The role of state constraints in turnpike phenomena for LQ problems

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based on joint works with Lars Grüne, University of Bayreuth

VIII Partial differential equations, optimal design and numerics

Benasque, August 27, 2019

Optimal control problem

Given $N > 0$ optimization horizon, minimize the cost functional

$$J_N(x,u) = \sum_{k=0}^{N-1} \ell(x(k), u(k))$$

over $u(k) \in U$, subject to the discrete-time control system

$$x(k+1) = f(x(k), u(k)), \quad k = 0, \ldots, N$$

with state and input constraints $x(k) \in \mathbb{X}$, $u(k) \in \mathbb{U}$

initial condition $x(0) = x_0 \in X$

$\ell : X \times U \to \mathbb{R}$ is a running cost,

f solution operator (ODE, PDE, or their numerical approx)

Brief notation: $x^+ = f(x,u)$

Outline

Turnpike property

Strict dissipativity

LQ problem

Turnpike - Interpretation

The turnpike property describes a behaviour of optimal trajectories for a finite horizon optimal control problem

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"There is a fastest route between any two points; and if origin and destination are close together and far from the turnpike, the best route may not touch the turnpike. But if origin and destination are far enough apart, it will always pay to get on to the turnpike and cover distance at the best rate of travel, even if this means adding a little mileage at either end."

Dorfman, Samuelson and Solow,
Linear Programming and Economic Analysis (1958)

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Informally: any optimal trajectory stays *near a steady-state* (x^e, u^e) most of the time

Turnpike - Example 1

Keep the state of the system

$$x^+ = 2x + u$$

inside the given interval $\mathbb{X} = [-2; 2]$ minimising the control effort

$$\ell(x; u) = u^2$$

with input constraints $\mathbb{U} = [-3; 3]$

Rmk: the closer the state is to $x^e = 0$, the cheaper it is to keep the system inside $\mathbb{X}$

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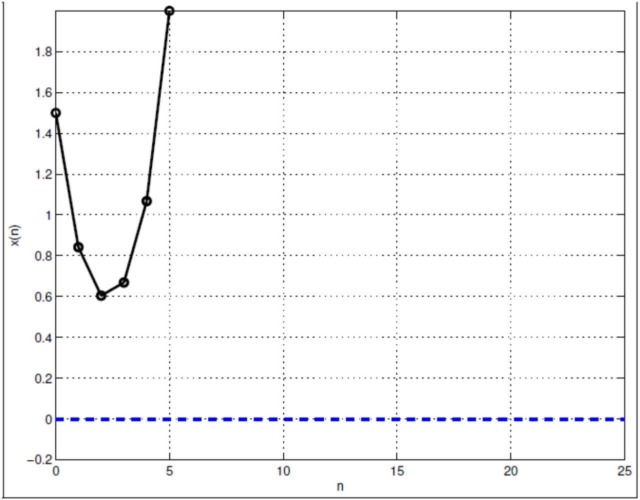
Rmk: the closer the state is to $x^e = 0$, the cheaper it is to keep the system inside $\mathbb{X}$

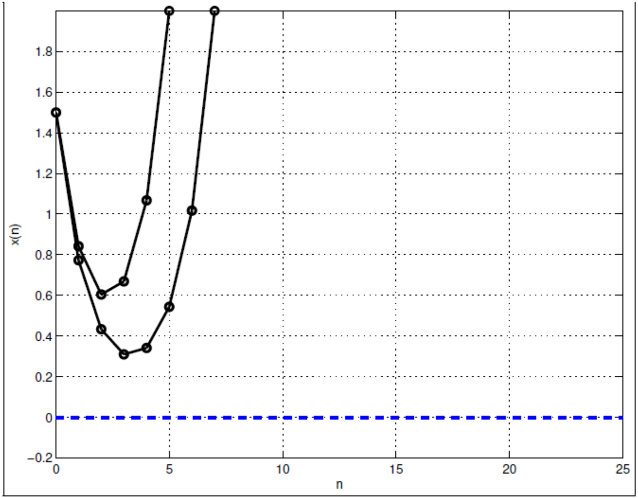
$\Longrightarrow$ optimal trajectories should stay near $x^e = 0$

Example 1 - Optimal trajectories



Optimal trajectory for $N = 5$

Example 1 - Optimal trajectories



Optimal trajectories for $N = 5, \ldots, 7$

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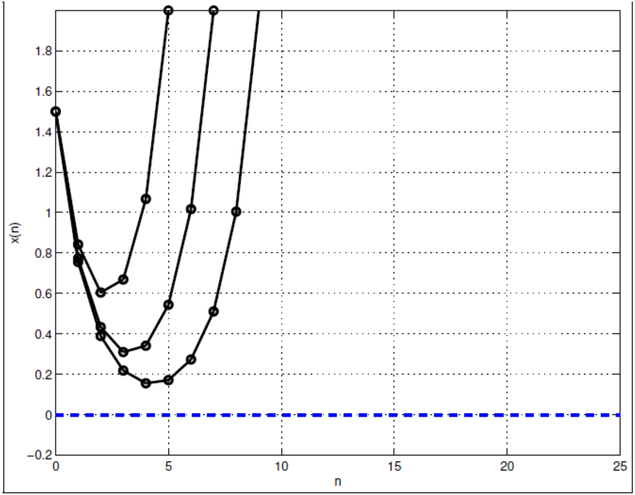

Optimal trajectories for $N = 5, \ldots, 9$

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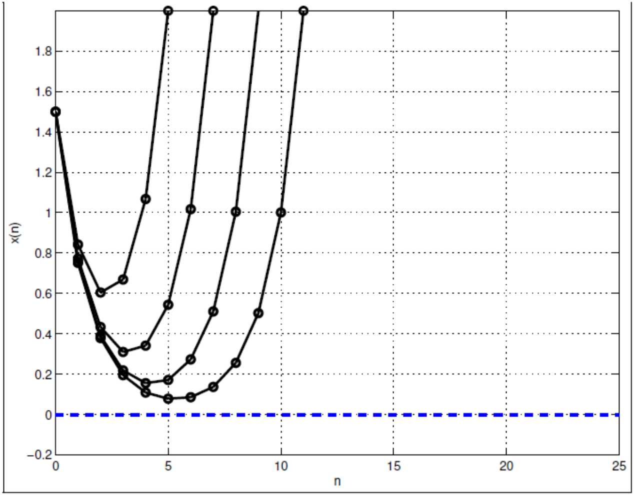

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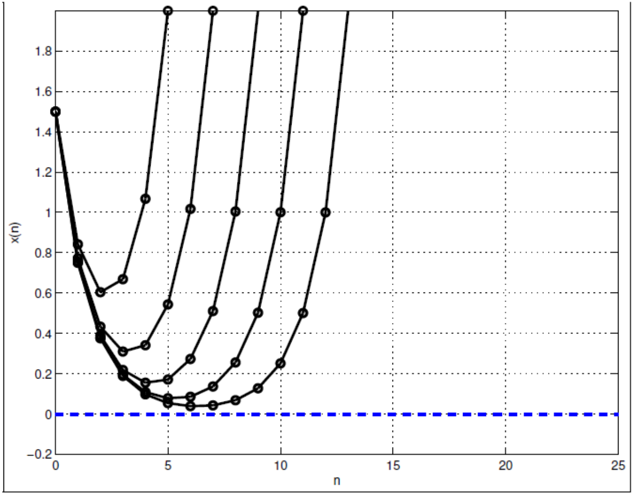

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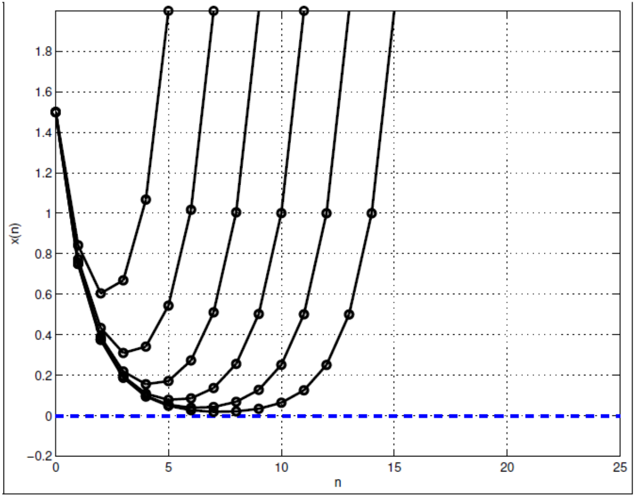

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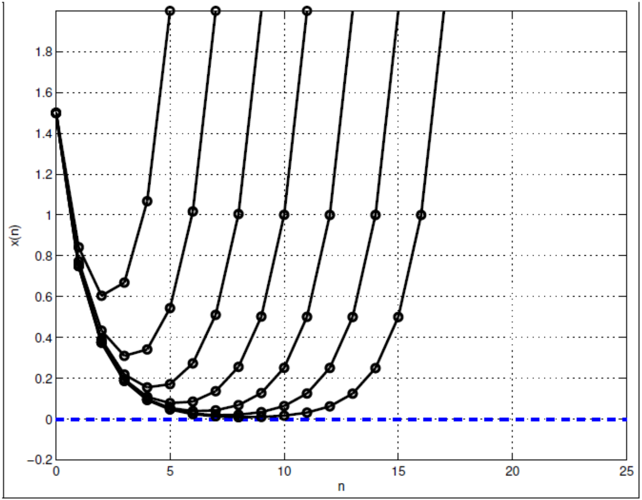

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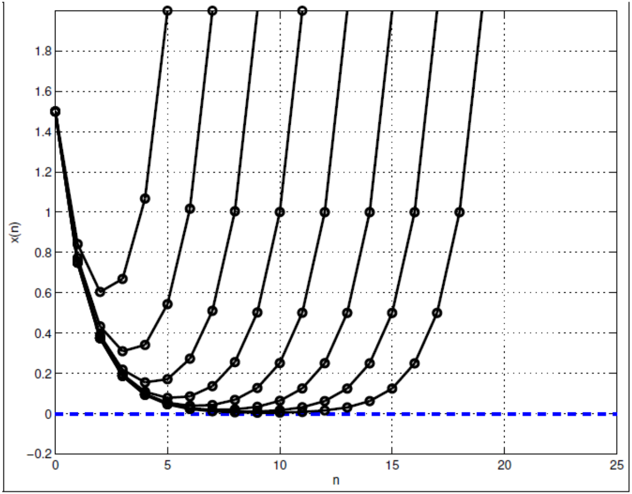

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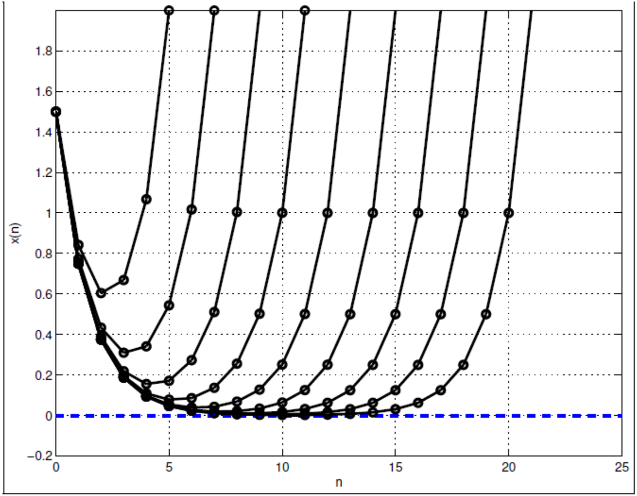

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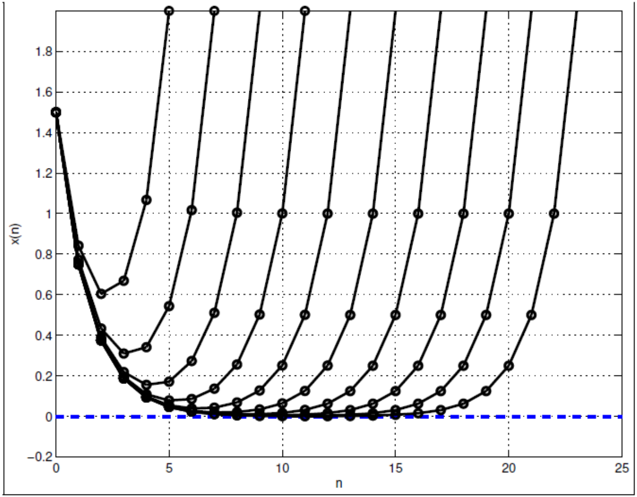

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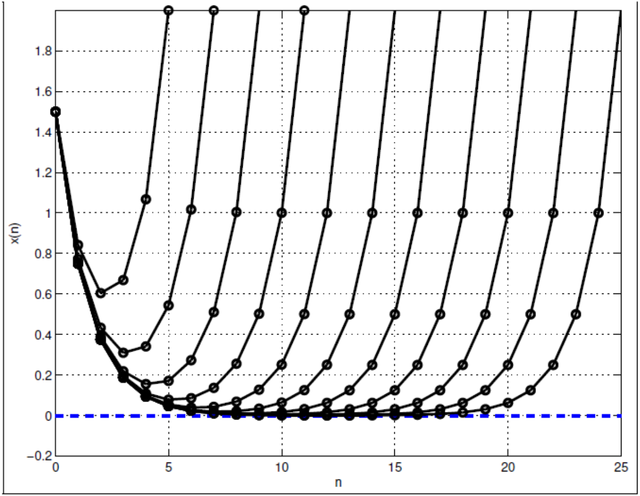

Optimal trajectories for $N = 5, \ldots, 25$

Turnpike - Example 2

Consider the $1D$ macroeconomic model by [Brock-Mirman '72]
Minimize the cost functional J with running cost

$$\ell(x,u) = -\ln(Ax^{\alpha} - u)\,, \quad A = 5\,,\ \alpha = 0.34$$

with dynamics

$$x^{+} = u$$

with input and state constraints $\mathbb{X} = \mathbb{U} = [0, 10]$

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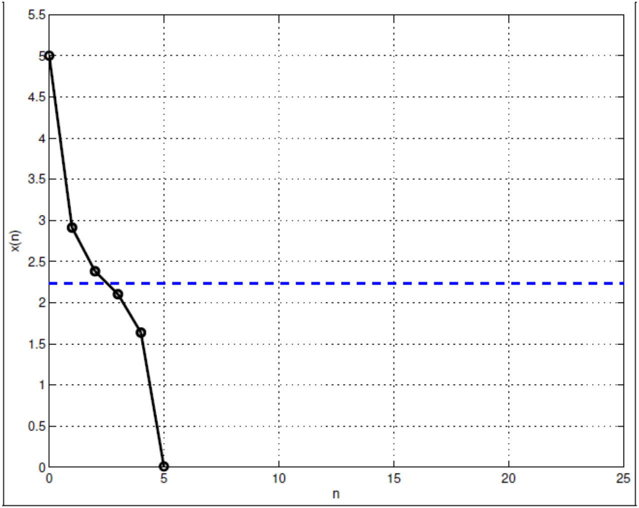
Here the optimal trajectories are less obvious
On infinite horizon, it is optimal to stay at the equilibrium

$$x^e \approx 2.2344 \quad \text{with } \ell(x^e;u^e) \approx 1.4673$$

One may thus expect that finite horizon optimal trajectories also stay for a long time near that equilibrium

Example 2 - Optimal trajectories



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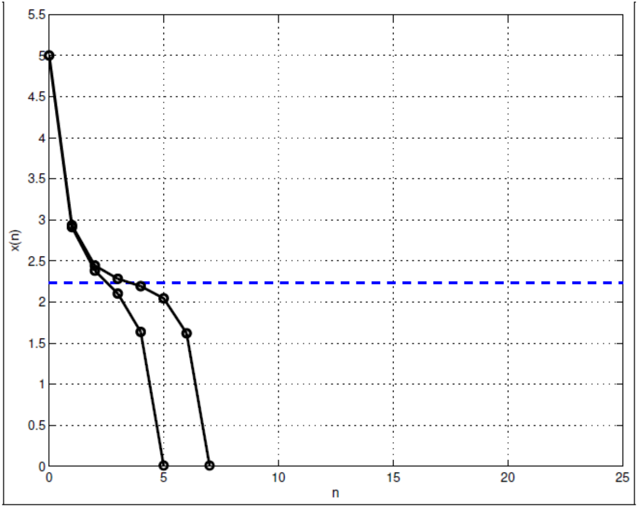

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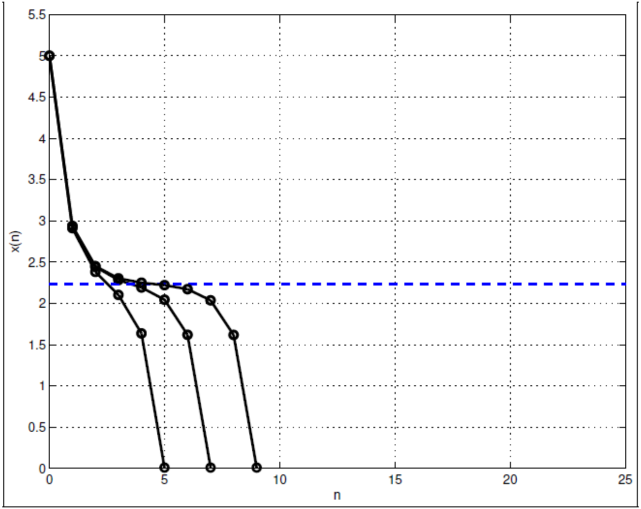

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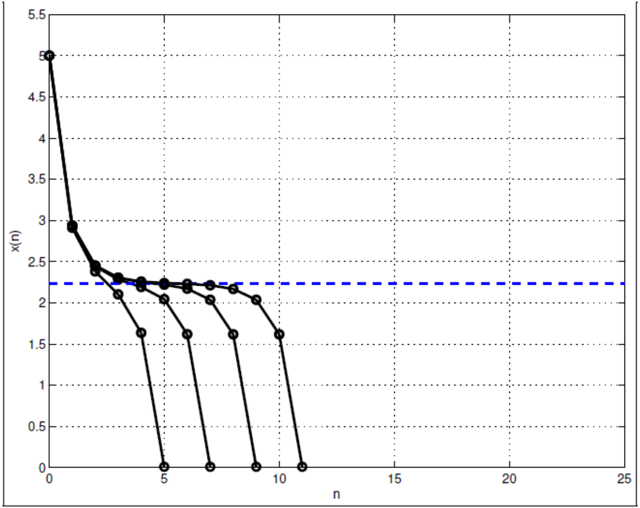

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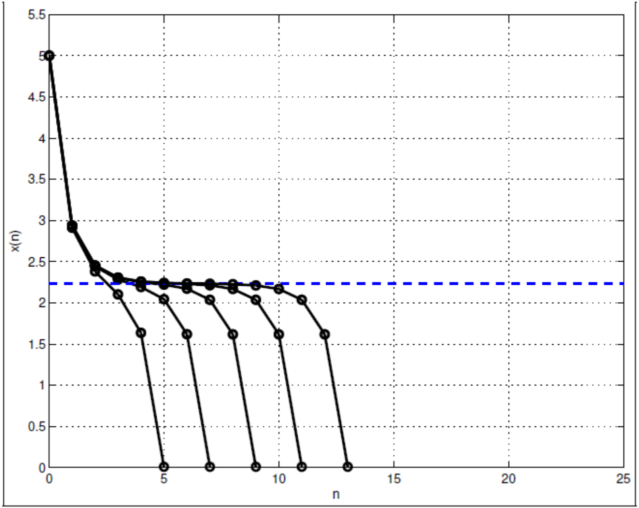

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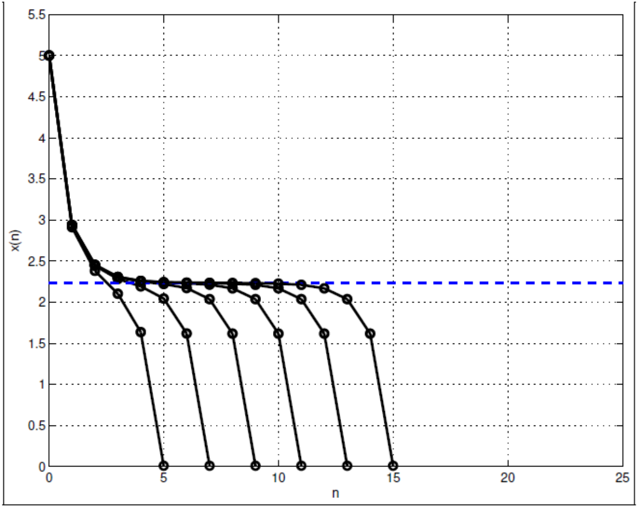

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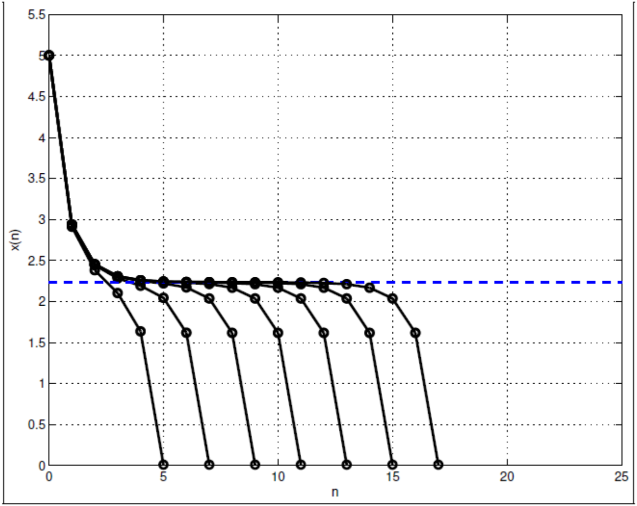

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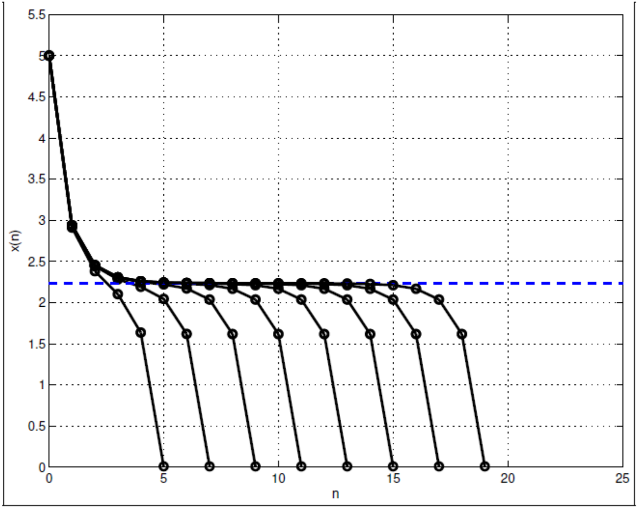

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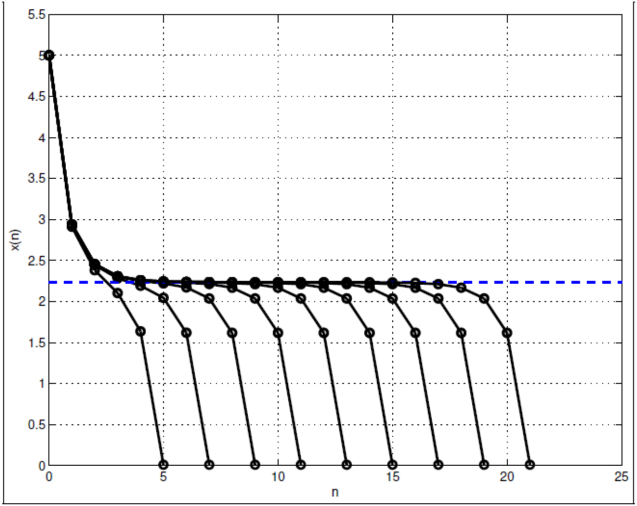

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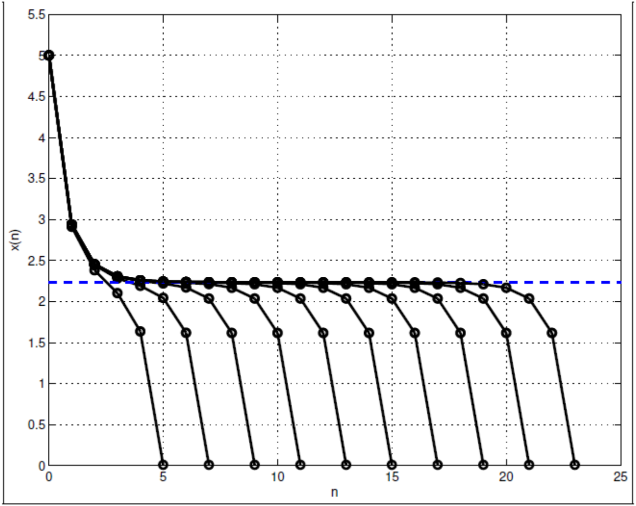

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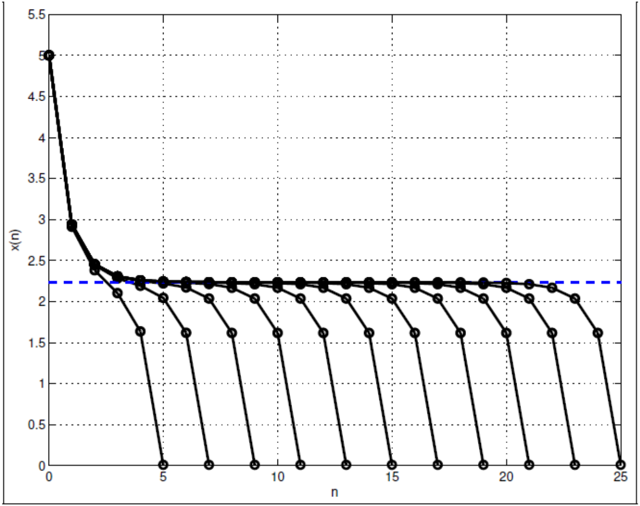

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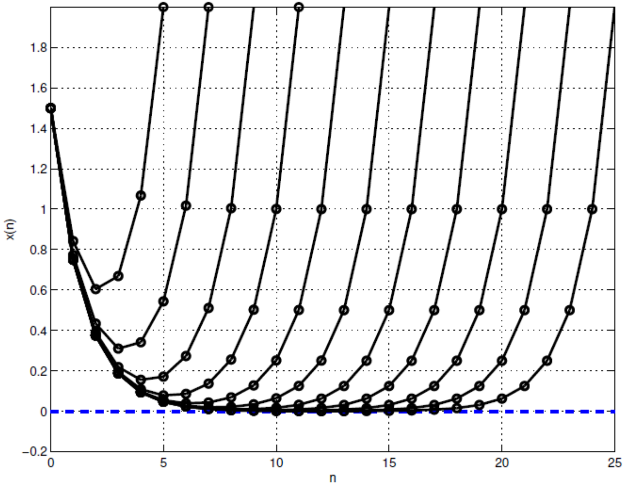
Turnpike - Essential feature



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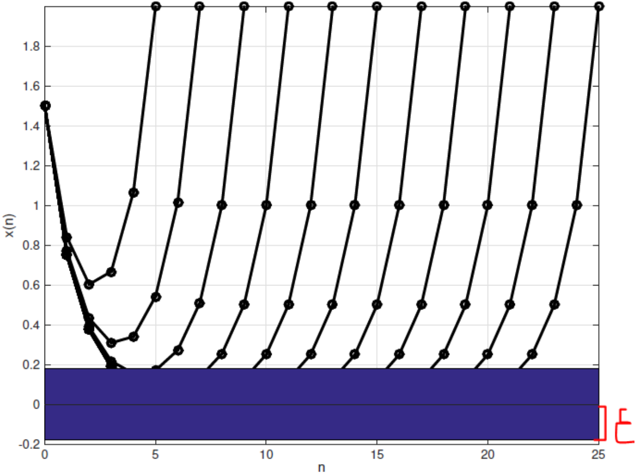

The number of points outside the blue neighborhood is bounded indipendently of the optimization horizon N

Definition of Turnpike

$(x^e, u^e) \in \mathbb{X} \times \mathbb{U}$ equilibrium for the control system, i.e., $f(x^e, u^e) = x^e$.

Turnpike property at (x^e, u^e) if for each $\varepsilon > 0$ and $\rho > 0$ there exists $C_{\rho,\varepsilon} > 0$ s.t. for all $N \in \mathbb{N}$ all optimal trajectories $x^\star$ starting in $B_\rho(x^e)$ satisfy

$$\#\left\{k \in \{0, \ldots, N\} \,\middle|\, \|x^\star(k) - x^e\| > \varepsilon\right\} \leq C_{\rho,\varepsilon}\,.$$

Near-equilibrium turnpike property at (x^e, u^e) if for each $\rho > 0$, $\varepsilon > 0$ and $\delta > 0$ there exists $C_{\rho,\varepsilon,\delta} > 0$ s.t. for all $x \in B_\rho(x^e)$ and all $N \in \mathbb{N}$, all trajectories $x_u(\cdot, x)$ with $J_N(x, u) \leq N\ell(x^e, u^e) + \delta$ for some $u \in \mathbb{U}$, satisfy the inequality

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$$\#\big\{k \in \{0, \dots, N\} \,\big|\, \|x_u(k, x) - x^e\| > \varepsilon\big\} \leq C_{\rho,\varepsilon,\delta}\,.$$

In words, the optimal/near equilibrium trajectories stay in an ε-neighbourhood of x^e for all but finitely many “exceptional” time instants whose number is bounded independently of the optimization horizon N.

Turnpike at (x^e, u^e) - References

First described in mathematical economy in the context of optimal growth strategies

Ramsey (1928), *A Mathematical Theory of Saving*

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Recent interest from the mathematical control community

Zaslavski (2014)

for ODEs, continuous time: Porretta, Zuazua (2013), Trélat, Zuazua (2015), Faulwasser et al. (2015), Grüne, RG (2019), ...

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Further works for turnpike towards non equilibrium points

Zanon et al (2017), Grüne, Pickelmann (2017), Berberich et al (2018), Trélat, Zhang, Zuazua (2019), ...

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Outline

Dissipativity - Main notions

$$x^{+} = f(x, u)$$

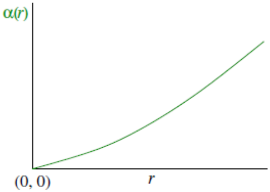
Introduce functions $s : X \times U \to \mathbb{R}$ and $\lambda : X \to \mathbb{R}$, where

$s(x, u)$ supply rate, measuring the (possibly negative) amount of energy supplied to the system via the input u in the next time step

$\lambda(x)$ storage function, measuring the amount of energy stored inside the system when it is in state x

Strict Dissipativity - Definition

$\mathcal{K} := \{\alpha : \mathbb{R}_0^+ \to \mathbb{R}_0^+,$ continuous, strictly increasing, $\alpha(0) = 0\}$



The system $x^+ = f(x, u)$ is called strictly pre-dissipative if there are $x^e \in \mathbb{X}$, $\alpha \in \mathcal{K}$ s.t. for all $x \in \mathbb{X}$, $u \in \mathbb{U}$, the inequality

$$\lambda(x^+) \leq \lambda(x) + s(x, u) - \alpha(\|x - x^e\|)$$

holds.

The system is called strictly dissipative if it is strictly pre-dissipative with λ bounded from below

Dissipativity - Physical interpretation

$$\lambda(x^+) \leq \lambda(x) + s(x,u) - \alpha(\|x - x^e\|)$$

Physical interpretation:

$\lambda(x) =$ energy stored in the system in state x
$s(x,u) =$ energy supplied to the system

Strict dissipativity:

- energy can not be generated inside the system
- a certain amount of energy $\alpha(\|x - x^e\|)$ must be dissipated

Outline

LQ Optimal Control Problem

We consider linear quadratic finite dimensional discrete-time optimal control problems

$$\min_{u \in \mathbb{U}^N(x_0)} J_N(x_0, u)$$

$$J_N(x_0, u) := \sum_{k=0}^{N-1} x(k)^T Q x(k) + u(k)^T R u(k) + s^T x(k) + v^T u(k) + c\,,$$

$$x(k+1) = Ax(k) + Bu(k)\,, \quad x(0) = x_0\,,$$

where

$N \in \mathbb{N}$ optimization horizon, $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$

$s \in \mathbb{R}^n$, $v \in \mathbb{R}^m$, $c \in \mathbb{R}$, $x_0 \in \mathbb{R}^n$

A, B, Q and R matrices of appropriate dimension,

Q and R symmetric, $Q \geq 0$ and $R > 0$

LQ Optimal Control Problem /2

$$x(k+1) = Ax(k) + Bu(k)\,, \qquad x(0) = x_0\,, \tag{1}$$

$x(k) \in \mathbb{X}$ state of the system at time t_k, $\mathbb{X}$ state constraints

$u(k) \in \mathbb{U}$ control acting on the system from t_k to t_{k+1},
$\mathbb{U}$ input constraints

$x_u(k, x_0)$ solution to (1) corresponding to IC x_0 and control u

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$f(x, u) = Ax + Bu$ solution operator

$\ell(x, u) = x^T Qx + u^T Ru + s^T x + v^T u + c$ running cost

$J_N(x_0, u) := \sum_{k=0}^{N-1} \ell(x(k), u(k))$ minimized over

$$\mathbb{U}^N(x_0) := \{u \in \mathbb{U}^N \mid x_u(k, x_0) \in \mathbb{X} \text{ for all } k = 0, \ldots, N\}$$

Characterization of strict dissipativity for LQ problems

Let (x^e, u^e) be an equilibrium on $\mathbb{X} \times \mathbb{U}$

Lemma: (i) For LQ problems, the supply rate can be chosen as $s(x, u) = \ell(x, u) - \ell(x^e, u^e)$, and the storage function as

$$\lambda(x) = x^T P x + q^T x \,. \tag{2}$$

(ii) The LQ problem is strictly pre-dissipative, with a storage function λ of the form (2), if and only if the matrix inequality

$$Q + P - A^T P A > 0$$

is satisfied.
Moreover, if $P > 0$, then the LQ problem is strict dissipative.

(iii) The LQ problem is strictly dissipative if and only if $P > 0$ or $\mathbb{X}$ is bounded

Spectral criteria for solvability of the matrix inequality

Lemma: The matrix inequality

$$Q + P - A^T PA > 0$$

has a solution P if and only if all unobservable eigenvalues satisfy $|\mu| \neq 1$.

Moreover, the solution satisfies $P > 0$ if and only if all unobservable eigenvalues μ of A satisfy $|\mu| < 1$, i.e., iff (A, C) is detectable

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Rmk: $x^+ = Ax + Bu\,, \qquad \ell(x, u) = x^T Qx + u^T Ru + s^T x + v^T u$ with $Q = C^T C \geq 0$ and $R > 0$.

We call an eigenvalue μ of A unobservable if the corresponding eigenvector v satisfies $Cv = 0$

Main result without state constraints

Theorem: Consider the LQ problem with (A, B) stabilizable, $Q = C^T C$, constraint sets $\mathbb{X} = \mathbb{R}^n$ and $\mathbb{U} \subset \mathbb{R}^m$. Then the following properties are equivalent

(i) The problem is strictly dissipative at an equilibrium $(x^e, u^e) \in int(\mathbb{X} \times \mathbb{U})$

(ii) The problem has the turnpike property at an equilibrium $(x^e, u^e) \in int(\mathbb{X} \times \mathbb{U})$

(iii) The pair (A, C) is detectable, i.e., all unobservable eigenvalues μ of A satisfy $|\mu| < 1$.

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(iii) The pair (A, C) is detectable, i.e., all unobservable eigenvalues μ of A satisfy $|\mu| < 1$.

Moreover, if one of these properties holds, then the equilibria in (i) and (ii) coincide, and the *exponential* turnpike property holds.

Main result with state constraints

Theorem: Consider the LQ problem with $Q = C^T C$, constraint sets $\mathbb{X} \subset \mathbb{R}^n$ bounded and $\mathbb{U} \subset \mathbb{R}^m$. Then the following properties are equivalent

(i) The problem is strictly pre-dissipative at an equilibrium $(x^e, u^e) \in int(\mathbb{X} \times \mathbb{U})$

(ii) The problem has the near equilibrium turnpike property at an equilibrium $(x^e, u^e) \in int(\mathbb{X} \times \mathbb{U})$

(iii) All unobservable eigenvalues μ of A satisfy $|\mu| \neq 1$.

Main result with state constraints

Theorem: Consider the LQ problem with $Q = C^T C$, constraint sets $\mathbb{X} \subset \mathbb{R}^n$ bounded and $\mathbb{U} \subset \mathbb{R}^m$. Then the following properties are equivalent

(i) The problem is strictly pre-dissipative at an equilibrium $(x^e, u^e) \in int(\mathbb{X} \times \mathbb{U})$

(ii) The problem has the near equilibrium turnpike property at an equilibrium $(x^e, u^e) \in int(\mathbb{X} \times \mathbb{U})$

(iii) All unobservable eigenvalues μ of A satisfy $|\mu| \neq 1$.

Moreover, if one of these properties holds, then the equilibria in (i) and (ii) coincide. If, in addition, (A, B) stabilizable, then the exponential turnpike property holds.

The role of state constraints

It is evident that the conditions in the state constrained case are significantly less restrictive:

- with bounded state constraints, all unobservable eigenvalues μ of A must satisfy $|\mu| \neq 1$, i.e., all unobservable uncontrolled solutions must converge to 0 or diverge to ∞ exponentially fast
- without bounded state constraints, all unobservable eigenvalues μ of A must satisfy $|\mu| < 1$, i.e., all unobservable uncontrolled solutions must converge to 0 exponentially fast

Back to Example 1

Cost function $\ell(x, u) = u^2$

Dynamics $x^+ = 2x + u$

Constraints $\mathbb{X} = [-2; 2]$, $\mathbb{U} = [-3; 3]$

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Cost function $\quad \ell(x,u) = u^2 \implies Q = C = 0$

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Hence, the turnpike property holds for bounded constraints, but it cannot hold for $\mathbb{X} = \mathbb{R}$

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Hence, the turnpike property holds for bounded constraints, but it cannot hold for $\mathbb{X} = \mathbb{R}$

Indeed, in this case all optimal solutions grow exponentially (unless $x_0 = 0$), because $u \equiv 0$ is clearly the optimal control

Example 3 - Necessity of the conditions

Cost function $\ell(x, u) = u^2$

Dynamics $x^+ = x + u$

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Example 3 - Necessity of the conditions

Cost function $\ell(x, u) = u^2 \implies Q = C = 0$

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The single eigenvalue $\mu = 1$ is unobservable

Hence, the near equilibrium turnpike property does not hold

Indeed, in this case all optimal trajectories are constant, thus the near equilibrium turnpike property at (x^e, u^e) does not hold whenever $x_0 \neq x^e$

The role of state constraints for turnpike phenomena

General principle for bounded constraints:

- The solutions corresponding to eigenvalues $|\mu| > 1$ become unbounded without control action
- The bounded constraints make the unbounded solutions expensive, because we need to counteract using expensive control action
- This forces the optimal trajectories to the turnpike
- Hence, state constraints help to enforce the turnpike property

Discrete-time, $\mathbb{X} = \mathbb{R}^d$, (A, B) stabilizable, $Q = C^T C$

(A, C) is detectable $\iff$ $\forall$ unobservable eigenvalues μ of A: $|\mu| < 1$

$\Updownarrow$ $\qquad\qquad$ $\Updownarrow$

strict dissipativity at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

$\Uparrow$ (without (A, B) stabilizable) $\qquad$ $\Updownarrow$

turnpike property at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

$\Downarrow$ (without $Q = C^T C$)

exponential turnpike at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

Discrete-time, $\mathbb{X} \subset \mathbb{R}^d$ bounded, $Q = C^T C$

$\forall$ unobservable eigenvalues μ of A: $|\mu| \neq 1$

$\Updownarrow$ (without boundedness of $\mathbb{X}$)

strict pre-dissipativity at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

(without boundedness of $\mathbb{X}$) $\Uparrow$ $\Downarrow$ (without $Q = C^T C$)

near equilibrium turnpike at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

$\Downarrow$ (A, B) stabilizable

turnpike property at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$ $\Longrightarrow$ exponential turnpike at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

Conclusions & Outlook

- Connection between

  Turnpike phenomena $\Longleftrightarrow$ Strict dissipativity

  (Including the presence of state and control constraints)

- Necessary and sufficient conditions for turnpike and near equilibrium turnpike properties in terms of

  i) spectral properties of the system matrices,

  ii) the notions of strict dissipativity and the newly introduce strict pre-dissipativity of a system at an equilibrium point.

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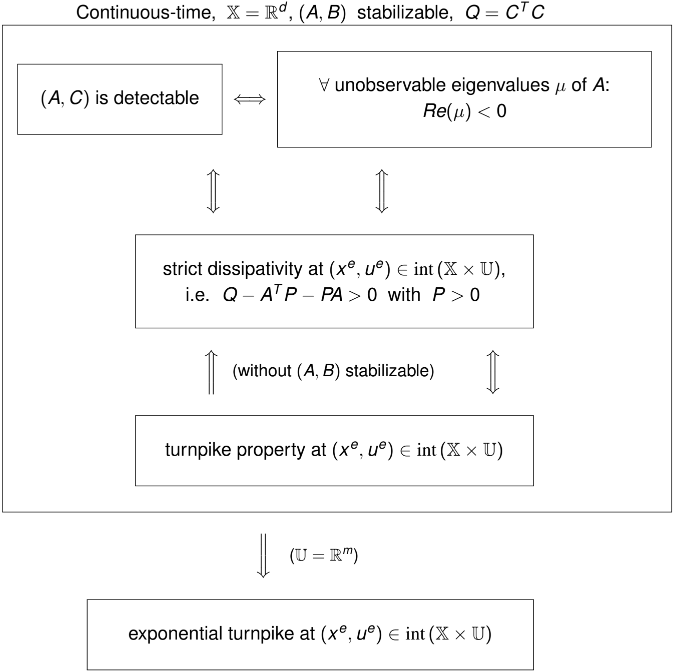
Outlook:

- Analyse these relations for infinite dimensional systems
- Analyse the case of equilibrium $(x^e, u^e) \in \partial(\mathbb{X} \times \mathbb{U})$
- Develop a dynamical system characterization of turnpike phenomena in the presence of constraints

Thank you

for your attention!


Continuous-time, $\mathbb{X} = \mathbb{R}^d$, (A, B) stabilizable, $Q = C^T C$
(A, C) is detectable
$\Longleftrightarrow$
$\forall$ unobservable eigenvalues μ of A: $Re(\mu) < 0$
strict dissipativity at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$, i.e. $Q - A^T P - PA > 0$ with $P > 0$
(without (A, B) stabilizable)
turnpike property at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$
$(\mathbb{U} = \mathbb{R}^m)$
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Continuous-time, $\mathbb{X} \subset \mathbb{R}^d$ bounded, $Q = C^T C$

$\forall$ unobservable eigenvalues μ of A: $Re(\mu) \neq 0$

$\Updownarrow$ (without boundedness of $\mathbb{X}$)

strict pre-dissipativity at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$,
i.e. $Q - A^T P - PA > 0$

(without boundedness of $\mathbb{X}$) $\Uparrow$ $\quad$ $\Downarrow$ (without $Q = C^T C$)

near equilibrium turnpike at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$

((A, B) stabilizable) $\Downarrow$

turnpike property at $(x^e, u^e) \in \operatorname{int}(\mathbb{X} \times \mathbb{U})$