tag{9}$$

Main problems

- the existence of the biorthogonal sequences $\left(\theta_{mn}^{1,\pm}\right)_n$ and $\left(\theta_{mn}^{2,\pm}\right)_m$ to the family $(e^{i\lambda_{mn}^{\pm}t})_{(m,n)}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$
- evaluation of the norms of $\left(\theta_{mn}^{1,\pm}\right)_n$ and $\left(\theta_{mn}^{2,\pm}\right)_m$

This estimates are needed to show the convergence of the series in (9) and to have a bound of the norms of v_n^1 and v_m^2.

S. Micu, L. Teresa, Asymptotic Analysis, 2010

$$\| \left(\theta_{mn}^{1,\pm}\right)_n \|_{L^2(0,T)} \leq C \qquad (n \geq m),$$
$$\| \left(\theta_{mn}^{2,\pm}\right)_m \|_{L^2(0,T)} \leq C \qquad (m > n).$$

A constructive way to obtain a biorthgonal sequence

- $(\Psi^{\pm}_{mn})_{(m,n)\in\mathbb{N}^*\times\mathbb{N}^*}$ entire functions.

H1 ▷ $|\Psi^{\pm}_{mn}(z)| \leq A e^{\frac{\mathbf{T}}{\mathbf{2}}|z|}$,

H2 ▷ $\Psi^{\pm}_{mn} \in L^2(\mathbb{R})$,

H3 ▷ $\begin{cases} \Psi^{+}_{mn}\left(\lambda^{+}_{mq}\right) = \delta_{nq} & \Psi^{+}_{mn}\left(\lambda^{-}_{mq}\right) = 0 \\ \Psi^{-}_{mn}\left(\lambda^{-}_{mq}\right) = \delta_{nq} & \Psi^{-}_{mn}\left(\lambda^{+}_{mq}\right) = 0 \end{cases}$

Paley–Wiener Theorem (1934)

$$\theta^{\pm}_{mn} \in L^2\left(-\frac{T}{2}, \frac{T}{2}\right) \text{ such that } \Psi^{\pm}_{mn}(z) = \int_{-\frac{T}{2}}^{\frac{T}{2}} \theta^{\pm}_{mn}(t) e^{-izt}\, dt.$$

Plancherel's Theorem (1910)

$$\int_{-\frac{T}{2}}^{\frac{T}{2}} \left|\theta^{\pm}_{mn}(t)\right|^2 dt = \frac{1}{2\pi} \int_{\mathbb{R}} \left|\Psi^{\pm}_{mn}(x)\right|^2 dx.$$

Finite differences for the 2-D wave equation

Let $J, K \in \mathbb{N}^*$, $h_1 = \frac{1}{J+1}$, $h_2 = \frac{1}{K+1}$ and $x_{j,k} = (jh_1, kh_2)$, $0 \le j \le J+1$, $0 \le k \le K+1$,
$\Gamma_h^1 = \{(0,k) \,|\, 0 \le k \le K+1\} \cup \{(j,0) \,|\, 0 \le j \le J+1\}$,
$\Gamma_h^0 = \{(J+1,k) \,|\, 0 \le k \le K+1\} \cup \{(j,K+1) \,|\, 0 \le j \le J+1\}$

$$\left\{\begin{array}{ll} u''_{pr}(t) - \frac{u_{j+1k}(t) - 2u_{jk}(t) + u_{j-1k}(t)}{h_1^2} - \frac{u_{jk+1}(t) - 2u_{jk}(t) + u_{jk-1}(t)}{h_2^2} = 0 & \\ \qquad\qquad t > 0,\ 1 \le j \le J,\ 1 \le k \le K & \\ u_{jk}(t) = 0 & t \in (0,T),\ (j,k) \in \Gamma_h^1 \\ u_{J+1,k}(t) = v_k^1(t) & t \in (0,T),\ (j,k) \in \Gamma_h^0 \\ u_{j,K+1}(t) = v_j^2(t) & t \in (0,T),\ (j,k) \in \Gamma_h^0 \\ u_{jk}(0) = u^0_{jk}, \quad u'_{jk}(0) = u^1_{jk} & 1 \le j \le J,\ 1 \le k \le K. \end{array}\right. \tag{10}$$

Discrete controllability problem: Given $T > 0$ and $\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = (u^0_{jk}, u^1_{jk})_{1 \le j \le J,\, 1 \le k \le K} \in \mathbb{C}^{2JK}$, there exists a control function $v_h = \begin{pmatrix} v^1_{h_1} \\ v^2_{h_2} \end{pmatrix} \in L^2(0,T)$ such that the corresponding solution $(u_{jk})_{1 \le j \le J,\, 1 \le k \le K}$ of (10) verifies

$$u_{jk}(T) = u'_{jk}(T) = 0 \qquad (1 \le j \le J,\ 1 \le k \le K) \tag{11}$$

Control of the projection of the solution

E. Zuazua, J. Math. pures et appl, 1999

- The constants on the boundary observability inequality blow-up as the mesh-size tends to zero.
  This is do to the largest eigenvalues of the corresponding adjoint system which are very different from the continuous ones (numerical spurious high eigenfrequencies).
- We recuperate the uniform observability inequality if we consider only the projections of the solutions over the space generated by the low frequencies:

$$\lambda \max\{h_1, h_2\} \leq 2\delta, \quad \delta \in (0,1).$$

  This is equivalent to the uniform controllability of the projection of the solution over this space.

The aim of this work is to show that we can guarantee the uniform controllability of the entire solution by filtering the high frequencies of the initial data only.

Spectral analysis

The eigenvalues are given by the family $(i\lambda_{mn}^{\pm}(h_1,h_2))_{\substack{1\le m\le J\\ 1\le n\le K}}$, where

$$\lambda_{mn}^{\pm}(h_1,h_2) = \pm\sqrt{\frac{4}{h_1^2}\sin^2\left(\frac{m\pi h_1}{2}\right)+\frac{4}{h_2^2}\sin^2\left(\frac{n\pi h_2}{2}\right)}, \tag{12}$$

and the corresponding eigenvectors are

$$\Phi_{mn}^{\pm}(h_1,h_2) = \sqrt{2}\begin{pmatrix} \frac{1}{i\lambda_{mn}^{\pm}(h_1,h_2)} \\ -1 \end{pmatrix}\left(\sin\left(m\pi p h_1\right)\sin\left(n\pi r h_2\right)\right)_{\substack{1\le p\le J\\ 1\le r\le K}} \tag{13}$$

Moreover, the vectors $(\Phi_{mn}^{\pm}(h_1,h_2))_{\substack{1\le p\le J\\ 1\le r\le K}}$ form an orthonormal basis in $\mathbb{C}^{2JK}$

Moment problem for the discrete problem

Theorem

The system (10) is null-controllable in time T if, and only if, for any initial data $\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} \in \mathbb{C}^{2JK}$ of the form

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \sum_{\substack{1\le p\le J\\ 1\le q\le K}} \alpha^{\pm}_{pq} \Phi^{\pm}_{pq}(h_1, h_2), \tag{14}$$

there exists $v_h \in L^2(0,T)$ such that, for any $1 \le m \le J$, $1 \le n \le K$, we have

$$\int_0^T \left(\frac{1}{h_1} \sin(m\pi J h_1)\, \tilde{v}^1_n(t) + \frac{1}{h_2} \sin(n\pi K h_2)\, \widetilde{v}^2_m(t) \right) e^{i\lambda^{\pm}_{mn} t} dt = \alpha^{\pm}_{mn}. \tag{15}$$

Discrete control

Let $\delta \in (0,1)$. If $\left(\theta^{1,\pm}_{mn}\right)_{1\leq n\leq K}$ is $(1,m)$-biorthogonal to the family $\left(e^{i\lambda^{\pm}_{mn}t}\right)_{1\leq n\leq K}$ and $\left(\theta^{2,\pm}_{mn}\right)_{1\leq m\leq J}$ is $(2,n)$-biorthogonal to the family $\left(e^{i\lambda^{\pm}_{mn}t}\right)_{1\leq m\leq J}$ in $L^2\left(-\frac{T}{2},\frac{T}{2}\right)$ then

$$\tilde{v}^1_n(t) = \begin{cases} \displaystyle\sum_{n<j<\delta(J+1)} \alpha^{\pm}_{jn} \frac{h_1}{(j\pi J h_1)} e^{-i\lambda^{\pm}_{jn}\frac{T}{2}} \theta^{2,\pm}_{jn}\left(\frac{T}{2}-t\right) \\ 0 \qquad \text{if} \;\; n \geq \delta(J+1), \end{cases}$$

$$\tilde{v}^2_m(t) = \begin{cases} \displaystyle\sum_{m\leq k<\delta(K+1)} \alpha^{\pm}_{mk} \frac{h_2}{\sin\left(k\pi K h_2\right)} e^{-i\lambda^{\pm}_{mk}\frac{T}{2}} \theta^{1,\pm}_{mk}\left(\frac{T}{2}-t\right) \\ 0 \qquad \text{if} \;\; m \geq \delta(K+1). \end{cases}$$

The biorthogonal sequences

Theorem

Let $\delta \in (0,1)$. There exist a time $T_1 > \frac{4\pi^2(\frac{h_2}{h_1}+\pi)}{(1-\delta)^2}$, $T_2 > \frac{4\pi^2(\frac{h_1}{h_2}+\pi)}{(1-\delta)^2}$, $h_1', h_2' > 0$ and a constant $C > 0$ such that for every $h_1 \in (0, h_1')$, $h_2 \in (0, h_2')$ $1 \leq m < \delta(J+1)$ and $1 \leq n < \delta(K+1)$ there exist two biorthogonal sequences $\left(\theta_{mn}^{1\pm}\right)_{1 \leq n \leq K}$ and $\left(\theta_{mn}^{2\pm}\right)_{1 \leq m \leq J}$ to the family of exponential functions $\left(e^{i\,\lambda_{mq}^{\pm}}\right)_{1 \leq q \leq K}$ in $L^2\left(-\frac{T_1}{2}, \frac{T_1}{2}\right)$, and $\left(e^{i\,\lambda_{pn}^{\pm}}\right)_{1 \leq p \leq J}$ in $L^2\left(-\frac{T_2}{2}, \frac{T_2}{2}\right)$, respectively, with the property that

$$\left\|\theta_{mn}^{1\pm}\right\|_{L^2\left(-\frac{T_1}{2}, \frac{T_1}{2}\right)} \leq C \qquad \left(m \leq n < \delta(K+1)\right), \tag{16}$$

$$\left\|\theta_{mn}^{2\pm}\right\|_{L^2\left(-\frac{T_2}{2}, \frac{T_2}{2}\right)} \leq C \qquad \left(n < m < \delta(J+1)\right). \tag{17}$$

- **A Weierstrass Product:**

$$(P1)\ P^{1,\pm}_{mn}(z) = \prod_{\substack{1\le q\le K\\ q\ne n}} \frac{z-\lambda^{\pm}_{mq}}{\lambda^{\pm}_{mn}-\lambda^{\pm}_{mq}} \prod_{1\le q\le K} \frac{z-\lambda^{\mp}_{mq}}{\lambda^{\pm}_{mn}-\lambda^{\mp}_{mq}} \qquad (1\le n\le K),$$

$$(P2)\ |P^{1,\pm}_{mn}(x)| \le C \begin{cases} \exp\left(\frac{1}{2}(\frac{h_2}{h_1}+\pi)\varphi_1(x)\right) & \left(|x|\le \frac{2}{h_1}\sin\frac{m\pi h_1}{2}\right) \\ 1 & \left(\frac{2}{h_1}\sin\frac{m\pi h_1}{2} < |x| < \sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right) \\ \exp\left(\varphi_2(x)\right) & \left(|x|\ge \sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right), \end{cases}$$

where

$$\varphi_1(x) = \sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2} - x^2},$$

$$\varphi_2(x) = \frac{2}{h_2}\ln\left(\frac{h_2}{2}\sqrt{x^2-\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}} + \sqrt{x^2\frac{h_2^2}{4}-\frac{h_2^2}{h_1^2}\sin^2\frac{m\pi h_1}{2}-1}\right)$$

The construction of the biorthogonal sequences

- **The multipliers:**

P. Lissy, I. Roventa, Math. Comp., 2019

$$|M_{mn}^{\pm}(x-\lambda_{mn}^{\pm})| \leq \begin{cases} \exp\left(-\varphi_2(x)\right) & \left(|x| \geq \sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right) \\ C & \left(|x| < \sqrt{\frac{4}{h_1^2}\sin^2\frac{m\pi h_1}{2}+\frac{4}{h_2^2}}\right) \end{cases}$$

S. Micu, L. Teresa, Asymptotic Analysis, 2010

$$|G_{mn}^{\pm}(x-\lambda_{mn}^{\pm})| \leq \begin{cases} \exp\left(-\frac{1}{2}(\frac{h_2}{h_1}+\pi)\varphi_1(x)\right) & \left(|x| \leq \frac{2}{h_1}\sin\frac{m\pi h_1}{2}\right) \\ C & \left(|x| > \frac{2}{h_1}\sin\frac{m\pi h_1}{2}\right) \end{cases}$$

The construction of the biorthogonal sequences

- **The entire function**
  $\Psi^{1,\pm}_{mn}(z) :=$
  $P^{1,\pm}_{mn}(z) M^{\pm}_{mn}(z-\lambda^{\pm}_{mn}) G^{\pm}_{mn}(z-\lambda^{\pm}_{mn}) \frac{\sin \epsilon(z-\lambda^{\pm}_{mn})}{\epsilon(z-\lambda^{\pm}_{mn})} \quad (z \in \mathbb{C}).$

- **Th. Paley-Wienner** $\Rightarrow$ $(\theta^{1,\pm}_{mn})_n = (\widehat{\Psi}^{1,\pm}_{mn})_n$ **biorthogonal**

Uniform boundedness of the sequence of controls

Theorem

Let $\delta \in (0,1)$. There exist h_1^0, h_2^0 and T_0 such that for any $h_1 < h_1^0$, $h_2 < h_2^0$, $T > T_0$ and initial data

$$\begin{pmatrix} U^0 \\ U^1 \end{pmatrix} = \sum_{\substack{1 \le m \le J \\ 1 \le n \le K}} \alpha_{mn}^{\pm} \Phi_{mn}^{\pm}(h_1, h_2) \in \mathbb{C}^{2JK}$$

there exists a control $v_h \in L^2\left((0,T); \mathbb{C}^{J+K}\right)$ for the problem (10) with the filtered initial data

$$\begin{pmatrix} \widetilde{U}^0 \\ \widetilde{U}^1 \end{pmatrix} = \sum_{\substack{1 \le m \le \delta J \\ 1 \le n \le \delta K}} \alpha_{mn}^{\pm} \Phi_{mn}^{\pm}(h_1, h_2),$$

such that the family $(v_h)_{h>0}$ is uniformly bounded in $L^2\left((0,T); \mathbb{C}^{J+K}\right)$. Moreover, there exists a subsequence which is weakly convergent to a control v of the continuous problem (1).

THANK YOU!!!!