Null Controllability for Parabolic Systems with Dynamic Boundary Conditions and Drift Terms

L. Maniar, Cadi Ayyad University, Marrakesh, Morocco

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In this talk, we study the null controllability of the following Parabolic equation with dynamic boundary conditions

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = \chi_\omega v(t,x) & \text{in } \Omega_T := (0,T)\times\Omega, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d\partial_\nu y + b(t,x)y_\Gamma = 0 & \text{on } \Gamma_T := (0,T)\times\Gamma, \\ y_\Gamma(t,x) = y_{|\Gamma}, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases}$$

- $\Omega \subset \mathbb{R}^N$ is a bounded domain with compact smooth boundary $\Gamma = \partial\Omega$, $N \geq 2$, and the control region ω is an *arbitrary* nonempty open subset such that $\overline{\omega} \subset \Omega$.
- The term $\partial_t y_\Gamma - \Delta_\Gamma y_\Gamma$ models the tangential diffusive flux on the boundary which is coupled to the equation on the bulk by the normal derivative $\partial_\nu y = \nu \cdot \nabla y|_\Gamma$.

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This type of dynamic boundary conditions arises for many known equations of mathematical physics and biology.
They are motivated by :

- problems in diffusion phenomena,
- Reaction-diffusion systems in phase-transition phenomena.
- Special flows in hydrodynamics (the flow of heat for a solid in contact with a fluid),
- Models in Dynamical populations, ...

References :
C. Gal, Favini, J. and G. Goldstein, Grasselli, Miranville, Meyries, Romanelli, Schnaubelt, Vazquez, Vitillaro, Warma, Zellik, ...

G. R. Goldstein, Derivation of dynamical boundary conditions, Adv. Differential Equations, 11 (2006), 457–480.

The operator Δ_Γ on Γ is given here by the surface divergence theorem

$$\int_\Gamma \Delta_\Gamma y \, z \, dS = -\int_\Gamma \langle \nabla_\Gamma y, \nabla_\Gamma z \rangle_\Gamma \, dS, \; y \in H^2(\Gamma), \; z \in H^1(\Gamma),$$

where ∇_Γ is the surface gradient.

Proposition

The operator $(\Delta_\Gamma, H^2(\Gamma))$ is self-adjoint and non positive on $L^2(\Gamma)$. Thus it generates an analytic C_0-semigroup on $L^2(\Gamma)$.

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Consider the following inhomogeneous parabolic problem with dynamic boundary conditions

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = f(t,x), & \text{in } \Omega_T, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + b(t,x)y_\Gamma = g(t,x), & \text{on } \Gamma_T \\ y_\Gamma = y|_\Gamma, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases} \tag{1}$$

$d > 0$ and $\delta > 0$.

On $\mathbb{L}^2 := L^2(\Omega) \times L^2(\Gamma)$, we consider the linear operator

$$A_0 = \begin{pmatrix} d\Delta & 0 \\ -d\partial_\nu & \delta\Delta_\Gamma \end{pmatrix}, \qquad D(A_0) = \mathbb{H}^2,$$

where $\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) : y|_\Gamma = y_\Gamma\}$, for $k \in \mathbb{N}$.

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where $\mathbb{H}^k := \{(y, y_\Gamma) \in H^k(\Omega) \times H^k(\Gamma) \,:\, y|_\Gamma = y_\Gamma\}$ for $k \in \mathbb{N}$, Our wellposedness and regularity results for the underlying evolution equations rely on this fact.

Proposition

The operator A_0 is densely defined, self-adjoint, non-positive and generates an analytic C_0-semigroup $(e^{tA_0})_{t\geq 0}$ on $\mathbb{L}^2$. We further have $(\mathbb{L}^2, \mathbb{H}^2)_{1/2,2} = \mathbb{H}^1$.

Let $a \in L^\infty((0,T)\times\Omega)$ and $b \in L^\infty((0,T)\times\Gamma)$. Then, The following perturbed system

$$\begin{cases} \partial_t y - d\Delta y + a(t,x)y = 0 & \text{in } \Omega_T := (0,T)\times\Omega, \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + b(t,x)y_\Gamma = 0 & \text{on } \Gamma_T := (0,T)\times\Gamma, \\ y_\Gamma = y|_\Gamma, \\ y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}, \end{cases}$$

has also a solution which is an evolution family $S(t,s)$ on $\mathbb{L}^2$ depending strongly continuously on $0 \le s \le t \le T$ such that

$$S(t,\tau)y_0 = e^{(t-\tau)A_0}y_0 - \int_\tau^t e^{(t-s)A_0}(a(s,\cdot), b(s,\cdot))S(s,\tau)y_0\, ds$$

Proposition

Let $f \in L^2(\Omega_T)$, $g \in L^2(\Gamma_T)$ and $(y_0, y_{0,\Gamma}) \in \mathbb{L}^2$.

(a) *There is a unique mild solution $y \in C([0,T];\mathbb{L}^2)$ of (1). Moreover, y belongs to $\mathbb{E}_1(\tau, T) := H^1(\tau, T; \mathbb{L}^2) \cap L^2(\tau, T; D(A_0))$ and solves (1) strongly on (τ, T) with initial $y(\tau)$, for all $\tau \in (0,T)$ and it is given by*

$$y(t) = S(t,0)y_0 + \int_0^t S(t,s)(f(s), g(s))\, ds, \qquad t \in [0,T],$$

(b) *If $y_0 \in \mathbb{H}^1$, then the mild solution y of (1) is the strong one, i.e., $y \in \mathbb{E}_1 := H^1(0,T;\mathbb{L}^2) \cap L^2(0,T;D(A_0))$ and solves (1) strongly on $(0,T)$ with initial data y_0.*

$$
\begin{aligned}
\partial_t y - d\Delta y + a(t,x)y &= v(t,x)1_\omega && \text{in } \Omega_T,\\
\partial_t y - \delta\Delta_\Gamma y + d\partial_\nu y + b(t,x)y &= 0 && \text{on } \Gamma_T, \qquad (2)\\
y(0,\cdot) &= y_0 && \text{in } \overline{\Omega},
\end{aligned}
$$

Definition

The system (2) is said to be null controllable at time $T > 0$ if for all given $y_0 \in L^2(\Omega)$ and $y_{0,\Gamma} \in L^2(\Gamma)$ we can find a control $v \in L^2((0,T) \times \omega)$ such that the solution satisfies

$$y(T,\cdot) = y_\Gamma(T,\cdot) = 0.$$

Some References

Static boundary conditions : Dirichlet, Neumann, Mixed boundary conditions (Robin or Fourier)

-Lebeau-Robbiano
- Fursikov-Imanuvilov

- Albano, Cannarsa, Zuazua, Yamamoto, Zhang, Guerrero, Fernandez-Cara, Puel, Benabdellah, Dermenjian, Le Rousseau, Ammar-Khodja, Gonzalez-Burgos, ...

Dynamic boundary conditions :

1. I.I. Vrabie, the approximate controllability : ($\omega = \Omega$).
2. D. Höomberg, K. Krumbiegel, J. Rehberg, Optimal Control : ($\omega = \Omega$.)
3. G. Nikel and Kumpf, Approximate controllability : (one-dimension heat equation with control at the boundary).

The solution of the linear system

$$\partial_t y - d\Delta y + a(t,x)y = v(t,x)1_\omega \qquad \text{in } \Omega_T, \qquad (3)$$

$$\partial_t y - \delta\Delta_\Gamma y + d\partial_\nu y + b(t,x)y = 0 \qquad \text{on } \Gamma_T, \qquad (4)$$

$$y(0,\cdot) = y_0 \qquad \text{in } \overline{\Omega}, \qquad (5)$$

can be written as

$$(y(T,\cdot), y_\Gamma(T,\cdot)) = S(T,0)y_0 + \mathcal{T}v,$$

$$\mathcal{T}v = \int_0^T S(T,\tau)(1_\omega v(\tau), 0)\, d\tau.$$

Null controllability $\iff R(S(T,0)) \subset R(\mathcal{T})$

$$\iff \exists C : \|S(T,0)^*\varphi_T\|_{\mathbb{L}^2} \le C\|\mathcal{T}^*\varphi_T\|_{\mathbb{L}^2}, \quad \varphi_T \in \mathbb{L}^2.$$

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Lemma

1. The function $\varphi(t) = S(T,t)^*\varphi_T$ *is the solution of the backward adjoint system*

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= 0 && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= 0 && \text{on } \Gamma_T \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned}$$

2. The adjoint of the operator $\mathcal{T}$ *is given by*

$$\mathcal{T}^*\varphi_T = \chi_\omega\varphi.$$

3. The estimate (3) *can be written as (Observability Ineq.)*

$$\|\varphi(0,\cdot)\|_{L^2}^2 + \|\varphi_\Gamma(0,\cdot)\|_{L^2}^2 \le C\int_0^T\int_\omega |\varphi|^2\,dt\,dx.$$

To show the above observabity inequality, we show first a Carleman estimate for the backward adjoint linear problem

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= f(t,x) && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= g(t,x) && \text{on } \Gamma_T \quad (6) \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned}$$

for given φ_T in $H^1(\Omega)$ or in $L^2(\Omega)$, $f \in L^2(\Omega_T)$ and $g \in L^2(\Gamma_T)$.

Lemma

Given a nonempty open set $\omega \Subset \Omega$, there is a function $\eta^0 \in C^2(\overline{\Omega})$ such that

$$\eta^0 > 0 \quad \text{in } \Omega, \qquad \eta^0 = 0 \quad \text{on } \Gamma, \qquad |\nabla \eta^0| > 0 \quad \text{in } \overline{\Omega \backslash \omega}.$$

Take $\lambda, m > 1$ and η^0 with respect to ω as in the lemma. We define the weight functions α and ξ by

$$\alpha(x,t) = (t(T-t))^{-1}\left(e^{2\lambda m \|\eta^0\|_\infty} - e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}\right), \quad x \in \overline{\Omega}$$
$$\xi(x,t) = (t(T-t))^{-1} e^{\lambda(m\|\eta^0\|_\infty + \eta^0(x))}, \quad x \in \overline{\Omega}.$$

Theorem

There are constants $C > 0$ and $\lambda_1, s_1 \geq 1$ such that, $\forall \lambda \geq \lambda_1, s \geq s_1$ and every mild solution φ of (6), we have

$$
\begin{aligned}
& s\lambda^2 \int_{\Omega_T} e^{-2s\alpha} \xi |\nabla \varphi|^2 \, dx \, dt + s^3 \lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt \\
& + s\lambda \int_{\Gamma_T} e^{-2s\alpha} \xi |\nabla_\Gamma \varphi_\Gamma|^2 + s^3 \lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\varphi_\Gamma|^2 \, dS \, dt \\
& + s\lambda \int_{\Gamma_T} e^{-2s\alpha} \xi |\partial_\nu \varphi|^2 \, dS \, dt \\
\leq \quad & C s^3 \lambda^4 \int_0^T \int_\omega e^{-2s\alpha} \xi^3 |\varphi|^2 \, dx \, dt \\
& + C \int_{\Omega_T} e^{-2s\alpha} |f|^2 \, dx \, dt + C \int_{\Gamma_T} e^{-2s\alpha} |g|^2 \, dS \, dt.
\end{aligned}
$$

Lemma

For $f = g = 0$, we obtain the following fundamental estimates

$$\begin{aligned}\int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Omega} |\varphi(t,x)|^2 \, dx \, dt + \int_{\frac{T}{4}}^{\frac{3T}{4}} \int_{\Gamma} |\varphi_\Gamma(t,x)|^2 \, dS \, dt \\ \leq \quad C \int_0^T \int_\omega |\varphi(t,x)|^2 \, dx \, dt\end{aligned}$$

and

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} + \|\varphi_\Gamma(0,\cdot)\|^2_{L^2(\Gamma)} \leq C \|\varphi(t,\cdot)\|^2_{\mathbb{L}^2}, \quad 0 \leq t \leq T.$$

Proposition

Let $T > 0$, a nonempty open set $\omega \Subset \Omega$ and $a \in L^\infty(\Omega_T)$ and $b \in L^\infty(\Gamma_T)$. Then there is a constant $C > 0$ (depending on $\Omega, \omega, \|a\|_\infty, \|b\|_\infty$) such that

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} + \|\varphi_\Gamma(0,\cdot)\|^2_{L^2(\Gamma)} \leq C \int_0^T \int_\omega |\varphi|^2 \, dx\, dt$$

for every mild solution φ of the homogeneous backward problem

$$\begin{aligned} -\partial_t\varphi - d\Delta\varphi + a(t,x)\varphi &= 0 && \text{in } \Omega_T, \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi + b(t,x)\varphi_\Gamma &= 0 && \text{on } \Gamma_T \\ \varphi(T,\cdot) &= \varphi_T && \text{in } \overline{\Omega}, \end{aligned}$$

Theorem

Let $T > 0$ and coefficients $d, \delta > 0$, $a \in L^\infty(\Omega_T)$ and $b \in L^\infty(\Gamma_T)$ be given. Then for each nonempty open set $\omega \Subset \Omega$ and for all data $y_0, y_{0,\Gamma}$, there is a control $v \in L^2((0,T) \times \omega)$ such that the mild solution y of (3)–(5) satisfies $y(T, \cdot) = y_\Gamma(T, \cdot) = 0$.

L. Maniar, M. Meyries, R. Schnaubelt, Null controllability for parabolic problems with dynamic boundary conditions of reactive-diffusive type, Evol. Equat. and Cont. Theo. 6 (2017), 381-407.

We consider now the controllability of a dynamic boundary Parabolic equation with drift terms

$$\begin{cases} \partial_t y - d\Delta y + C(x).\nabla y + a(x)y = v1_{\omega} & \text{in } \Omega_T \\ \partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d\partial_\nu y + D(x).\nabla_\Gamma y_\Gamma + b(x)y_\Gamma = 0 & \text{on } \Gamma_T, \\ y_{|\Gamma}(t;x) = y_\Gamma(t;x) & \text{on } \Gamma_T, \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) & \text{in } \Omega \times \Gamma, \end{cases} \tag{7}$$

$a \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma)$, $C \in L^\infty(\Omega)^N$ and $D \in L^\infty(\Gamma)^N$.

Fursikov-Immanuvilov, Fernandez-Cara, Guerrero, Gonzalez-Burgos, Puel : Dirichlet and Fourier boundary conditions.

For the wellposedness of the above parabilic equation, consider the operators

$$A_0 = \begin{pmatrix} d\Delta & 0 \\ -d\partial_\nu & \delta\Delta_\Gamma \end{pmatrix}, \quad D(A_0) = \mathbb{H}^2,$$

$$B = \begin{pmatrix} -C.\nabla - a & 0 \\ 0 & -D.\nabla_\Gamma - b \end{pmatrix}, \quad D(B_1) = \mathbb{H}^1,$$

Proposition

1. B is an A_0-bounded operator, with null A_0-bound.
That is
(i)

$$\begin{cases} D(A_0) \subset D(B), \\ \exists \alpha, \beta \in \mathbb{R}_+, \forall (f,g) \in D(A_0), \\ \|B(f,g)\|_{\mathbb{L}^2} \leq \alpha \|A_0(f,g)\|_{\mathbb{L}^2} + \beta \|(f,g)\|_{\mathbb{L}^2}, \end{cases} \tag{8}$$

(ii) $\inf\{\alpha \geq 0 :$ *there exists* $\beta \in \mathbb{R}_+,$ *such that* (8) *holds* $\} = 0.$
2. The operator $A = A_0 + B$ generates an analytic semigroup.
3. The interpolation result $(\mathbb{L}^2, \mathbb{H}^2)_{1/2,2} = \mathbb{H}^1$ holds.
4. The equation (7) is wellposed and has maximal regularity.

For the controllability of the dynamic boundary parabolic equation with drift terms (7), we need to consider its backward adjoint equation

$$\begin{cases} -\partial_t\psi - d\Delta\psi + a\psi - div(\psi C(x)) = f, \\ -\partial_t\psi_\Gamma - \delta\Delta_\Gamma\psi_\Gamma + d\partial_\nu\psi + b\psi_\Gamma + \psi C.\nu - div_\Gamma(\psi_\Gamma D(x)) = g \\ \psi(T,\cdot) = \psi_T \quad \text{on } \Omega, \\ \psi_\Gamma(T,\cdot) = \psi_{T,\Gamma}, \end{cases} \tag{9}$$

$a \in L^\infty(\Omega)$, $b \in L^\infty(\Gamma)$, $C \in L^\infty(\Omega)^N$ and $D \in L^\infty(\Gamma)^N$.

- $\mathbb{H}^{-k}$ the topological dual of $\mathbb{H}^k$, with pivot space $\mathbb{L}^2$.
- $\mathbb{X}_1 := \mathbb{H}^2$, equipped with the graph norm $\|.\|_{A_0}$.
- $\mathbb{X}_{-1}$ the dual space of $\mathbb{X}_1$ with pivot space $\mathbb{L}^2$.
- $\mathbb{X}_{-1}$ is the completed space of $\mathbb{L}^2$ for the norm

$$\|\cdot\|_{\lambda,-1} = \|(\lambda - A_0)^{-1}\cdot\|_{\mathbb{L}^2}.$$

$$\mathbb{H}^2 \hookrightarrow \mathbb{H}^1 \hookrightarrow \mathbb{L}^2 \hookrightarrow \mathbb{H}^{-1} \hookrightarrow \mathbb{X}_{-1} \hookrightarrow \mathbb{H}^{-2}.$$

Let $A_{0,-1} : \mathbb{L}^2 \longrightarrow \mathbb{X}_{-1}$ be the adjoint operator of A_0

(in the sense of duality) with pivot space $\mathbb{L}^2$.

$A_{0,-1}$ is also the unique continuous extension of A_0 to an operator from $\mathbb{L}^2$ to $\mathbb{X}_{-1}$.

$A_{0,-1}$ is called the extrapolated operator of A_0.

$A_{0,-1}$ generates an analytic semigroup $(S_{0,-1}(t))_{t\geq 0}$ on $\mathbb{X}_{-1}$

where $S_{0,-1}(t)$ is just the extension to $\mathbb{X}_{-1}$ of $S_0(t)$ (the semigroup generated by A_0).

Let the perturbation operator

$$B(u, u_\Gamma) = \begin{bmatrix} div(u.C) - au \\ -uC.\nu + div_\Gamma(u_\Gamma.D) - bu_\Gamma \end{bmatrix},$$

where for $F \in L^2(\Omega)^N$ and $F_\Gamma \in L^2(\Gamma)^N$,

$$div(F) : H^1(\Omega) \to \mathbb{R}, v \mapsto -\int_\Omega F.\nabla v dx + \langle F.\nu, v_{|\Gamma} \rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)}.$$

$$div_\Gamma(F_\Gamma) : H^1(\Gamma) \to \mathbb{R}, v_\Gamma \longmapsto -\int_\Gamma F_\Gamma.\nabla_\Gamma v_\Gamma d\sigma.$$

Theorem

1. B is a bounded linear operator from $\mathbb{L}^2$ to $(\mathbb{X}_{-1}, \mathbb{L}^2)_{1/2,2} = \mathbb{H}^{-1}$.

2. The operator $(A_{0,-1} + B)_{|\mathbb{L}^2}$ with domain $D((A_{0,-1} + B)_{|\mathbb{L}^2}) = \{U \in \mathbb{L}^2 : A_{0,-1}U + BU \in \mathbb{L}^2\}$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on $\mathbb{L}^2$, given by the variation of parameters formula

$$T(t) = S_0(t) + \int_0^t S_{0,-1}(t-r)B_{-1}T(r)dr.$$

3. The adjoint equation (9) is wellposed.

A. Khoutaibi, L. Maniar, D. Mungolo and A. Rhandi.

For the null controllability, we establish a Carleman estimate for the backward adjoint equation

$$\begin{cases} -\partial_t\varphi - d\Delta\varphi = -a(x)\varphi + div(\varphi.C(x)) \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi = -b(x)\varphi_\Gamma - \varphi C.\nu + div_\Gamma(\varphi_\Gamma.D(x)) \\ \varphi(T,\cdot) = \varphi_T \in L^2(\Omega) \\ \varphi_\Gamma(T,\cdot) = \varphi_{T,\Gamma} \in L^2(\Gamma). \end{cases}$$

For this, consider the intermediate system, for $F \in L^2(\Omega_T)$ and $F_\Gamma \in L^2(\Gamma_T)$.

$$\begin{cases} -\partial_t\varphi - d\Delta\varphi = F_0 + div(F) \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi_{|\Gamma} = F_{0,\Gamma} - F\nu + div_\Gamma(F_\Gamma) \end{cases} \tag{10}$$

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Lemma

$\exists \lambda_1 \geqslant 1$, $\sigma > 0$ et $C > 0$ s.t, $\forall \lambda \geq \lambda_1$ $\forall s \geq s_0 = \sigma(T + T^2)$ such that for all $\Phi = (\varphi, \varphi_\Gamma) \in L^2(0, T; \mathbb{H}^1) \cap H^1(0, T; \mathbb{L}^2)$ solution of (10)

$$s\lambda^2 \int_{\Omega_T} \xi e^{-2s\alpha} |\nabla \varphi|^2 dtdx + s^3\lambda^4 \int_{\Omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 dtdx$$
$$+ s\lambda^2 s \int_{\Gamma_T} \xi e^{-2s\alpha} |\nabla_\Gamma \varphi_\Gamma|^2 dtd\sigma + s^3\lambda^3 \int_{\Gamma_T} e^{-2s\alpha} \xi^3 |\varphi_\Gamma|^2 dtd\sigma.$$
$$\leq s^3\lambda^4 \int_{\omega_T} e^{-2s\alpha} \xi^3 |\varphi|^2 dtdx + \int_{\Omega_T} e^{-2s\alpha} |F_0|^2 dtdx$$
$$+ s^2\lambda^2 \int_{\Omega_T} e^{-2s\alpha} \xi^2 |F|^2 dtdx + \int_{\Gamma_T} e^{-2s\alpha} |F_{0,\Gamma}|^2 dtd\sigma$$
$$+ s^2\lambda^2 \int_{\Gamma_T} e^{-2s\alpha} \xi^2 |F_\Gamma|^2 dtd\sigma.$$

Come back to the adjoint parabolic equation

$$\begin{cases} -\partial_t\varphi - d\Delta\varphi = -a(x)\varphi + div(\varphi.C(x)) \\ -\partial_t\varphi_\Gamma - \delta\Delta_\Gamma\varphi_\Gamma + d\partial_\nu\varphi = -b(x)\varphi_\Gamma - \varphi C.\nu + div_\Gamma(\varphi_\Gamma.D(x)) \\ \varphi(T,\cdot) = \varphi_T \in L^2(\Omega) \\ \varphi_\Gamma(T,\cdot) = \varphi_{T,\Gamma} \in L^2(\Gamma) \end{cases}$$

With $F_0 = -a\varphi$, $F = \varphi C$, $F_{0,\Gamma} = -b\varphi_\Gamma$, $F_\Gamma = \varphi_\Gamma D$, we obtain the following Carleman estimate for the adjoint problem

$$s\lambda^2\int_{\Omega_T}\xi e^{-2s\alpha}|\nabla\varphi|^2 dtdx + s^3\lambda^4\int_{\Omega_T} e^{-2s\alpha}\xi^3|\varphi|^2 dtdx$$

$$+ s\lambda^2 s\int_{\Gamma_T}\xi e^{-2s\alpha}|\nabla_\Gamma\varphi_\Gamma|^2 dtd\sigma + s^3\lambda^3\int_{\Gamma_T} e^{-2s\alpha}\xi^3|\varphi_\Gamma|^2 dtd\sigma.$$

$$\leq s^3\lambda^4\int_{\omega_T} e^{-2s\alpha}\xi^3|\varphi|^2 dtdx.$$

Theorem

1. *There exists a constant $C_T > 0$ s.t the unique mild solution φ to the backward system (9) satisfies observability inequality*

$$\|\varphi(0,\cdot)\|^2_{L^2(\Omega)} + \|\varphi_\Gamma(0,\cdot)\|^2_{L^2(\Gamma)} \leq C_T \int_{\omega_T} |\varphi|^2 dtdx.$$

$$Log(C_T) = C(1 + 1/T + \|c\|_\infty^{2/3} + \|\ell\|_\infty^{2/3}) \\ + CT(\|B\|^2_\infty + \|b\|^2_\infty + \|c\|_\infty + \|B\|_\infty + \|\ell\|_\infty + \|b\|_\infty).$$

2. *The parabolic system with dynamic boundary conditions and drift terms is nul controllable.*

A. Khoutaibi and L. Maniar : Null controllability for a heat equation with dynamic boundary condition and drift terms, **JEEC, accepted.**

$$\partial_t y - d\Delta y + F(y, \nabla y) = v1_\omega, \quad \text{in } \Omega_T, \tag{11}$$
$$\partial_t y_\Gamma - \delta\Delta_\Gamma y_\Gamma + d(\partial_\nu y)|_\Gamma + G(y_\Gamma, \nabla_\Gamma y_\Gamma) = 0, \quad \text{on } \Gamma_T, \tag{12}$$
$$y(0,\cdot) = y_0, y_\Gamma(0,\cdot) = y_{0,\Gamma}. \tag{13}$$

Theorem

Let $T > 0$, $\omega \Subset \Omega$ be open and nonempty, and $F, G \in C^1(\mathbb{R} \times \mathbb{R}.)$ satisfy

$$F(0,0) = G(0,0) = 0 \quad \textit{and} \quad |F(s,\xi)|+|G(s,\xi)| \leq C(1+|s|+|\xi|).$$

Then for all $y_0 \in \mathbb{H}^1$, there is $v \in L^2(\omega_T)$ such that (11)–(13) has a unique strong solution $y \in \mathbb{E}_1$ with $y(T,\cdot) = y_\Gamma(T,\cdot) = 0$.

– **Blowing up semilinear dynamic boundary equations.** Under preparation.

$F, G : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz-continuous, $F(0,0) = G(0,0) = 0$ and

$$\lim_{|(s,p)|\to+\infty} \frac{|F_1(s,\xi)| + |G_1(s,\xi)|}{(\ln(1+|s|+|\xi|)^{\frac{3}{2}}} = 0, \quad \lim_{|(s,p)|\to+\infty} \frac{|F_2(s,\xi)| + |G_2(s,\xi)|}{\ln(1+|s|+|\xi|)^{\frac{1}{2}}} = 0$$

Fernandez-Cara, Gonzalez-Burgos, Guerrero, Zuazua.

– The presence of Lapalce Beltrami in the second equation ($\delta \neq 0$) was necessary to get rid of bad boundary terms.

– In the case $\delta = 0$?

$$\begin{cases} \partial_t y - d\Delta y + C(x).\nabla y + a(t,x)y = v1_\omega & \text{in } \Omega_T \\ \partial_t y_\Gamma + d\partial_\nu y + D(x).\nabla_\Gamma y_\Gamma + b(t,x)y_\Gamma = 0 & \text{on } \Gamma_T, \\ y_{|\Gamma}(t,x) = y_\Gamma(t,x) & \text{on } \Gamma_T, \\ (y, y_\Gamma)|_{t=0} = (y_0, y_{0,\Gamma}) & \text{in } \Omega \times \Gamma, \end{cases}$$

– $N = 1$.

– Cornilleau-Guerrero : Boundary Carleman estimate.

– With F. Ammar-Khodja : Internal Carleman estimate.

– $N \geq 2$: ? ? Could one obtain a **uniform Carleman estimate** with respect δ and tends δ to 0.

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Thank you for your attention